Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 109 (1984), No. 3, 261--265

Persistent URL: `http://dml.cz/dmlcz/108442`

Časopis pro pěstování matematiky, roč. 109 (1984), Praha

CONNECTEDNESS AND STRONG SEMI-CONTINUITY

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(Received April 20, 1983)

1. INTRODUCTION

Let S be a subset of a topological space (X, T). We denote the closure of S and the interior of S with respect to T by $T\operatorname{cl} S$ and $T\operatorname{int} S$ respectively, although we may suppress the T when there is no possibility of confusion.

Definition 1. A subset S of (X, T) is called
(i) an *α-set* if $S \subset T\operatorname{int}(T\operatorname{cl}(T\operatorname{int} S))$,
(ii) a *semi-open set* if $S \subset T\operatorname{cl}(T\operatorname{int} S)$,
(iii) a *preopen set* if $S \subset T\operatorname{int}(T\operatorname{cl} S)$.

These three concepts were introduced by Njåstad [6], Levine [3], and Mashhour et al [5], respectively. Njåstad used the term β-set for a semi-open set. Any open set in (X, T) is an α-set, and each α-set is semi-open and preopen, but the separateconverses are false. Lemma 1 below shows that a subset of (X, T) is an α-set if and only if it is semi-open and preopen.

Following Njåstad [6] we denote the family of all α-sets in (X, T) by T^α, rather than by the notation $\alpha(X)$ of [4] and [7]. The families of all semi-open sets and of all preopen sets in (X, T) are denoted by $SO(X)$ and $PO(X)$, respectively. Njåstad [6, Proposition 2] proved that T^α is a topology on X. It is unusual for either $SO(X)$ or $PO(X)$ to be a topology on X. Proposition 7 of Njåstad [6] shows that $SO(X)$ is a topology on X if and only if (X, T) is extremally disconnected. The complement of an α-set in (X, T) is called an α-closed set, and semi-closed and preclosed subsets of (X, T) are similarly defined.

Recently, Noiri [7] has introduced the concept of strong semi-continuity of functions between topological spaces.

Definition 2. A function $f:(X, T) \to (Y, U)$ is called *strongly semi-continuous* (abbreviated hereafter as s.s.c.) if the inverse image $f^{-1}(V)$, of any open set V in (Y, U), is an α-set in (X, T).

*) The first author acknowledges the support of the University of Auckland Research fund.

One purpose of this paper is to indicate that the distinction made by Noiri [7] between the concepts of continuity and strong semi-continuity, must be interpreted strictly. In fact, we observe (in Theorem 1 below) that if the domain space of an s.s.c. function f is retopologized in an obvious way, then the function f is simply a continuous mapping.

Our main result, Theorem 2, shows that connectedness is a topological property which is shared by any space and its α-topology. Together with Theorem 1, this enables us to see Noiri's work in its proper setting, namely as a particular case of the preservation of connectedness by continuous functions. We are also able to extend Noiri's result [7, Theorem 3.6] for open connected subsets to the class of semi-open connected subsets.

2. RELATIONSHIPS

Theorem 1. *The function $f : (X, T) \to (Y, U)$ is s.s.c. if and only if $f : (X, T^\alpha) \to (Y, U)$ is continuous.*

Proof. We have $f : (X, T) \to (Y, U)$ is s.s.c. if and only if $f^{-1}(V) \in T^\alpha$ for all $V \in U$, that is if and only if $f : (X, T^\alpha) \to (Y, U)$ is continuous. □

The observation of Noiri [7] that s.s.c. is a weak form of continuity, that is that continuity implies s.s.c., is immediate from the containment $T \subset T^\alpha$. Taking the topology on X as fixed, Example 2.3 of [7] shows that the notions of continuity and s.s.c. are distinct. Theorem 1 shows that these concepts coincide if one is willing to change the topology on X in the appropriate fashion. Then [7, Example 2.3] can be regarded as showing that the set $C((X, T), Y)$ of continuous functions from (X, T) to Y is properly contained in $C((X, T^\alpha), Y)$.

Lemma 1. For any topological space (X, T), $SO(X) \cap PO(X) = T^\alpha$.

Proof. One implication, namely $T^\alpha \subset SO(X) \cap PO(X)$, is clear since closure and interior respect inclusion.

Conversely, let S be semi-open and preopen. Then since S is semi-open we have $S \subset \operatorname{cl}(\operatorname{int} S)$, so that $\operatorname{cl} S \subset \operatorname{cl}(\operatorname{cl}(\operatorname{int} S)) = \operatorname{cl}(\operatorname{int} S)$, and hence $\operatorname{int}(\operatorname{cl} S) \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$. But since S is preopen, $S \subset \operatorname{int}(\operatorname{cl} S)$ so that $S \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$, that is, S is an α-set. □

Definition 3. A function $f : (X, T) \to (Y, U)$ is called

(i) *semi-continuous* [3] (abbreviated as s.c.) if the inverse image of each open set in Y is semi-open in X,

(ii) *precontinuous* [5] (abbreviated as p.c.) if the inverse image of each open set in Y is preopen in X.

It is worth noting that the concept of precontinuity has been in the literature for some considerable time. In 1922, Blumberg [1] defined the notion of a real valued function on a Euclidean space being *densely approached* at a point in its domain. More recently, Husain [2] has generalized this idea to arbitrary topological spaces. The function $f : (X, T) \to (Y, U)$ is said to be *almost continuous at* $x \in X$ if for each open set V in Y containing $f(x)$, the T closure of $f^{-1}(V)$ is a neighbourhood of x. If f is almost continuous at each point of X, then f is called *almost continuous* in the sense of Husain. This is clearly equivalent to the condition that for each open set V in Y, $f^{-1}(V) \subset \operatorname{int} \operatorname{cl} f^{-1}(V)$.

Noiri [7] has observed that s.s.c. implies s.c. but not conversely. Lemma 1 allows us to provide the answer as to when the converse holds.

Proposition 1. *The function* $f : (X, T) \to (Y, U)$ *is s.s.c. if and only if it is s.c. and p.c.*

Proof. That f is s.s.c. implies f is s.c. and f is p.c. follows immediately from the definitions.

Conversely, let f be s.c. ano p.c., and let V be an open set in Y. Then $f^{-1}(V) \in SO(X) \cap PO(X)$, so that $f^{-1}(V) \in T^\alpha$ by Lemma 1 and hence f is s.s.c.

Definition 4. The function $f : (X, T) \to (Y, U)$ is called

(i) *irresolute* if the inverse image of each semi-open set in Y is semi-open in X,
(ii) *α-irresolute* [4] if the inverse image of every α-set in Y is an α-set in X.

From this definition it is clear that f is α-irresolute (irresolute) implies f is s.s.c. (s.c.), and that $f : (X, T) \to (Y, U)$ is α-irresolute if and only if $f : (X, T^\alpha) \to (Y, U^\alpha)$ is continuous.

Example 1. Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$, and define topologies $T = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $U = \{\phi, Y, \{x\}\}$. We define $f : X \to Y$ by $f(a) = x$, $f(b) = y$, $f(c) = f(d) = z$. Note that $T^\alpha = T$ and $U^\alpha = \{\phi, X, \{x\}, \{x, y\}, \{x, z\}\}$. Then f is s.s.c., but not α-irresolute since $f^{-1}(\{x, y\}) = \{a, b\} \notin T^\alpha$. Define $j : (Y, U) \to (X, T)$ by $j(x) = b$, $j(y) = c$ and $j(z) = d$. Then j is s.s.c. since it is α-irresolute, but j is not irresolute since $j^{-1}(\{a, d\}) = \{z\} \notin SO(Y)$.

3. CONNECTEDNESS

Here we prove that the property of connectedness is shared by any topological space and its α-topology.

Theorem 2. *If* (X, T) *is a topological space, then* (X, T) *is disconnected if and only if* (X, T^α) *is disconnected.*

Proof. If (X, T) is disconnected, then $T \subset T^\alpha$ implies that (X, T^α) is disconnected.

Conversely, suppose (X, T^α) is disconnected. Then $X = A \cup B$ where A and B are non-empty, T^α open sets such that $A \cap B = \emptyset$. Hence $\operatorname{int} A \cap \operatorname{int} B = \emptyset$, so that $\operatorname{int} A \cap \operatorname{cl}(\operatorname{int} B) = \emptyset$. [All closures and interiors are in (X, T).] Therefore $\operatorname{int} A \cap \operatorname{int}(\operatorname{cl}(\operatorname{int} B)) = \emptyset$ which implies that $\operatorname{cl}(\operatorname{int} A) \cap \operatorname{int}(\operatorname{cl}(\operatorname{int} B)) = \emptyset$, so that we have $\operatorname{int}(\operatorname{cl}(\operatorname{int} A)) \cap \operatorname{int}(\operatorname{cl}(\operatorname{int} B)) = \emptyset$. But $A, B \in T^\alpha$ so that $A \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$ and similarly for B. Thus $X = A \cup B = \operatorname{int}(\operatorname{cl}(\operatorname{int} A) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int} B))$, and hence (X, T) is disconnected. □

As a corollary we have Noiri's main result [7, Theorem 3.1].

Theorem 3. *If $f : (X, T) \to (Y, U)$ is a s.s.c. surjection and (X, T) is connected, then (Y, U) is connected.*

Proof. By Theorem 2, (X, T^α) is connected. Thus by Theorem 1, (Y, U) is the image of the connected space (X, T^α) under the continuous function $f : (X, T^\alpha) \to (Y, U)$, and so is connected. □

The other major result of Noiri's paper [7, Theorem 3.6] is that the s.s.c. images of open connected sets are connected. We provide a significant generalization of this theorem by replacing open sets by semi-open sets. First we need a lemma.

Lemma 2. *If A is semi-open and B is an α-set in (X, T), then $A \cap B$ is an α-set in the subspace $(A, T \mid A)$.*

Proof. We note that

(i) If $M \subset A$ then $T| A \operatorname{cl} M = (T\operatorname{cl} M) \cap A$ and $T| A \operatorname{int} M \supset T\operatorname{int} M$,

and (ii) if G is T open then $G \cap T\operatorname{cl} H \subset T\operatorname{cl}(G \cap H)$ for any $H \subset X$.

We have that $A \subset T\operatorname{cl}\, T\operatorname{int} A$ and $B \subset T\operatorname{int}\, T\operatorname{cl}\, T\operatorname{int} B$, and we want to establish $A \cap B \subset T| A \operatorname{int}\, T| A \operatorname{cl}\, T| A \operatorname{int}(A \cap B)$. Note that we suppress many of the parentheses we could use in this proof. Now $A \cap B \subset A \cap T\operatorname{int}\, T\operatorname{cl}\, T\operatorname{int} B$, which being open in A,

$= T| A \operatorname{int}(A \cap T\operatorname{int}\, T\operatorname{cl}\, T\operatorname{int} B)$,
$\subset T| A \operatorname{int}(T\operatorname{cl}\, T\operatorname{int} A \cap T\operatorname{int}\, T\operatorname{cl}\, T\operatorname{int} B)$, which by (ii),
$\subset T| A \operatorname{int}\, T\operatorname{cl}(T\operatorname{int} A \cap T\operatorname{int}\, T\operatorname{cl}\, T\operatorname{int} B)$, which by (i),
$= T| A \operatorname{int}\, T| A \operatorname{cl}(T\operatorname{int} A \cap T\operatorname{int}\, T\operatorname{cl}\, T\operatorname{int} B)$, which by (i) and the equality $T\operatorname{int}\, T\operatorname{int} A = \operatorname{int} A$,
$\subset T| A \operatorname{int}\, T| A \operatorname{cl}\, T| A \operatorname{int}(T\operatorname{int} A \cap T\operatorname{cl}\, T\operatorname{int} B)$, which by (ii) and (i)
$\subset T| A \operatorname{int}\, T| A \operatorname{cl}\, T| A \operatorname{int}\, T| A \operatorname{cl}(T\operatorname{int} A \cap T\operatorname{int} B)$, which by (i)
$\subset T| A \operatorname{int}\, T| A \operatorname{cl}\, T| A \operatorname{int}\, T| A \operatorname{cl}\, T| A \operatorname{int}(A \cap B)$
$= T| A \operatorname{int}\, T| A \operatorname{cl}\, T| A \operatorname{int}(A \cap B)$, since $\operatorname{int} \operatorname{cl} \operatorname{int} \operatorname{cl} W = \operatorname{int} \operatorname{cl} W$

for any subset W of an arbitrary topological space. □

Proposition 2. *If* $f:(X, T) \to (Y, U)$ *is s.s.c. and* $A \in SO(X)$, *then* $f \mid A:$ $:(A, T \mid A) \to (Y, U)$ *is s.s.c.*

Proof. If V is open in (Y, U), then $f^{-1}(V) \in T^{\alpha}$. Now $(f \mid A)^{-1}(V) = A \cap f^{-1}(V)$, which is an α-set in $(A, T \mid A)$ by Lemma 2. Hence $f \mid A:(A, T \mid A) \to (Y, U)$ is s.s.c. □

Theorem 4. *If* $f:(X, T) \to (Y, U)$ *is s.s.c., then* $f(A)$ *is connected for any semi-open connected subset* A *of* X.

Proof. By Proposition 2, $f \mid A:(A, T \mid A) \to (Y, U)$ is s.s.c. Hence $f \mid A:$ $:(A, T \mid A) \to (f(A), U \mid f(A))$ is a s.s.c. surjection and A is connected so that $f(A)$ is connected by Theorem 3. □

References

[1] *H. Blumberg:* New properties of all real functions. Trans. Amer. Math. Soc. *24* (1922), 113—128.

[2] *T. Husain:* Almost continuous mappings. Prace Mat. *10* (1966), 1—7.

[3] *N. Levine:* Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly *70* (1963), 36—41.

[4] *S. N. Maheshwari, S. S. Thakur:* On α-irresolute mappings. Tamkang J. Math. *11* (1980), 209—214.

[5] *A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Debb:* On precontinuous and weak precontinuous mappings. Proc. Math. and Phys. Soc. Egypt *51* (1981).

[6] *O. Njåstad:* On some classes of nearly open sets, Pacific J. Math. *15* (1965), 961—970.

[7] *T. Noiri:* A function which preserves connected spaces. Čas. pěst. mat. *107* (1982), 393—396.

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