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THIRD BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION I

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INTRODUCTORY REMARKS AND NOTATIONS

We shall deal with the heat potential in the Euclidean plane R^2. Points in R^2 will be denoted by $[x, t]$, $[\xi, \tau]$ etc.

Fix $a, b \in R^1$, $a < b$ and let $\varphi : \langle a, b\rangle \to R^1$ be a continuous function on the interval $\langle a, b\rangle$. Denote

$$E = \{[x, t] \in R^2;\ t \in (a, b),\ x > \varphi(t)\}\ , \tag{0.1}$$

$$K = K_\varphi = \{[x, t] \in R^2;\ t \in \langle a, b\rangle,\ x = \varphi(t)\}\ . \tag{0.2}$$

For $\alpha \in R^1$ let

$$R_\alpha = \{[x, t] \in R^2;\ t < \alpha\}$$

and for $\alpha, \beta \in R^1$, $\alpha < \beta$ we denote

$$R_{\alpha\beta} = R_\beta - \bar{R}_\alpha\ .$$

By $\mathscr{D}$ we denote the class of all infinitely differentiable functions with compact support in R^2; the support of a function ψ will by denoted by spt ψ. For $\alpha \in R^1$ let

$$\mathscr{D}_\alpha = \{\psi \in \mathscr{D};\ \operatorname{spt} \psi \subset R_\alpha\}\ .$$

Let G stand for the heat kernel in R^2, that is, $G(x, t) = 0$ for $t \leqq 0$,

$$G(x, t) = (\pi t)^{-1/2} \exp\left(-\frac{x^2}{4t}\right)$$

for $t > 0$.

By the term measure we shall always mean a finite signed Borel measure usually in R^1. The set of all finite signed Borel measures in $\langle a, b\rangle$ (that is, measures in R^1 with supports contained in $\langle a, b\rangle$) will be denoted by $\mathscr{B}'(\langle a, b\rangle)$ or simply $\mathscr{B}'$. For $\mu \in \mathscr{B}'(\langle a, b\rangle)$ let μ^+, μ^- and $|\mu|$ be the positive, the negative and the total variation (respectively) of the measure μ. Then

$$\mu = \mu^+ - \mu^- , \quad |\mu| = \mu^+ + \mu^- .$$

The set $\mathscr{B}'$ is known to be a Banach space if it is equipped with the norm

$$\|\mu\| = |\mu| (\langle a, b \rangle) .$$

For $\mu \in \mathscr{B}'(\langle a, b \rangle)$ let us define the heat potential U_μ by

$$U_\mu(x, t) = U_\mu^\varphi(x, t) = \int_a^b G(x - \varphi(\tau), \ t - \tau) \, \mathrm{d}\mu(\tau) \tag{0.3}$$

for those $[x, t] \in R^2$ for which the integral in (0.3) exists. The potential U_μ is certainly well defined on the set $R^2 - K$ and solves the heat equation on this set.

One can easily calculate that the following inequalities hold for $\alpha, \beta \in R^1$, $\alpha < \beta$;

$$\iint_{R_{\alpha\beta}} G(x, t) \, \mathrm{d}x \, \mathrm{d}t \leqq 2(\beta - \alpha) , \tag{0.4}$$

$$\iint_{R_{\alpha\beta}} \left| \frac{\partial G}{\partial x} (x, t) \right| \mathrm{d}x \, \mathrm{d}t \leqq \frac{4}{\sqrt{\pi}} \sqrt{(\beta - \alpha)} \tag{0.5}$$

(see also (10), (11) in [15]). Putting

$$E_{\alpha\beta} = E \cap R_{\alpha\beta}$$

we obtain from (0.4), (0.5) that for any $\mu \in \mathscr{B}'(\langle a, b \rangle)$ the following estimates hold:

$$\iint_{E_{\alpha\beta}} |U_\mu(x, t)| \, \mathrm{d}x \, \mathrm{d}t \leqq 2\|\mu\| (\beta - \alpha) , \tag{0.6}$$

$$\iint_{E_{\alpha\beta}} \left| \frac{\partial}{\partial x} U_\mu(x, t) \right| \mathrm{d}x \, \mathrm{d}t \leqq \frac{4}{\sqrt{\pi}} \|\mu\| \sqrt{(\beta - \alpha)} \tag{0.7}$$

(see also (13), (14) in [15]). The validity of these inequalities allows us to define for $\mu \in \mathscr{B}'(\langle a, b \rangle)$ a distribution H_μ on $\mathscr{D}_b$ by

$$\langle \psi, H_\mu \rangle = - \iint_E \left(\frac{\partial U_\mu}{\partial x} \frac{\partial \psi}{\partial x} - U_\mu \frac{\partial \psi}{\partial t} \right) \mathrm{d}x \, \mathrm{d}t , \quad (\psi \in \mathscr{D}_b) . \tag{0.8}$$

Let us suppose for a while that the function φ has a bounded variation on $\langle a, b \rangle$ and that U_μ and $\partial U_\mu / \partial x$ can be extended continuously from E to $\bar{E}$. Then the term $\langle \psi, H_\mu \rangle$ can be expressed in the following way. Let $\Phi(t) = [\varphi(t), t]$ $(t \in \langle a, b \rangle)$,

$$\vec{F} = \left[- \psi U_\mu, - \psi \frac{\partial U_\mu}{\partial x} \right] .$$

In virtue of the fact that the potential U_μ satisfies the heat equation we have

$$\operatorname{rot} \vec{F} = -\frac{\partial \psi}{\partial x}\frac{\partial U_\mu}{\partial x} + \frac{\partial \psi}{\partial t} U_\mu$$

on E. Since, by assumption, $\operatorname{spt} \psi \subset R_b$, $\operatorname{spt} \psi$ is compact, and since $U_\mu(x, t) = 0$ for $t \leqq a$, we obtain from the Green theorem that

$$\langle \psi, H_\mu \rangle = \iint_E \operatorname{rot} \vec{F} \,\mathrm{d}x\,\mathrm{d}t = -\int_a^b \vec{F}\,\mathrm{d}\Phi =$$

$$= \int_a^b \psi(\varphi(t), t)\, U_\mu(\varphi(t), t)\,\mathrm{d}\varphi'(t) + \int_a^b \psi(\varphi(t), t) \frac{\partial U_\mu}{\partial x}(\varphi(t), t)\,\mathrm{d}t$$

($\psi \in \mathscr{D}_b$). In this sense one can view the distribution H_μ as a weak characterization of the term

$$\frac{\partial U_\mu}{\partial x} + U_\mu \,.\, \lambda_0$$

considered on K, where λ_0 is a measure on $\langle a, b\rangle$ derived from the function φ ($\mathrm{d}\lambda_0(t) = \mathrm{d}\varphi(t)$; note that any measure on $\langle a, b\rangle$ can be considered a measure in R^2 with support contained in the set K – see [7], for instance).

In the following, $\mathscr{C} = \mathscr{C}(\langle a, b\rangle)$ and $\mathscr{C}_0^{\sim} = \mathscr{C}_0^{\sim}(\langle a, b\rangle)$ will denote the spaces of all continuous functions on $\langle a, b\rangle$ and of all continuous functions f on $\langle a, b\rangle$ such that $f(b) = 0$, respectively. Both $\mathscr{C}$ and $\mathscr{C}_0^{\sim}$ will be endowed with the supremum norm (these spaces are then Banach spaces). Then $\mathscr{B}'(\langle a, b\rangle)$ is the dual space of the space $\mathscr{C}(\langle a, b\rangle)$. The dual space of the space $\mathscr{C}_0^{\sim}(\langle a, b\rangle)$ is the space

$$\mathscr{B}'_0 = \mathscr{B}'_0(\langle a, b\rangle) = \{\mu \in \mathscr{B}'(\langle a, b\rangle);\ \mu(\{b\}) = 0\}\,.$$

We shall show in what follows that under a certain condition on φ (namely a geometrical condition on the boundary of E) the distribution H_μ can be represented by a measure from $\mathscr{B}'_0$ and the equation

$$H_\mu = \nu \tag{0.9}$$

(where ν is given and μ unknown) has a unique solution in $\mathscr{B}'_0$ for each $\nu \in \mathscr{B}'_0$.

For a given $\lambda \in \mathscr{B}'_0$ we shall further define an operator A_μ,

$$A_\mu = H_\mu + L_\mu\,,$$

where

$$\langle \psi, L_\mu \rangle = \int_a^b \psi(\varphi(t), t)\, U_\mu(\varphi(t), t)\,\mathrm{d}\lambda(t)$$

($\psi \in \mathscr{D}_b$). It will be also shown that under certain conditions on the function φ and on the measure λ the operator A_μ can be represented by a measure from $\mathscr{B}'_0$ and that the equation

$$(0.10) \qquad A_\mu = \nu$$

has a unique solution in $\mathscr{B}'_0$ for each $\nu \in \mathscr{B}'_0$. If μ is the solution of (0.9) or of (0.10) then the potential U_μ solves a certain third boundary value problem of the heat equation on the set E with a prescribed condition on K.

1. THE OPERATORS W AND $\tilde{W}$

Before starting the study of the operator H let us consider the operators $\tilde{W}$, W. For $[\xi, \tau] \in R^2$, $\psi \in \mathscr{D}$ let us define $\tilde{W}\psi(\xi, \tau)$ by

$$(1.1) \qquad \tilde{W}\psi(\xi, \tau) =$$

$$= -\iint_E \left(\frac{\partial}{\partial x} G(x - \xi, t - \tau) \frac{\partial \psi}{\partial x}(x, t) - G(x - \xi, t - \tau) \frac{\partial \psi}{\partial t}(x, t)\right) \mathrm{d}x\, \mathrm{d}t\,.$$

Let us introduce the following notation. Let $\eta > 0$, $r > 0$, $\tau \geqq a$, $r + \tau \leqq b$. Then we put

$$\boldsymbol{H}^1_{\xi,\tau}(\eta, r) = \left\{\left[\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right];\ \varrho > 0,\ \frac{\varrho^2}{4\eta} < r\right\},$$

$$\boldsymbol{H}^{-1}_{\xi,\tau}(\eta, r) = \left\{\left[\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right];\ \varrho > 0,\ \frac{\varrho^2}{4\eta} < r\right\}.$$

A point $a \in \boldsymbol{H}^\sigma_{\xi,\tau}(\eta, r)$ ($\sigma = \pm 1$) is called a hit of $\boldsymbol{H}^\sigma_{\xi,\tau}(\eta, r)$ on E if, for each $\varepsilon > 0$,

$$H_1(\boldsymbol{H}^\sigma_{\xi,\tau}(\eta, r) \cap E \cap \Omega_\varepsilon(a)) > 0\,,$$

$$H_1((\boldsymbol{H}^\sigma_{\xi,\tau}(\eta, r) - E) \cap \Omega_\varepsilon(a)) > 0$$

(where H_1 is the linear Hasudorff measure in R^2, $\Omega_\varepsilon(a) = \{z \in R^2;\ |z - a| < \varepsilon\}$). The number (finite or infinite) of the hits of $\boldsymbol{H}^\sigma_{\xi,\tau}(\eta, r)$ on E is denoted by $\tilde{n}^\sigma_{\xi,\tau}(\eta, r)$. Further, let

$$\tilde{n}_{\xi,\tau}(\eta, r) = \tilde{n}^1_{\xi,\tau}(\eta, r) + \tilde{n}^{-1}_{\xi,\tau}(\eta, r)\,.$$

1.1. Lemma. *Let $r > 0$, $\tau \geqq a$, $r + \tau \leqq b$, $\xi \in R^1$. Then the function $\tilde{n}_{\xi,\tau}(\eta, r)$ is a Baire function of the variable η on the interval $(0, \infty)$. If we denote*

$$(1.2) \qquad \tilde{v}^r(\xi, \tau) = \int_0^\infty \frac{1}{2\sqrt{\eta}} e^{-\eta}\, \tilde{n}_{\xi,\tau}(\eta, r)\, \mathrm{d}\eta = \int_0^\infty e^{-\alpha^2}\, \tilde{n}_{\xi,\tau}(\alpha^2, r)\, \mathrm{d}\alpha\,,$$

$$\mathscr{D}_1 = \{\psi \in \mathscr{D}_{\tau+r};\ [\xi, \tau] \notin \operatorname{spt} \psi,\ \|\psi\| \leqq 1\}\,,$$

then

$$\sup\{\tilde{W}\psi(\xi,\tau);\ \psi\in\mathscr{D}_1\} = \frac{2}{\sqrt{\pi}}\,\tilde{v}^r(\xi,\tau)\,. \tag{1.3}$$

Proof. The proof is analogous to those of Lemma 1.6 in [15] and Proposition 1.8 in [15].

Let $a \leqq \tau < b$, $\xi \in R^1$ and let $E_\tau = E - \bar{R}_\tau$,

$$E_\tau^+ = \{[x,t]\in E_\tau;\ x > \xi\}\,,\quad E_\tau^- = \{[x,t]\in E_\tau;\ x < \xi\}\,.$$

For $\psi \in \mathscr{D}$ we then have

$$\tilde{W}\psi(\xi,\tau) =$$

$$= \iint_{E_\tau^+} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)\left\{\frac{x-\xi}{2\sqrt{(\pi(t-\tau)^3)}}\frac{\partial\psi}{\partial x}(x,t) + \frac{1}{\sqrt{(\pi(t-\tau))}}\frac{\partial\psi}{\partial t}(x,t)\right\}\mathrm{d}x\,\mathrm{d}t +$$

$$+ \iint_{E_\tau^-} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)\left\{\frac{x-\xi}{2\sqrt{(\pi(t-\tau)^3)}}\frac{\partial\psi}{\partial x}(x,t) + \frac{1}{\sqrt{(\pi(t-\tau))}}\frac{\partial\psi}{\partial t}(x,t)\right\}\mathrm{d}x\,\mathrm{d}t =$$

$$= I_1 + I_2\,.$$

In the integral I_1 we employ the change of variables

$$\frac{(x-\xi)^2}{4(t-\tau)} = \eta\,,\quad x - \xi = \varrho$$

(that is,

$$x = \xi + \varrho\,,\quad t = \tau + \frac{\varrho^2}{4\eta}\,;$$

let us denote this mapping by S^+). In the integral I_2 we employ the change of variables

$$\frac{(x-\xi)^2}{4(t-\tau)} = \eta\,,\quad x - \xi = -\varrho$$

(that is,

$$x = \xi - \varrho\,,\quad t = \tau + \frac{\varrho^2}{4\eta}\,;$$

let us denote this mapping by S^-). If we write

$$E_+^* = (S^+)^{-1}(E_\tau^+)\,,\quad E_-^* = (S^-)^{-1}(E_\tau^-)$$

then

$$E_+^* = \left\{[\varrho,\eta];\ \varrho > 0,\ \eta > 0,\ \frac{\varrho^2}{4\eta} < b - \tau,\ \varphi\left(\tau + \frac{\varrho^2}{4\eta}\right) < \xi + \varrho\right\},$$

$$E_-^* = \left\{[\varrho,\eta];\ \varrho > 0,\ \eta > 0,\ \frac{\varrho^2}{4\eta} < b - \tau,\ \varphi\left(\tau + \frac{\varrho^2}{4\eta}\right) < \xi - \varrho\right\}.$$

Denoting further

$$\mathscr{U}_\eta^+ = \left\{\varrho;\ 0 < \varrho < 2\sqrt{(\eta(b-\tau))},\ \varphi\left(\tau + \frac{\varrho^2}{4\eta}\right) < \xi + \varrho\right\},$$

$$\mathscr{U}_\eta^- = \left\{\varrho;\ 0 < \varrho < 2\sqrt{(\eta(b-\tau))},\ \varphi\left(\tau + \frac{\varrho^2}{4\eta}\right) < \xi - \varrho\right\}$$

we obtain by the change of variables and by the Fubini theorem

$$I_1 = \iint_{E_+^*} \frac{1}{\sqrt{(\pi\eta)}} e^{-\eta}\left(\frac{\partial\psi}{\partial x}\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right) + \frac{\varrho}{2\eta}\frac{\partial\psi}{\partial t}\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right) \mathrm{d}\varrho\, \mathrm{d}\eta =$$

$$= \iint_{E_+^*} \frac{1}{\sqrt{(\pi\eta)}} e^{-\eta} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\, \mathrm{d}\eta =$$

$$= \int_0^\infty \left\{\frac{1}{\sqrt{(\pi\eta)}} e^{-\eta} \int_{\mathscr{U}_\eta^+} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\right\} \mathrm{d}\eta\,,$$

$$I_2 = \iint_{E_-^*} \frac{1}{\sqrt{(\pi\eta)}} e^{-\eta}\left(-\frac{\partial\psi}{\partial x}\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right) + \frac{\varrho}{2\eta}\frac{\partial\psi}{\partial t}\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right) \mathrm{d}\varrho\, \mathrm{d}\eta =$$

$$= \iint_{E_-^*} \frac{1}{\sqrt{(\pi\eta)}} e^{-\eta} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\, \mathrm{d}\eta =$$

$$= \int_0^\infty \left\{\frac{1}{\sqrt{(\pi\eta)}} e^{-\eta} \int_{\mathscr{U}_\eta^-} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\right\} \mathrm{d}\eta\,.$$

Now we can conclude that (if $\tau \in \langle a, b)$, $\xi \in R^1$, $\psi \in \mathscr{D}$)

$$\text{(1.4)}\qquad \tilde{W}\psi(\xi, \tau) = \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}} e^{-\eta} \left\{\int_{\mathscr{U}_\eta^+} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho + \right.$$

$$\left. + \int_{\mathscr{U}_\eta^-} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\right\} \mathrm{d}\eta\,.$$

It is easy to verify that

$$\sup_{\psi\in\mathscr{D}_1}\left\{\int_{\mathscr{U}_\eta^+} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho + \int_{\mathscr{U}_\eta^-} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\right\} =$$

$$= \sup_{\psi\in\mathscr{D}_1}\left\{\int_{\mathscr{U}_\eta^+} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi + \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\right\} + \sup_{\psi\in\mathscr{D}_1}\left\{\int_{\mathscr{U}_\eta^-} \frac{\partial}{\partial\varrho}\left[\psi\left(\xi - \varrho, \tau + \frac{\varrho^2}{4\eta}\right)\right] \mathrm{d}\varrho\right\}.$$

Now we could complete the proof in a way quite similar to the proof of Proposition 1.8 in [15] (see also Lemma 1.3 in [39]). The crucial point of the proof consists in using Lemma 1.3 from [15] (we also use 1.9 from [13]).

1.2. Remark. We have defined $\tilde{v}^r(\xi, \tau)$ in the case $\tau \in \langle a, b)$, $r > 0$, $\tau + r \leqq \leqq b$ ($\xi \in R^1$). If $\tau + r > b$ (including the case $r = +\infty$) then we write

$$\tilde{v}^r(\xi, \tau) = \tilde{v}^{b-\tau}(\xi, \tau) .$$

In the case $\tau \geqq b$ we put

$$\tilde{v}^r(\xi, \tau) = 0$$

for each $r > 0$. Further, we shall also write $\tilde{v}(\xi, \tau) = \tilde{v}^\infty(\xi, \tau)$ ($\tau \geqq a$). The term $\tilde{v}(\xi, \tau)$ is called the adjoint parabolic variation of E (or of the function φ) at the point $[\xi, \tau]$. We shall see below what is the connection between the adjoint parabolic variation and the parabolic variation defined in [3].

1.3. Let us introduce the following notation. Let $[\xi, \tau] \in R^2$, $\tau \in \langle a, b)$ and suppose that $\tilde{v}(\xi, \tau) < \infty$. Then

$$\tilde{n}_{\xi,\tau}(\eta, b - \tau) < \infty$$

for almost all $\eta > 0$.

For $\eta > 0$ with $\tilde{n}_{\xi,\tau}(\eta, b - \tau) < \infty$, for $0 < \varrho < 2\sqrt{(\eta(b - \tau))}$ and for $\varkappa = \pm 1$ let us put

$$s^\varkappa_{\xi,\tau}(\varrho, \eta) = \sigma \quad (= \pm 1)$$

if there is a $\delta > 0$ such that

$$H_1\left(\left\{\left[\xi + \varkappa(\varrho + \sigma u),\ \tau + \frac{(\varrho + \sigma u)^2}{4\eta}\right];\ u \in (0, \delta)\right\} \cap E\right) = 0 ,$$

$$H_1\left(\left\{\left[\xi + \varkappa(\varrho - \sigma u),\ \tau + \frac{(\varrho - \sigma u)^2}{4\eta}\right];\ u \in (0, \delta)\right\} - E\right) = 0 .$$

Further, put

$$s^\varkappa_{\xi,\tau}(0, \eta) = -1$$

if there is a $\delta > 0$ such that

$$H_1\left(\left\{\left[\xi + \varkappa u,\ \tau + \frac{u^2}{4\eta}\right];\ u \in (0, \delta)\right\} - E\right) = 0 .$$

In all the other cases put

$$s^\varkappa_{\xi,\tau}(\varrho, \eta) = 0 , \quad (\varrho \geqq 0,\ \eta > 0) .$$

Let f be a bounded Baire function on $\langle a, b\rangle$. Then for $\eta > 0$ we define

$$\Sigma_f(\eta) = \sum_{\varkappa = \pm 1} \left(\sum_{\varrho > 0} f\left(\tau + \frac{\varrho^2}{4\eta}\right) s^\varkappa_{\xi,\tau}(\varrho, \eta)\right) . \tag{1.5}$$

For $\psi \in \mathscr{D}_b$ we define

$$\Sigma_\psi(\eta) = \Sigma_f(\eta) , \tag{1.6}$$

where

$$f(t) = \psi(\varphi(t), t), \quad (t \in \langle a, b\rangle).$$

Further, we define for $\psi \in \mathscr{D}_b$ (and for the above given $[\xi, \tau]$)

$$(1.7) \qquad \tilde{\Sigma}_\psi(\eta) = \Sigma_\psi(\eta) + \psi(\xi, \tau)\,(s^1_{\xi,\tau}(0, \eta) + s^{-1}_{\xi,\tau}(0, \eta)).$$

It follows from (1.4) that for $\psi \in \mathscr{D}_b$

$$(1.8) \qquad \tilde{W}\psi(\xi, \tau) = \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}}\, e^{-\eta}\, \tilde{\Sigma}_\psi(\eta)\, \mathrm{d}\eta.$$

Especially, it is seen from here that the function $\tilde{\Sigma}_\psi(\eta)$ is a measurable function of the variable η on the interval $(0, \infty)$ (if $\psi \in \mathscr{D}_b$) and also that $\Sigma_\psi(\eta)$ is a measurable function on $(0, \infty)$ if $\psi \in \mathscr{D}_b$ is such that $\psi(\xi, \tau) = 0$. Since the term $\Sigma_f(\eta)$ is independent of $f(\tau)$ and $f(b)$, we can obtain (by means of limit processes) that the function $\Sigma_f(\eta)$ is a measurable function of the variable η on $(0, \infty)$ for any bounded Baire function f on $\langle a, b\rangle$. It follows from (1.7) that also the term

$$(s^1_{\xi,\tau}(0, \eta) + s^{-1}_{\xi,\tau}(0, \eta))$$

is a measurable function of the variable η on $(0, \infty)$.

Let now f be a Baire bounded function on $\langle a, b\rangle$ (the set of all bounded Baire functions on $\langle a, b\rangle$ will be denoted by $\mathscr{B} = \mathscr{B}(\langle a, b\rangle)$) and let $|f| \leqq k$ (say). Then certainly

$$|\Sigma_f(\eta)| \leqq k\, \tilde{n}_{\xi,\tau}(\eta, b - \tau).$$

As $\tilde{v}(\xi, \tau)$ is supposed to be finite the integral

$$(1.9) \qquad Wf(\xi, \tau) = \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}}\, e^{-\eta}\, \Sigma_f(\eta)\, \mathrm{d}\eta$$

converges and

$$(1.10) \qquad |Wf(\xi, \tau)| \leqq \frac{2}{\sqrt{\pi}}\, k\, \tilde{v}(\xi, \tau).$$

We obtain from (1.3) and (1.10) that

$$(1.11) \qquad \sup\{Wf(\xi, \tau);\ f \in \mathscr{C}_0^{\sim}(\langle a, b\rangle),\ \|f\| \leqq 1\} = \sup\{Wf(\xi, \tau);\ f \in \mathscr{B},\ \|f\| \leqq 1\} = \frac{2}{\sqrt{\pi}}\, \tilde{v}(\xi, \tau).$$

Moreover, we find that if $f_n \in \mathscr{B}$, $|f_n| \leqq k$, $f_n \to f$ pointwise on $\langle a, b\rangle$, then

$$(1.12) \qquad \lim_{n\to\infty} Wf_n(\xi, \tau) = Wf(\xi, \tau),$$

as

$$\left|\Sigma_{f_n}(\eta)\right| \leqq k\, \tilde{n}_{\xi,\tau}(\eta, b - \tau)$$

and (1.12) follows from the Lebesgue theorem.

Before finding another expression for $Wf(\xi, \tau)$ let us introduce the following assertion.

1.4. Proposition. *Let* $[x_i, t_i] \in R^2$ $(i = 1, 2, 3)$ *be the points in the general position (that is,* $[x_i, t_i]$ *are not situated on a single line) and such that* $t_i \in \langle a, b)$ $(i = 1, 2, 3)$. *If*

$$b > t_0 > \max\{t_1, t_2, t_3\}$$

and if

$$\tilde{v}(x_i, t_i) < \infty\,, \quad (i = 1, 2, 3)$$

then

$$\operatorname{var}[\varphi; \langle t_0, b\rangle] < \infty\,. \tag{1.13}$$

Proof. We shall not prove this assertion in detail since the proof is quite analogous to that of Proposition 2.3 from [39].

Let $q \in R^1$, $q > \max\{\varphi(t); t \in \langle a, b\rangle\}$. Using the same arguments as in the proof of Proposition 2.3 from [39] one can show that the perimeter of the set

$$Q_{t_0} = \{[x, t] \in E - \bar{R}_{t_0};\ x < q\}$$

is finite. But this implies that (1.13) is fulfilled.

1.5. We shall suppose henceforth that the condition

$$\operatorname{var}[\varphi; \langle a, b\rangle] < \infty \tag{1.14}$$

is fulfilled.

1.6. Now we shall show another way how to express the term $Wf(\xi, \tau)$. For this purpose we shall recall some definitions from [3] and introduce the following notation.

We shall denote $\tilde{\xi} = -\xi$, $\tilde{\tau} = -\tau$, $\tilde{t} = -t$,

$$\tilde{\varphi}(\tilde{t}) = -\varphi(t)\,, \quad \tilde{f}(\tilde{t}) = f(t)\,, \quad (\tilde{t} \in \langle -b, -a\rangle = \langle \tilde{b}, \tilde{a}\rangle,\ f \in \mathscr{B}(\langle a, b\rangle))\,,$$

$$\tilde{K} = \{[\tilde{\varphi}(\tilde{t}), \tilde{t}];\ \tilde{t} \in \langle \tilde{b}, \tilde{a}\rangle\}\,.$$

The notion of the parabolic variation of the function φ (or of the set K) has been introduced in [3] (Definition 1.1). Now we shall apply the definition to the function $\tilde{\varphi}$ (that is, to the set $\tilde{K}$), denoting the parabolic variation of $\tilde{K}$ by $V_{\tilde{K}}$ in accordance with the notation from [3]. Under our notation we get (see the equality (1.10) from [3]) that for $[\tilde{\xi}, \tilde{\tau}] \in R^2$, $\tilde{\tau} \in (\tilde{b}, \tilde{a}\rangle$,

$$(1.15)\qquad V_{\tilde{K}}(\tilde{\xi}, \tilde{\tau}) = \int_{\tilde{b}}^{\tilde{\tau}} \exp\left(-\frac{(\tilde{\xi} - \tilde{\varphi}(\tilde{t}))^2}{4(\tilde{\tau} - \tilde{t})}\right) \mathrm{d}\,\mathrm{var}_{\tilde{t}} \left[\frac{\tilde{\xi} - \tilde{\varphi}(\tilde{t})}{2\sqrt{(\tilde{\tau} - \tilde{t})}}\right].$$

Further, an operator T has been defined in [3] which we shall again consider here *not* for the function φ but for the function $\tilde{\varphi}$ (that is, not for the set K but for the set $\tilde{K}$). By $T\tilde{f}(\tilde{\xi}, \tilde{\tau})$ for $\tilde{f} \in \mathscr{B}(\langle \tilde{b}, \tilde{a} \rangle)$, $[\tilde{\xi}, \tilde{\tau}] \in R^2$, $\tilde{\tau} \in (\tilde{b}, \tilde{a}\rangle$ we thus mean (see [3], Definition 2.1)

$$(1.16)\qquad T\tilde{f}(\tilde{\xi}, \tilde{\tau}) = \frac{2}{\sqrt{\pi}} \int_{\tilde{b}}^{\tilde{\tau}} \tilde{f}(\tilde{t}) \exp\left(-\frac{(\tilde{\xi} - \tilde{\varphi}(\tilde{t}))^2}{4(\tilde{\tau} - \tilde{t})}\right) \mathrm{d}_{\tilde{t}} \frac{\tilde{\xi} - \tilde{\varphi}(\tilde{t})}{2\sqrt{(\tilde{\tau} - \tilde{t})}}$$

(if this integral converges, which, for instance, is fulfilled for each $f \in \mathscr{B}(\langle \tilde{b}, \tilde{a} \rangle)$ provided $V_{\tilde{K}}(\tilde{\xi}, \tilde{\tau}) < \infty$).

1.7. Lemma. *Let* $[\xi, \tau] \in R^2$, $\tau \in \langle a, b)$. *Then*

$$(1.17)\qquad \tilde{v}(\xi, \tau) = V_{\tilde{K}}(\tilde{\xi}, \tilde{\tau}).$$

If, in addition, $\tilde{v}(\xi, \tau) < \infty$ *then for any function* $f \in \mathscr{B}(\langle a, b \rangle)$

$$(1.18)\qquad Wf(\xi, \tau) = \frac{2}{\sqrt{\pi}} \int_{\tau}^{b} f(t) \exp\left(-\frac{(\varphi(t) - \xi)^2}{4(t - \tau)}\right) \mathrm{d}_t \frac{\varphi(t) - \xi}{2\sqrt{(t - \tau)}} = -T\tilde{f}(\tilde{\xi}, \tilde{\tau}).$$

Proof. Let $f \in \mathscr{B}(\langle a, b \rangle)$ and suppose first that f is of the form

$$f(t) = \psi(\varphi(t), t), \quad (t \in \langle a, b \rangle),$$

where $\psi \in \mathscr{D}_b$, $[\xi, \tau] \notin \operatorname{spt} \psi$. Denote

$$\vec{F} = \left[-G\psi, -\frac{\partial G}{\partial x}\psi\right], \quad \Phi(t) = [\varphi(t), t].$$

Then

$$(1.19)\qquad Wf(\xi, \tau) = \tilde{W}\psi(\xi, \tau) = \iint_E \operatorname{rot} \vec{F} \,\mathrm{d}x\,\mathrm{d}t =$$

$$= \int_a^b G(\varphi(t) - \xi, t - \tau)\, \psi(\varphi(t), t) \,\mathrm{d}\varphi(t) + \int_a^b \frac{\partial G}{\partial x}(\varphi(t) - \xi, t - \tau)\, \psi(\varphi(t), t) \,\mathrm{d}t =$$

$$= \int_\tau^b f(t) \frac{1}{\sqrt{(\pi(t - \tau))}} \exp\left(-\frac{(\varphi(t) - \xi)^2}{4(t - \tau)}\right) \mathrm{d}\varphi(t) -$$

$$- \int_\tau^b f(t) \frac{\varphi(t) - \xi}{2\sqrt{(\pi(t - \tau)^3)}} \exp\left(-\frac{(\varphi(t) - \xi)^2}{4(t - \tau)}\right) \mathrm{d}t =$$

$$= \frac{2}{\sqrt{\pi}} \int_\tau^b f(t) \exp\left(-\frac{(\varphi(t) - \xi)^2}{4(t - \tau)}\right) \mathrm{d}_t \frac{\varphi(t) - \xi}{2\sqrt{(t - \tau)}} =$$

$$= -\frac{2}{\sqrt{\pi}} \int_{\tilde{b}}^{\tilde{\tau}} \tilde{f}(\tilde{t}) \exp\left(-\frac{(\tilde{\xi} - \tilde{\varphi}(\tilde{t}))^2}{4(\tilde{\tau} - \tilde{t})}\right) \mathrm{d}_{\tilde{t}} \frac{\tilde{\xi} - \tilde{\varphi}(\tilde{t})}{2\sqrt{(\tilde{\tau} - \tilde{t})}} = -T\tilde{f}(\tilde{\xi}, \tilde{\tau}).$$

(1.19) is valid without the assumption $\tilde{v}(\xi, \tau) < \infty$ (f being of the mentioned special form).

Now we obtain the equality (1.17) from (1.3), (1.19) and [3] (2.9). Furthermore, we see that the equality (1.18) holds for each $f \in \mathscr{B}(\langle a, b \rangle)$ of the form $f(t) = \psi(\varphi(t), t)$ $(t \in \langle a, b \rangle)$, $\psi \in \mathscr{D}_b$, $[\xi, \tau] \notin \operatorname{spt} \psi$. If we suppose $\tilde{v}(\xi, \tau) < \infty$ then (1.12) holds for any bounded sequence of functions $f_n \in \mathscr{B}(\langle a, b \rangle)$ such that $f_n \to f$ (pointwise on $\langle a, b \rangle$). But as (1.17) holds we have $V_R(\tilde{\xi}, \tilde{\tau}) < \infty$ and we can use the same limit process in the second and third terms in (1.18) (as in the first term in (1.18)) and thus (1.18) is valid for each $f \in \mathscr{B}(\langle a, b \rangle)$.

1.8. From (1.8), (1.9) and (1.7) we obtain the following relation between Wf and $\tilde{W}f$. Let $\psi \in \mathscr{D}_b$, $f \in \mathscr{C}(\langle a, b \rangle)$ be such that $f(t) = \psi(\varphi(t), t)$ $(t \in \langle a, b \rangle)$. Then (for $[\xi, \tau] \in R^2$, $\tau \in \langle a, b)$)

$$
\begin{aligned}
(1.20) \qquad Wf(\xi, \tau) &= \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}}\, e^{-\eta}\, \Sigma_f(\eta)\, \mathrm{d}\eta = \\
&= \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}}\, e^{-\eta}[\tilde{\Sigma}_\psi(\eta) - \psi(\xi, \tau)\,(s^1_{\xi,\tau}(0, \eta) + s^{-1}_{\xi,\tau}(0, \eta))]\, \mathrm{d}\eta = \\
&= \tilde{W}\psi(\xi, \tau) - \psi(\xi, \tau) \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}}\, e^{-\eta}(s^1_{\xi,\tau}(0, \eta) + s^{-1}_{\xi,\tau}(0, \eta))\, \mathrm{d}\eta = \\
&= \tilde{W}\psi(\xi, \tau) + 2\psi(\xi, \tau)\, \mathscr{P}^{\sim}_E(\xi, \tau),
\end{aligned}
$$

where we put

$$
(1.21) \qquad \mathscr{P}^{\sim}_E(\xi, \tau) = -\int_0^\infty \frac{1}{2\sqrt{(\pi\eta)}}\, e^{-\eta}(s^1_{\xi,\tau}(0, \eta) + s^{-1}_{\xi,\tau}(0, \eta))\, \mathrm{d}\eta\,.
$$

The term $\mathscr{P}^{\sim}_E(\xi, \tau)$ is called the parabolic density of the set E at the point $[\xi, \tau]$. If $[\xi, \tau] \in R^2 - \bar{E}$ then $s^1_{\xi,\tau}(0, \eta) = s^{-1}_{\xi,\tau}(0, \eta) = 0$ for every $\eta > 0$ and thus $\mathscr{P}^{\sim}_E(\xi, \tau) = 0$. In the case $[\xi, \tau] \in E$ we have $s^1_{\xi,\tau}(0, \eta) = s^{-1}_{\xi,\tau}(0, \eta) = -1$ for every $\eta > 0$ and

$$
\mathscr{P}^{\sim}_E(\xi, \tau) = \int_0^\infty \frac{1}{\sqrt{(\pi\eta)}}\, e^{-\eta}\, \mathrm{d}\eta = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\alpha^2}\, \mathrm{d}\alpha = 1\,.
$$

Let us now show how the parabolic density $\mathscr{P}^{\sim}_E(\xi, \tau)$ can be expressed in the case $\tau \in \langle a, b)$, $\xi = \varphi(\tau)$. Let g be a function on $^*R^1 = R^1 \cup \{-\infty, +\infty\}$ such that $g(-\infty) = 0$,

$$
g(t) = \int_{-\infty}^t e^{-\alpha^2}\, \mathrm{d}\alpha \quad \text{for} \quad t > -\infty\,.
$$

Suppose that $\tilde{v}(\xi, \tau) < \infty$. Then $V_R(\tilde{\xi}, \tilde{\tau}) < \infty$ as well and the limit

$$
\alpha_0(\tau) = \lim_{t \to \tau+} \frac{\varphi(t) - \xi}{2\sqrt{(t - \tau)}} = \lim_{\tilde{t} \to \tilde{\tau}-} \frac{\tilde{\xi} - \tilde{\varphi}(\tilde{t})}{2\sqrt{(\tilde{\tau} - \tilde{t})}}
$$

exists (see [3], Remark 2.2; the limit can also take the values $\pm\infty$). Let us show the following simple assertion.

1.9. Proposition. *Let* $\tau \in \langle a, b)$, $\xi = \varphi(\tau)$ *and suppose that* $\tilde{v}(\xi, \tau) < \infty$. *Then*

$$\mathscr{P}_E^{\sim}(\xi, \tau) = 1 - \frac{1}{\sqrt{\pi}}\, g(\alpha_0(\tau))\,. \tag{1.22}$$

Proof. Let us distinguish the following three cases:

$$1)\quad \alpha_0(\tau) = +\infty\,, \qquad 2)\quad \alpha_0(\tau) = -\infty., \qquad 3)\quad \alpha_0(\tau) \in R^1\,.$$

The following result holds in the case 1): for any $k > 0$ there is a $\delta > 0$ such that, for each $t \in (\tau, \tau + \delta)$,

$$\frac{\varphi(t) - \xi}{2\sqrt{(t - \tau)}} > k\,, \quad \text{that is}\,, \quad \varphi(t) > \xi + 2k\sqrt{(t - \tau)}\,.$$

It follows that $s^1_{\xi,\tau}(0, \eta) = s^{-1}_{\xi,\tau}(0, \eta) = 0$ for each $\eta > 0$, that is $\mathscr{P}_E^{\sim}(\xi, \tau) = 0$.

In the case 2) we have: for any $k < 0$ there is a $\delta > 0$ such that, for each $t \in \in (\tau, \tau + \delta)$,

$$\frac{\varphi(t) - \xi}{2\sqrt{(t - \tau)}} < k\,, \quad \text{that is}\,, \quad \varphi(t) < \xi + 2k\sqrt{(t - \tau)}\,.$$

We see that $s^1_{\xi,\tau}(0, \eta) = s^{-1}_{\xi,\tau}(0,\eta) = -1$ for each $\eta > 0$ and thus $\mathscr{P}_E^{\sim}(\xi, \tau) = 1$.

Suppose now that the case 3) takes place. Then we have: for each $\varepsilon > 0$ there is a $\delta > 0$ such that, for each $t \in (\tau, \tau + \delta)$,

$$\alpha_0(\tau) - \varepsilon < \frac{\varphi(t) - \xi}{2\sqrt{(t - \tau)}} < \alpha_0(\tau) + \varepsilon\,,$$

that is,

$$\xi + (\alpha_0(\tau) - \varepsilon)\, 2\sqrt{(t - \tau)} < \varphi(t) < \xi + (\alpha_0(\tau) + \varepsilon)\, 2\sqrt{(t - \tau)}\,. \tag{1.23}$$

If $\alpha_0(\tau) = 0$ then we obtain that $s^1_{\xi,\tau}(0, \eta) = -1$, $s^{-1}_{\xi,\tau}(0, \eta) = 0$ for each $\eta > 0$ and thus $\mathscr{P}_E^{\sim}(\xi, \tau) = \frac{1}{2}$.

Now let $\alpha_0(\tau) > 0$. We easily see that $s^{-1}_{\xi,\tau}(0, \eta) = 0$ for each $\eta > 0$. Let $0 < \varepsilon < < \alpha_0(\tau)$ and put

$$\varrho = (\alpha_0(\tau) - \varepsilon)\, 2\sqrt{(t - \tau)}\,; \quad \text{then} \quad t = \tau + \frac{\varrho^2}{4(\alpha_0(\tau) - \varepsilon)^2}\,.$$

The first inequality from (1.23) takes the following form:

$$\xi + \varrho < \varphi\left(\tau + \frac{\varrho^2}{4(\alpha_0(\tau) - \varepsilon)^2}\right).$$

It follows that

$$s^1_{\xi,\tau}(0, \eta) = 0 \quad \text{for} \quad \eta \leqq (\alpha_0(\tau) - \varepsilon)^2 .$$

Similarly we get

$$s^1_{\xi,\tau}(0, \eta) = -1 \quad \text{for} \quad \eta \geqq (\alpha_0(\tau) + \varepsilon)^2 .$$

As $\varepsilon > 0$ was arbitrary we have

$$s^1_{\xi,\tau}(0, \eta) = 0 \quad \text{for} \quad 0 < \eta < (\alpha_0(\tau))^2 ,$$
$$s^1_{\xi,\tau}(0, \eta) = -1 \quad \text{for} \quad \eta > (\alpha_0(\tau))^2 .$$

Now it is seen that

$$\mathscr{P}_E^{\sim}(\xi, \tau) = \int_{(\alpha_0(\tau))^2}^{\infty} \frac{1}{2\sqrt{(\pi\eta)}} e^{-\eta} \, d\eta = \frac{1}{\sqrt{\pi}} \int_{\alpha_0(\tau)}^{\infty} e^{-\alpha^2} \, d\alpha .$$

In the case $\alpha_0(\tau) < 0$ we similarly get that $s^1_{\xi,\tau}(0, \eta) = -1$ for each $\eta > 0$,

$$s^{-1}_{\xi,\tau}(0, \eta) = -1 \quad \text{for} \quad 0 < \eta < (\alpha_0(\tau))^2 ,$$
$$s^{-1}_{\xi,\tau}(0, \eta) = 0 \quad \text{for} \quad \eta > (\alpha(\tau))^2$$

and hence

$$\mathscr{P}_E^{\sim}(\xi, \tau) = \int_0^{\infty} \frac{1}{2\sqrt{(\pi\eta)}} e^{-\eta} \, d\eta + \int_0^{(\alpha_0(\tau))^2} \frac{1}{2\sqrt{(\pi\eta)}} e^{-\eta} \, d\eta = \frac{1}{\sqrt{\pi}} \int_{\alpha_0(\tau)}^{\infty} e^{-\alpha^2} \, d\alpha .$$

In any case we thus have

$$\mathscr{P}_E^{\sim}(\xi, \tau) = \frac{1}{\sqrt{\pi}} \int_{\alpha_0(\tau)}^{\infty} e^{-\alpha^2} \, d\alpha = \frac{1}{\sqrt{\pi}} (g(+\infty) - g(\alpha_0(\tau))) = \frac{1}{\sqrt{\pi}} (\sqrt{\pi} - g(\alpha_0(\tau))) .$$

The assertion is proved.

2. THE OPERATOR H

For $\mu \in \mathscr{B}'$ we have defined by (0.8) a distribution H_μ on $\mathscr{D}_b$. In the case that for each $\mu \in \mathscr{B}'$ the distribution H_μ can be represented by a measure on $\langle a, b \rangle$ (see below), the term H can be regarded as an operator on $\mathscr{B}'$. Let us now study some properties of this operator.

Let δ_τ stand for the Dirac measure on R^1 concentrated at the point $\tau \in R^1$.

2.1. Remark. The following indentity holds for $\mu \in \mathscr{B}'(\langle a, b \rangle)$, $\psi \in \mathscr{D}_b$:

$$(2.1) \quad \langle \psi, H_\mu \rangle = -\iint_E \left(\frac{\partial U_\mu}{\partial x}(x, t) \frac{\partial \psi}{\partial x}(x, t) - U_\mu(x, t) \frac{\partial \psi}{\partial t}(x, t) \right) dx \, dt =$$
$$= -\iint_E \left\{ \int_a^b \frac{\partial G}{\partial x}(x - \varphi(\tau), t - \tau) \, d\mu(\tau) \frac{\partial \psi}{\partial x}(x, t) - \right.$$

$$- \int_a^b G(x - \varphi(\tau), t - \tau)\, \mathrm{d}\mu(\tau) \frac{\partial \psi}{\partial t}(x, t)\Big\}\, \mathrm{d}x\, \mathrm{d}t =$$

$$= -\iint_E \Big\{ \int_a^b \Big(\frac{\partial G}{\partial x}(x - \varphi(\tau), t - \tau) \frac{\partial \psi}{\partial x}(x, t) - G(x - \varphi(\tau), t - \tau) \frac{\partial \psi}{\partial t}(x, t) \Big) \mathrm{d}\mu(\tau) \Big\} \,.$$

$$. \,\mathrm{d}x\, \mathrm{d}t = \int_a^b \Big\{ - \iint_E \Big(\frac{\partial G}{\partial x}(x - \varphi(\tau), t - \tau) \frac{\partial \psi}{\partial x}(x, t) -$$

$$- G(x - \varphi(\tau), t - \tau) \frac{\partial \psi}{\partial t}(x, t) \Big) \mathrm{d}x\, \mathrm{d}t \Big\} \mathrm{d}\mu(\tau) = \int_a^b \langle \psi, H_{\delta_\tau} \rangle\, \mathrm{d}\mu(\tau) \,.$$

At the same time we have

$$\langle \psi, H_{\delta_\tau} \rangle = \tilde{W}\psi(\varphi(\tau), \tau)\,, \quad (\psi \in \mathscr{D}_b)\,. \tag{2.2}$$

With respect to (2.2) and (1.3) one can prove analogously to the proof of Lemma 1.9 in [15] that H_{δ_τ} can be represented by a measure if and only if $\tilde{v}(\varphi(\tau), \tau) < \infty$. H_{δ_τ} is a functional defined on $\mathscr{D}_b$ and if $\tilde{v}(\varphi(\tau), \tau) < \infty$ then H_{δ_τ} can be regarded as a measure in R^2. This measure is then uniquely determined by the condition

$$|H_{\delta_\tau}| (R^2 - R_b) = 0 \,.$$

It is easily seen that the support of H_{δ_τ} is contained in the set $K(= \{[\varphi(t), t];\ t \in \in \langle a, b \rangle\})$. This implies that H_{δ_τ} can be regarded as a measure on $\langle a, b \rangle$ – if $f \in \mathscr{B}$ then there is a Baire (bounded) function ψ on R^2 such that $f(t) = \psi(\varphi(t), t)$ and then we put

$$\langle f, H_{\delta_\tau} \rangle = \langle \psi, H_{\delta_\tau} \rangle \,.$$

The measure H_{δ_τ} as a measure on $\langle a, b \rangle$ is then uniquely determined provided

$$H_{\delta_\tau}(\{b\}) = 0$$

(which corresponds to the fact that the set of all functions $f \in \mathscr{C}_0^\sim(\langle a, b \rangle)$ of the form $f(t) = \psi(\varphi(t), t)$ $(t \in \langle a, b \rangle)$, $\psi \in \mathscr{D}_b$, is dense in $\mathscr{C}_0^\sim(\langle a, b \rangle)$). Recall that the space

$$\mathscr{B}_0' = \{\mu \in \mathscr{B}';\ \mu(\{b\}) = 0\}$$

is the dual space of the space $\mathscr{C}_0^\sim(\langle a, b \rangle)$. We can assert that if $\tilde{v}(\varphi(\tau), \tau) < \infty$ then the distribution H_{δ_τ} can be represented by a unique measure from $\mathscr{B}_0'$. In the sequel, we shall identify H_{δ_τ} with that measure.

2.2. Let ν_b be the zero measure on $\langle a, b \rangle$. Let $\tau \in \langle a, b)$ and let $\tilde{v}(\varphi(\tau), \tau) < \infty$. Define a measure ν_τ on $\langle a, b \rangle$ by putting

$$\nu_\tau(M) = \frac{2}{\sqrt{\pi}} \int_{M \cap (\tau, b)} \exp\Big(- \frac{(\varphi(t) - \varphi(\tau))^2}{4(t - \tau)} \Big) \mathrm{d}_t \frac{\varphi(t) - \varphi(\tau)}{2\sqrt{(t - \tau)}} \tag{2.3}$$

for any Borel set $M \subset \langle a, b\rangle$ (the integral on the right hand side of (2.3) is considered a Lebesgue-Stieltjes integral). Let $\psi \in \mathscr{D}_b$, $f(t) = \psi(\varphi(t), t)$ $(t \in \langle a, b\rangle)$. It is seen from (1.18) and (1.20) that

$$(2.4)\quad \langle f, H_{\delta_\tau}\rangle = \langle \psi, H_{\delta_\tau}\rangle = \tilde{W}\psi(\varphi(\tau), \tau) = \int_a^b f(t)\,\mathrm{d}\nu_\tau(t) - 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau)\,\delta_\tau(f)\,.$$

Until now we have not defined the parabolic density $\mathscr{P}_E^{\sim}(\xi, \tau)$ in the case $\tau \geqq b$; in this case let us put $\mathscr{P}_E^{\sim}(\xi, \tau) = 0$. Then (2.4) is valid for each $\tau \in \langle a, b\rangle$ for which $\tilde{v}(\varphi(\tau), \tau) < \infty$. Now it is already seen that if $\tilde{v}(\varphi(\tau), \tau) < \infty$ (and H_{δ_τ} is regarded as a measure from $\mathscr{B}_0'$) then

$$(2.5)\qquad H_{\delta_\tau} = \nu_\tau - 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau)\,\delta_\tau\,.$$

As ν_τ is a non-atomic measure, we have

$$(2.6)\qquad \|H_{\delta_\tau}\| = \|\nu_\tau\| + 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau) = \frac{2}{\sqrt{\pi}}\,\tilde{v}(\varphi(\tau), \tau) + 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau)\,.$$

With regard to the fact that H_{δ_b} is the zero measure we can and will in the sequel deal with the space $\mathscr{B}_0'$ instead of $\mathscr{B}'$. The following assertion is valid.

2.3. Theorem. *The distribution H_μ can be represented by a measure for each $\mu \in \mathscr{B}_0'$ if and only if*

$$(2.7)\qquad \tilde{V}_K = \sup\{\tilde{v}(\varphi(\tau), \tau);\ \tau \in \langle a, b\rangle\} < \infty\,.$$

If $\tilde{V}_K < \infty$ then for each $\mu \in \mathscr{B}_0'$ one can identify H_μ with a unique measure from $\mathscr{B}_0'$. The operator

$$H : \mu \mapsto H_\mu\,,\quad (H : \mathscr{B}_0' \to \mathscr{B}_0')$$

is then a bounded operator on $\mathscr{B}_0'$ and

$$(2.8)\qquad \|H\| = \sup\left\{\frac{2}{\sqrt{\pi}}\,\tilde{v}(\varphi(\tau), \tau) + 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau);\ \tau \in \langle a, b\rangle\right\}.$$

Proof. Again we shall not prove this assertion in detail since the proof is quite similar, for instance, to that of Theorem 1.11 from [15]. In the proof we should use the equality

$$\langle \psi, H_\mu\rangle = \int_a^b \langle \psi, H_{\delta_\tau}\rangle\,\mathrm{d}\mu(\tau)\,,\quad (\psi \in \mathscr{D}_b,\ \mu \in \mathscr{B}_0')\,.$$

If H_μ is a measure for each $\mu \in \mathscr{B}_0'$ then the condition $\tilde{V}_K < \infty$ follows from the uniform boundedness principle. (2.8) follows from (2.5).

2.4. In what follows we shall always suppose that the condition (2.7) is fulfilled. Note that then for any function $f \in \mathscr{B}(\langle a, b\rangle)$ and any measure $\mu \in \mathscr{B}'_0(\langle a, b\rangle)$ we have

$$\langle f, H_\mu \rangle = \int_a^b \langle f, H_{\delta_\tau} \rangle \, \mathrm{d}\mu(\tau). \tag{2.9}$$

2.5. If the condition (2.7) is fulfilled then also

$$\sup \{ V_{\tilde{K}}(\tilde{\xi}, \tilde{\tau});\ [\tilde{\xi}, \tilde{\tau}] \in \tilde{K} \} < \infty$$

and for each $\tau \in \langle a, b\rangle$ and each $f \in \mathscr{C}_0^{\sim}(\langle a, b\rangle)$ the limit (see [3], Theorem 2.1)

$$\overline{W}_- f(\tau) = \lim_{\substack{[\xi', \tau'] \to [\varphi(\tau), \tau] \\ \tau' \in \langle a, b\rangle,\, \xi' < \varphi(\tau')}} W(\xi', \tau') = - \lim_{\substack{[\tilde{\xi}', \tilde{\tau}'] \to [\tilde{\varphi}(\tilde{\tau}), \tilde{\tau}] \\ \tilde{\tau}' \in \langle \tilde{b}, \tilde{a}\rangle,\, \tilde{\xi}' > \tilde{\varphi}(\tilde{\tau}')}} T\tilde{f}(\tilde{\xi}', \tilde{\tau}') =$$

$$= - \left[T\tilde{f}(\tilde{\varphi}(\tilde{\tau}), \tilde{\tau}) + 2\tilde{f}(\tilde{\tau}) \left(1 - \frac{1}{\sqrt{\pi}}\, g(\alpha_0(\tau)) \right) \right] = Wf(\varphi(\tau), \tau) - 2f(\tau)\, \mathscr{P}_E^{\sim}(\varphi(\tau), \tau)$$

exists ($\alpha_0(\tau)$ has the same meaning as in 1.8). $\overline{W}_-(\overline{W}_- : f \mapsto \overline{W}_- f)$ can be regarded as an operator on $\mathscr{C}_0^{\sim}(\overline{W}_- : \mathscr{C}_0^{\sim} \to \mathscr{C}_0^{\sim})$. Then we have for $f \in \mathscr{C}_0^{\sim}$, $\mu \in \mathscr{B}'_0$ (see (2.4))

$$\langle f, H_\mu \rangle = \int_a^b \langle f, H_{\delta_\tau} \rangle \, \mathrm{d}\mu(\tau) = \tag{2.10}$$

$$= \int_a^b \{ Wf(\varphi(\tau), \tau) - 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau) f(\tau) \} \, \mathrm{d}\mu(\tau) = \langle \overline{W}_- f, \mu \rangle.$$

Thus we see that the operators H (on $\mathscr{B}'_0$) and $\overline{W}_-$ (on $\mathscr{C}_0^{\sim}$) are adjoint to each other. Consider now operators H_1, $\overline{W}_1$:

$$H_1 = H + I, \quad \overline{W}_1 = \overline{W}_- + I$$

(where I denotes the identity operator on $\mathscr{B}'_0$ resp. on $\mathscr{C}_0^{\sim}$). Then, of course, the operators H_1, $\overline{W}_1$ are adjoint to each other as well. Using the notation from [4] (see (1.4), (1.7) [4]) we can write for $f \in \mathscr{C}_0^{\sim}$, $\tau \in \langle a, b\rangle$

$$\overline{W}_1 f(\tau) = - \left[T\tilde{f}(\tilde{\varphi}(\tilde{\tau}), \tilde{\tau}) + 2\tilde{f}(\tilde{\tau}) \left(1 - \frac{1}{\sqrt{\pi}}\, g(\alpha_0(\tau)) \right) - \tilde{f}(\tilde{\tau}) \right] = \tag{2.11}$$

$$= - [\tilde{T}_+ \tilde{f}(\tilde{\tau}) - \tilde{f}(\tilde{\tau})] = - \overline{T}_0 \tilde{f}(\tilde{\tau}).$$

The Fredholm radius of the operator $\overline{T}_0$ has been evaluated in [4] and it has been shown there under which condition there exists an inverse operator to the operator $\tilde{T}_+$, hence (as is easily seen) also to the operator $\overline{W}_-$ and also to the operator H

(see Theorem 1.1 [4], Lemma 2.1 [4], Theorem 2.1 [4]). This condition is

$$\omega \tilde{T}_0 = \lim_{r\to 0+} \sup_{\tilde{\tau}\in(\tilde{b},\tilde{a}\rangle} \left(\frac{2}{\sqrt{\pi}} V_{\tilde{K}}(r;\tilde{\varphi}(\tilde{\tau}),\tilde{\tau}) + \alpha_{\tilde{K}}(\tilde{\tau})\right) =$$

$$= \lim_{r\to 0+} \sup_{\tau\in\langle a,b)} \left(\frac{2}{\sqrt{\pi}} \tilde{v}^r(\varphi(\tau),\tau) + \left|2\mathscr{P}_{\tilde{E}}(\varphi(\tau),\tau) - 1\right|\right) = \omega \overline{W}_1 < 1$$

($(\omega W_1)^{-1}$ is then the Fredholm radius of the operator $\overline{W}_1$ (resp. $\tilde{T}_0$)).
Now we can formulate the following assertion.

2.6. Theorem. *Let*

$$\lim_{r\to 0+} \sup_{\tau\in\langle a,b)} \left(\frac{2}{\sqrt{\pi}} \tilde{v}^r(\varphi(\tau),\tau) + \left|2\mathscr{P}_{\tilde{E}}(\varphi(\tau),\tau) - 1\right|\right) < 1\,.$$

Then for any measure $\nu \in \mathscr{B}'_0$ the equation

$$H_\mu = \nu \qquad (1.12)$$

has a unique solution $\mu \in \mathscr{B}'_0$.

2.7. Remark. Let us notice that if μ is the solution of the equation (2.12) then according to the introductory remarks the heat potential U_μ (considered on E) is an integral expression of a solution of a special case of the third boundary value problem for the heat equation on E with the boundary condition

$$\frac{\partial U_\mu}{\partial x} + \lambda_0 \,.\, U_\mu = \nu$$

on K (where λ_0 is a measure on $\langle a, b\rangle$, $\mathrm{d}\lambda_0(t) = \mathrm{d}\varphi(t)$ – see the introductory remarks). At the same time the operators H and $\overline{W}_-$ are adjoint to each other. Hence we can say that the above mentioned special type of the third boundary value problem for the heat equation on E is adjoint (in the sense of integral equations) to the first boundary value problem for the heat equation on the set $\tilde{E}_+ = \{[x, t] \in R^2;\ t \in (\tilde{b}, \tilde{a}),\ x > \tilde{\varphi}(t)\}$ with a boundary condition prescribed on the set $\tilde{K} = \{[x, t] \in R^2;\ t \in \langle \tilde{b}, \tilde{a}\rangle,\ x = \tilde{\varphi}(t)\}$ (and also to the first boundary value problem for the adjoint heat equation on the set $E_- = \{[x, t] \in R^2;\ t \in (a, b),\ x < \varphi(t)\}$ with a boundary condition prescribed on the set K). It is very well known that for the Laplace equation the interior Dirichlet problem and the exterior Neumann problem are adjoint to each other (again in the sense of integral equations). J. Král has shown in [15] that in R^{n+1} (roughly speaking) the interior second boundary value problem for the heat equation and the first exterior boundary value problem for the adjoint heat equation are adjoint to each other (in a similar sense) provided the sets considered are of the form $C = D \times (T_1, T_2)$, where $D \subset R^n$. But we now see that in the case of the time moving boundary the situation is rather more complicated.

As we have just noted, only a special type of the third boundary value problem can be solved by solving the integral equation (1.12). An investigation of a rather more general type of the third boundary value problem for the heat equation on the domain E will appear in another paper.

References

[1] *G. Anger:* Funktionalanalytische Betrachtungen bei Differentialgleichungen unter Verwendung von Methoden der Potentialtheorie I. Akademie-Verlag, Berlin 1967.

[2] *E. A. Бадерко:* Применение метода поуаболических потенциалов к решению одной краевой задачи контактной теплопроводности. Дифф. уравнения *6* (1970), 2200—2213.

[3] *M. Dont:* On a heat potential. Czech. Math. J. *25* (1975), 84—109.

[4] *M. Dont:* On a boundary value problem for the heat equation. Czech. Math. J. *25* (1975), 110—133.

[5] *M. Dont:* A note on a heat potential and the parabolic variation. Čas. Pěst. Mat. *101* (1976) 28—44.

[6] *M. Dont:* The heat and adjoint heat potentials. Čas. Pěst. Mat. *105* (1980), 199—203.

[7] *M. Dont:* On the continuity of the heat potentials, Čas. Pěst. Mat. *106* (1981), 156—167.

[8] *A. Friedman:* Partial differential equations of parabolic type. Prentice-Hall, 1964.

[9] *M. Gevrey:* Sur les équations aux dérivées du type parabolique J. Math. Pures Appl., *9* (1913), 305—471; *10* (1914), 105—148.

[10] *J. Král:* Theory of potential I (Czech). SPN Praha 1965.

[11] *J. Král:* On the logarithmic potential of the double distribution. Czech. Math. J. *14* (89) 1964, 306—321.

[12] *J. Král:* The Fredholm radius of an operator in potential theory. Czech. Math. J. *15* (90) 1965, 454—473; 565—588.

[13] *J. Král:* The Fredholm method in potential theory. Trans. A. M. S. *125* (1966), 511—547,

[14] *J. Král:* Flows of heat. Atti Accad. Naz. Lincei, Rend. Cl. fis. mat. e nat. *46* (1969), fasc. 2 60—63.

[15] *J. Král:* Flows of heat and the Fourier problem. Czech. Math. J. *20* (1970), 556—598.

[16] *J. Král:* A note on the Robin problem in potential theory. Comment. Math. Univ. Carolinae *14* (1973), 767—771.

[17] *J. Král:* Potentials and boundary value problems, 5. Tagung über Probleme und Methoden der Mathematischen Physik. Wissenschaftliche Schriftenreihe der Technischen Hochschule Karl-Marx-Stadt 1975, 484—500.

[18] *J. Král:* Heat sources and heat potentials. Preprint.

[19] *J. Král, J. Lukeš:* On the modified logarithmic potential. Czech. Math. J. *21* (1971), 76—98.

[20] *J. Král, J. Lukeš:* Integrals of the Cauchy type. Czech. Math. J. *22* (1972), 663—682.

[21] *J. Král, I. Netuka:* Contractivity of C. Neumann's operator in potential theory. J. Math, Anal. Appl. *61* (1977), 607—619.

[22] *J. Král, I. Netuka, J. Veselý:* Theory of potential II, III, IV (Czech). SPN, Praha 1972, 1976, 1977.

[23] *C. T. Михнин:* Интегралные уравнения и их приложения к некоторым проблемам механики, математической физики и техники. ГИТТЛ, Москва, 1949.

[24] *G. Miranda:* Integral equation solution of the first initial-boundary value problem for the heat equation in domains with non-smooth boundary. Comm. Pure. Appl. Math. *23* (1970), 757—765.

[25] *S. Mrzena:* Continuity of heat potentials (Czech). Praha, 1976.
[26] *Ch. H. Müntz:* Zum dynamischen Wärmeleitusproblem. Math. Z. *38* (1933), 323—337.
[27] *I. Netuka:* The Robin problem in potential theory. Comment. Math. Univ. Carolinae *12* (1971), 205—211.
[28] *I. Netuka:* Generalized Robin problem in potential theory. Czech. Math. J. *22* (1972), 312—324.
[29] *I Netuka:* An operator connected with the third boundary value problem in potential theory. Czech. Math. J. *22* (1972), 462—489.
[30] *I. Netuka:* The third boundary value problem in potential theory. Czech. Math. J. *22* (1972), 554—580.
[31] *I. Netuka:* A mixed boundary value problem for heat potentials. Comment. Math. Univ. Carolinae *19* (1978), 207—211.
[32] *I. Netuka:* Heat potentials and the mixed boundary value problem for the heat equation (Czech). Praha, 1977.
[33] *W. Pogorzelski:* Sur la solution de l'équation intégrale dans le probléme de Fourier. Ann. Soc. Polon. Math. *24* (1951), 56—74.
[34] *W. Pogorzelski:* Integral equations and their applications. Pergamon Press, Oxford, 1966.
[35] *F. Riesz, B. Sz. Nagy:* Lecons d'analyse fonctionnelle. Budapest, 1952.
[36] *А. Тихонов:* Об уравнении теплопроводности для нескольких переменных. Бюлетин Моск. Гос. Унив. *1* (1938), 1—45.
[37] *A. Tichonov, A. Samarskij:* Equations of mathematical physics (Czech). Praha, 1955.
[38] *J. Veselý:* On the heat potential of the double distribution. Čas. Pěst. Mat. *98* (1973), 181—198.
[39] *J. Veselý:* On a generalized heat potential. Czech. Math. J. *25* (1975), 404—423.

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