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WEAKLY ALMOST PERIODIC SOLUTIONS OF LINEAR EQUATIONS IN BANACH SPACES

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In this paper are given some conditions on linear operator A in a Banach space B, under which all bounded solutions of the equation

$$x'(t) = A\,x(t) \tag{1}$$

are weakly almost periodic. The received results generalize the results of paragraph 3 of paper [4]. We refer on this paper for more detailed information about this matter.

Throughout all paper we suppose, that is given a complex Banach space B and a linear operator A on B, which satisfies the following conditions:

$$\begin{aligned} &1)\quad \overline{D(A)} = B\,, \\ &2)\quad \overline{D(A^*)} = B^*\,. \end{aligned} \tag{2}$$

We shall consider only continuous solutions of the equation (1) even if the main results holds also for more general solutions.

1. Definition 1. A continuous function x on R with values in B is called *to be solution of the equation* (1), if it holds

$$\int_{-\infty}^{+\infty} [(x(t), x^*)\,f'(t) + (x(t), A^*x^*)\,f(t)]\,\mathrm{d}t = 0 \tag{3}$$

for any $x^* \in D(A^*)$ and for any $f \in \mathscr{D}(R)$.

Definition 2. Let $f \in L_\infty(R)$. By *spectrum of f* we mean the set of real numbers, denoted by $\sigma(f)$, such that $\lambda \in \sigma(f)$ iff the function $\exp_{i\lambda}$ ($\exp_{i\lambda}(t) = e^{i\lambda t}$ for $t \in R$) belongs to the smallest invariant L_1-closed subspace of $L_\infty(R)$, which contains the function f.

Definition 3. Let x be a bounded continuous function on R with values in B. Then *the spectrum of* x is the set of reals defined by

$$\sigma(x) = \bigcup_{x^* \in B^*} \sigma((x, x^*))$$

(where $(x, x^*)(t) = (x(t), x^*)$ for $t \in R$).

2. Lemma 1. *Let x be a bounded solution of an equation* (1) *and let* $\mu \in \varrho(A)$ (i.e. *let* $(\mu E - A)^{-1}$ *exist*). *Then the function* y_μ *defined by*

$$y_\mu(t) = (\mu E - A)^{-1} x(t) \quad (t \in R) \tag{4}$$

has the following expressions:

$$y_\mu(t) = \int_0^{+\infty} e^{-\mu s} x(t + s) \, ds \qquad \textit{for} \quad \mathrm{Re}\, \mu > 0 \tag{5}$$

$$y_\mu(t) = -\int_{-\infty}^{0} e^{-\mu s} x(t + s) \, ds \qquad \textit{for} \quad \mathrm{Re}\, \mu < 0$$

$$y_\mu(t) = e^{\mu t}(\mu E - A)^{-1} x(0) - \int_0^t e^{\mu(t-s)} x(s) \, ds \quad \textit{for} \quad \mathrm{Re}\, \mu = 0$$

Proof. Let for instance $\mathrm{Re}\, \mu > 0$ and let we denote $\tilde{y}_\mu$ the function: $\tilde{y}_\mu(t) = \int_0^{+\infty} e^{-\mu s} x(t + s) \, ds = \int_t^{+\infty} e^{\mu(t-s)} x(s) \, ds \ (t \in R)$. Let us note, that the function y_μ and $\tilde{y}_\mu$ are bounded on R. Let $x^* \in D(A^*)$ and let $f \in \mathscr{D}(R)$. Then we obtain:

$$\int_{-\infty}^{+\infty} (y_\mu(t) - \tilde{y}_\mu(t), x^*) f'(t) \, dt = \int_{-\infty}^{+\infty} (x(t), ((\mu E - A)^{-1})^* x^*) f'(t) \, dt -$$

$$- \int_{-\infty}^{+\infty} \left(\int_t^{+\infty} e^{\mu(t-s)} x(s) \, ds, x^* \right) f'(t) \, dt = - \int_{-\infty}^{+\infty} (x(t), A^*(\bar{\mu} E - A^*)^{-1} x^*) f(t) \, dt +$$

$$+ \int_{-\infty}^{+\infty} \left(\mu \int_t^{+\infty} e^{\mu(t-s)} x(s) \, ds - x(t), x^* \right) f(t) \, dt =$$

$$= - \int_{-\infty}^{+\infty} (x(t), (-E + \bar{\mu}((\mu E - A)^{-1})^*) x^*) f(t) \, dt +$$

$$+ \mu \int_{-\infty}^{+\infty} (\tilde{y}_\mu(t), x^*) f(t) \, dt - \int_{-\infty}^{+\infty} (x(t), x^*) f(t) \, dt =$$

$$= -\mu \int_{-\infty}^{+\infty} (y_\mu(t) - \tilde{y}_\mu(t), x^*) f(t) \, dt$$

From the above follows that $(y_\mu(t) - \tilde{y}_\mu(t), x^*) = c(x^*) e^{\mu t}$. As the function $y_\mu - \tilde{y}_\mu$ is bounded, we obtain that $c(x^*) = 0$. Hence $y_\mu = \tilde{y}_\mu$ because of $D(A^*)$ is dense in B^*.

The similar calculation prove our assertion also for Re $\mu < 0$ and for Re $\mu = 0$.

Lemma 2. *Let x be a bounded solution of an equation (1) and let $\mu \in \varrho(A)$. Then the function y_μ, defined by (4), is uniformly continuous solution of the equation (1).*

Proof. It follows from (5) that the function y_μ is uniformly continuous on R. For $x^* \in D(A^*)$ and for $f \in \mathscr{D}(R)$ we have further

$$\int_{-\infty}^{+\infty} (y_\mu(t), x^*)\, f'(t)\, dt = \int_{-\infty}^{+\infty} (x(t), ((\mu E - A)^{-1})^* x^*)\, f'(t)\, dt =$$

$$= -\int_{-\infty}^{+\infty} (x(t), A^*((\mu E - A)^{-1})^* x^*)\, f(t)\, dt = -\int_{-\infty}^{+\infty} (y_\mu(t), A^* x^*)\, f(t)\, dt$$

and so y_μ is a solution of the equation (1).

Theorem 1. *Let B be a complex Banach space and let A be a linear operator in B satisfying the conditions (2). Let moreover $\varrho(A) \neq \emptyset$. Then for any bounded solution x of the equation (1) the functions (x, x^*) are uniformly continuous on R for all $x^* \in B^*$.*

Proof. Let $\mu \in \varrho(A)$ and let x be a bounded solution of the equation (1). Let y_μ be defined by (4). Then y_μ is uniformly continuous on R by lemma 2 and because of the equality $(y_\mu, x^*) = (x, (\bar{\mu} E - A^*)^{-1} x^*)$ $(x^* \in B^*)$, the functions (x, x^*) are uniformly continuous on R for any $x^* \in R((\bar{\mu} E - A^*)^{-1}) = D(A^*)$.

Theorem 2. *Let B be a complex Banach space and let A be a linear operator in B satisfying the conditions (2). Let x be a bounded solution of the equation (1). Then $i\sigma(x) \subset \sigma(A)$.*

Proof. Let $x^* \in B^*$ and let us define a function g of complex variable z by

$$g(z) = \int_0^{+\infty} e^{-izs}(x, x^*)(s)\, ds \quad \text{for} \quad \operatorname{Im} z < 0$$

$$g(z) = -\int_{-\infty}^{0} e^{-izs}(x, x^*)(s)\, ds \quad \text{for} \quad \operatorname{Im} z > 0$$

Let λ be real number such that $i\lambda \in \varrho(A)$. Then for all z from some neighbourhood of λ with Im $z \neq 0$ we have by lemma 1:

$$g(z) = ((izE - A)^{-1} x(0), x^*)$$

and so g has analytic continuation on the whole neighbourhood of λ. From the theorem (XI, 4, 24) of [1] follows that $\lambda \notin \sigma((x, x^*))$. So we have implication: $\lambda \in R$, $i\lambda \in \varrho(A) \Rightarrow \lambda \notin \sigma(x)$, which proves our assertion.

3. We are now able to prove main theorem:

Theorem 3. *Let B be a complex Banach space and let A be a linear operator in B satisfying the conditions* (2). *Let the set* $-i\sigma(A) \cap R$ *be residual*[1]). *Then any bounded solution of the equation* (1) *is weakly almost periodic.*

Proof. Let x be a bounded solution of the equation (1). For any $x^* \in B^*$, the function (x, x^*) is uniformly continuous by theorem 1 and has residual spectrum by theorem 2 and by the assumptions of the theorem. Hence the function (x, x^*) is almost periodic by theorem 5 of [3].

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[1]) i.e. includes no non-null perfect subset.