Časopis pro pěstování matematiky

Bohdan Zelinka

A remark on isotopies of digraphs and permutation matrices

Časopis pro pěstování matematiky, Vol. 102 (1977), No. 3, 230--233

Persistent URL: `http://dml.cz/dmlcz/108463`

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

A REMARK ON ISOTOPIES OF DIGRAPHS AND PERMUTATION MATRICES

BOHDAN ZELINKA, Liberec

(Received January 6, 1976)

In [1], [2], [3] the concepts of isotopy and autotopy of a digraph are studied. Here we shall make a remark on applications of permutation matrices in investigating these concepts.

Let G and G' be two digraphs, let V be the vertex set of G, let V' be the vertex set of G'. The isotopy of G onto G' is an ordered pair $\langle f_1, f_2 \rangle$ of bijections of V onto V' with the property that for any two vertices u, v of G the edge $\overrightarrow{f_1(u) f_2(v)}$ exists in G' if and only if the edge $\overrightarrow{uv}$ exists in G. Two digraphs G and G' are called isotopic, if there exists an isotopy of G onto G'. An autotopy of a digraph is an isotopy of G again onto G.

Here we shall consider digraphs in which loops may exist as well as various edges with the same initial vertex and the same terminal vertex. For these graphs we adapt the definition of the isotopy so that the number of edges going from $f_1(u)$ into $f_2(v)$ in G' is equal to the number of edges going from u into v in G.

If G is a finite digraph with n vertices $u_1, u_2, \ldots, u_n$, then its adjacency matrix $\mathbf{A}_G$ is the $n \times n$ matrix in which the term in the i-th row and the j-th column is equal to the number of edges going from u_i into u_j in G.

Now consider a permutation π of the set of numbers $\{1, 2, \ldots, n\}$. The matrix of the permutation π is the $n \times n$ matrix $\mathbf{P}(\pi)$ in which the term in the i-th row and the j-th column is equal to the Kronecker delta $\delta_i^{p(j)}$. Each matrix which is the matrix of a certain permutation is called a permutation matrix.

We shall recall some well-known properties of permutation matrices.

Proposition 1. *A square matrix* $\mathbf{M}$ *is a permutation matrix, if and only if exactly one term in each row and exactly one term in each column of* $\mathbf{M}$ *is equal to* 1 *and all other terms of* $\mathbf{M}$ *are equal to* 0.

Proposition 2. *Let* π_1 *and* π_2 *be two permutations of the set of numbers* $\{1, 2, \ldots, n\}$. *Then*

$$\mathbf{P}(\pi_1)\,\mathbf{P}(\pi_2) = \mathbf{P}(\pi_2\pi_1)\,.$$

Proposition 3. *Let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. Then the transposed matrix to the matrix $\mathbf{P}(\pi)$ is the inverse matrix to $\mathbf{P}(\pi)$ and is equal to $\mathbf{P}(\pi^{-1})$.*

Now let $\mathbf{M}$ be a matrix with n rows, let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. To perform π on the rows of $\mathbf{M}$ means to construct a matrix with n rows in which $\pi(i)$-th row is equal to the i-th row of $\mathbf{M}$. For a matrix $\mathbf{M}$ with n columns we define analogously the meaning of "to perform a permutation on the columns of $\mathbf{M}$".

Proposition 4. *Let $\mathbf{M}$ be a matrix with n rows, let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. The product $\mathbf{P}(\pi^{-1})\,\mathbf{M}$ is the matrix obtained from $\mathbf{M}$ by performing the permutation π on its rows.*

Proposition 5. *Let $\mathbf{M}$ be a matrix with n columns, let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. The product $\mathbf{M}\,\mathbf{P}(\pi)$ is the matrix obtained from $\mathbf{M}$ by performing the permutation π on its columns.*

Now consider the adjacency matrix $\mathbf{A}_G$ of a digraph G with n vertices.

Theorem 1. *Let G and G' be two finite digraphs with n vertices, let $\mathbf{A}_G$ and $\mathbf{A}_{G'}$ be their adjacency matrices, respectively. The graphs G and G' are isotopic, if and only if there exist permutation $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$ such that*

$$\mathbf{A}_G\mathbf{P} = \mathbf{Q}\mathbf{A}_{G'} .$$

Proof. Let G and G' be isotopic, let $\langle f_1, f_2\rangle$ be an isotopy of G onto G'. The vertices of G are $u_1, \ldots, u_n$, the vertices of G' are $u'_1, \ldots, u'_n$ in the notation corresponding to the adjacency matrices $\mathbf{A}_G$, $\mathbf{A}_{G'}$. The mappings f_1, f_2 are bijections of the vertex set V of G onto the vertex set V' of G'. Let π_1, π_2 be such permutations of the set of numbers $\{1, 2, \ldots, n\}$ that $f_1(u_i) = u'_{\pi_1(i)}, f_2(u_i) = u'_{\pi_2(i)}$ for each $i \in \{1, 2, \ldots, n\}$. Then the term of $\mathbf{A}_{G'}$ in the $\pi_1(i)$-th row and the $\pi_2(j)$-th column is equal to the term of $\mathbf{A}_G$ in the i-th row and the j-th column. This means that $\mathbf{A}_{G'}$ is obtained from $\mathbf{A}_G$ by performing π_1 on its rows and π_2 on its columns. But this means

$$\mathbf{P}(\pi_1^{-1})\,\mathbf{A}_G\,\mathbf{P}(\pi_2) = \mathbf{A}_{G'}$$

and thus

$$\mathbf{A}_G\,\mathbf{P}(\pi_2) = \mathbf{P}(\pi_1)\,\mathbf{A}_{G'} .$$

Putting $\mathbf{P}(\pi_2) = \mathbf{P}$, $\mathbf{P}(\pi_1) = \mathbf{Q}$ we obtain the required result. The converse assertion can be proved so that we determine π_1, π_2 from $\mathbf{P}, \mathbf{Q}$ and then f_1, f_2.

Corollary 1. *Let G be a digraph with n vertices $u_1, \ldots, u_n$, let $\mathbf{A}_G$ be its adjacency matrix. Let f_1, f_2 be two permutations of the vertex set of G. Let π_1, π_2 be two permutations of the set of numbers $\{1, 2, \ldots, n\}$ such that $f_1(u_i) = u_{\pi_1(i)}, f_2(u_i) = u_{\pi_2(i)}$ for each $i \in \{1, 2, \ldots, n\}$. Then $\langle f_1, f_2\rangle$ is an autotopy of G, if and only if*

$$\mathbf{P}(\pi_1)\,\mathbf{A}_G = \mathbf{A}_G\,\mathbf{P}(\pi_2) .$$

As mentioned in [1], an isomorphism of a digraph G onto a digraph G' can be considered as a particular case of an isotopy. If $\langle f_1, f_2 \rangle$ is an isotopy of G onto G' and $f_1 \equiv f_2$, then f_1 is an isomorphism of G onto G' and vice versa. Thus we have the following corollaries.

Corollary 2. *Let G and G' be two digraphs with n vertices, let $\mathbf{A}_G$ and $\mathbf{A}_{G'}$ be their adjacency matrices, respectively. The graphs G and G' are isomorphic, if and only if there exists a permutation $n \times n$ matrix $\mathbf{P}$ such that*

$$\mathbf{A}_G \mathbf{P} = \mathbf{P} \mathbf{A}_{G'} .$$

Corollary 3. *Let G be a digraph with n vertices $u_1, \ldots, u_n$, let $\mathbf{A}_G$ be its adjacency matrix. Let f be a permutation of the vertex set of G. Let π be the permutation of the set of numbers $\{1, 2, \ldots, n\}$ such that $f(u_i) = u_{\pi(i)}$ for each $i \in \{1, 2, \ldots, n\}$. Then f is an automorphism of G, if and only if*

$$\mathbf{P}(\pi)\, \mathbf{A}_G = \mathbf{A}_G\, \mathbf{P}(\pi) .$$

Now we shall consider products of digraphs. If G_1 and G_2 are two digraphs with the same vertex set V, then the product $G_1 . G_2$ is the digraph whose vertex set is V and such that for any two vertices u, v of V the number of edges going from u into v is equal to the number of directed paths in the union of G_1 and G_2 of length 2 and with the property that the first edge of such a path belongs to G_1 and the second to G_2. It is well-known that for the adjacency matrix $\mathbf{A}_{G_1 . G_2}$ of the digraph $G_1 . G_2$ the equality $\mathbf{A}_{G_1 . G_2} = \mathbf{A}_{G_1} \mathbf{A}_{G_2}$ holds.

Theorem 2. *Let G_1 and G_2 be two digraphs with the same vertex set V. Let f_1, f_2, f_3 be three permutations of the set V such that $\langle f_1, f_2 \rangle$ is an autotopy of G_1 and $\langle f_2, f_3 \rangle$ is an autotopy of G_2. Then $\langle f_1, f_3 \rangle$ is an autotopy of $G_1 . G_2$.*

Proof. Let $V = \{u_1, \ldots, u_n\}$, let π_1, π_2, π_3 be the permutations of $\{1, 2, \ldots, n\}$ such $f_j(u_i) = u_{\pi_j(i)}$ for each $i \in \{1, 2, \ldots, n\}$ and $j = 1, 2, 3$. As $\langle f_1, f_2 \rangle$ is an autotopy of G_1, Corollary 1 yields

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1} = \mathbf{A}_{G_1}\, \mathbf{P}(\pi_2) .$$

As $\langle f_2, f_3 \rangle$ is an autotopy of G_2, we have

$$\mathbf{P}(\pi_2)\, \mathbf{A}_{G_2} = \mathbf{A}_{G_2}\, \mathbf{P}(\pi_3) .$$

We multiply the first equation from the right by $\mathbf{A}_{G_2}$; we obtain

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1} \mathbf{A}_{G_2} = \mathbf{A}_{G_1}\, \mathbf{P}(\pi_2)\, \mathbf{A}_{G_2} .$$

We substitute for $\mathbf{P}(\pi_2)\, \mathbf{A}_{G_2}$ from the second equation:

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1} \mathbf{A}_{G_2} = \mathbf{A}_{G_1} \mathbf{A}_{G_2}\, \mathbf{P}(\pi_3) .$$

As mentioned above, $\mathbf{A}_{G_1}\mathbf{A}_{G_2} = \mathbf{A}_{G_1.G_2}$ and thus

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1.G_2} = \mathbf{A}_{G_1.G_2}\, \mathbf{P}(\pi_3)\,.$$

Therefore $\langle f_1, f_3 \rangle$ is an autotopy of $G_1 \,.\, G_2$.

Corollary 4. *Let G_1 and G_2 be two digraphs with the same vertex set V. Let f_1, f_2 be two permutations of the set V such that f_1 is an automorphism of G_1 and $\langle f_1, f_2 \rangle$ is an autotopy of G_2. Then $\langle f_1, f_2 \rangle$ is an autotopy of $G_1 \,.\, G_2$.*

Corollary 4'. *Let G_1 and G_2 be two digraphs with the same vertex set V. Let f_1, f_2 be two permutations of the set V such that $\langle f_1, f_2 \rangle$ is an autotopy of G_1 and f_2 is an automorphism of G_2. Then $\langle f_1, f_2 \rangle$ is an autotopy of $G_1 \,.\, G_2$.*

The next theorem will concern digraphs with regular adjacency matrices.

Theorem 3. *Let G be a finite digraph whose adjacency matrix $\mathbf{A}_G$ is regular. Let f_1 be a permutation of the vertex set of G. Then there exists at most one permutation f_2 of V such that $\langle f_1, f_2 \rangle$ is an autotopy of G.*

Proof. Let $\langle f_1, f_2 \rangle$ be an autotopy of G, let π_1, π_2 be defined as in the proof of Theorem 1. Then

$$\mathbf{P}(\pi_1)\, \mathbf{A}_G = \mathbf{A}_G\, \mathbf{P}(\pi_2)\,.$$

As $\mathbf{A}_G$ is regular, we have

$$\mathbf{P}(\pi_2) = \mathbf{A}_G^{-1}\, \mathbf{P}(\pi_1)\, \mathbf{A}_G\,.$$

Thus if $\mathbf{A}_G^{-1}\, \mathbf{P}(\pi_1)\, \mathbf{A}_G$ is a permutation matrix, there exists exactly one f_2 to the given f_1. If it is not so, there exists no f_2 with the property that $\langle f_1, f_2 \rangle$ is an autotopy of G.

Theorem 3'. *Let G be a finite digraph whose adjacency matrix $\mathbf{A}_G$ is regular. Let f_2 be a permutation of the vertex set of G. Then there exists at most one permutation f_1 of V such that $\langle f_1, f_2 \rangle$ is an autotopy of G.*

Proof is analogous to that of Theorem 3.

The results of this paper may be used for finding the group of autotopies or automorphisms of a given digraph or for finding the digraphs which have a given autotopy or automorphism.

References

[1] *B. Zelinka:* Isotopy of digraphs. Czech. Math. J. *22* (1972), 353—360.

[2] *B. Zelinka:* The group of autotopies of a digraph. Czech. Math. J. *21* (1971), 619—624.

[3] *B. Zelinka:* Antiisotopy of directed graphs. Sborník věd. prací VŠST Liberec *9* (1970), 15—24.

Author's address: 460 01 Liberec 1, Komenského 2 (katedra matematiky VŠST).