Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 109 (1984), No. 3, 304--322

Persistent URL: http://dml.cz/dmlcz/108434

SEMIELLIPTIC SINGULARITIES

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(Received December 21, 1983)

INTRODUCTION

Let $m = [m_1, \ldots, m_N]$ be a fixed N-tuple of positive integers. Given a multiindex $\alpha = [\alpha_1, \ldots, \alpha_N]$ we put

$$|\alpha : m| = \sum_{k=1}^{N} \alpha_k / m_k ;$$

for the sake of brevity we put

$$b = |\mathbf{1} : m| , \tag{0}$$

where $\mathbf{1} = [1, \ldots, 1] \in \mathbb{R}^N$ has all components equal to 1. Let U be an open set in the Euclidean N-space $\mathbb{R}^N$. We shall suppose that with each multiindex α satisfying

$$|\alpha : m| \leqq 1 \tag{1}$$

is associated an infinitely differentiable function a_α in U. Writing ∂_k for the partial derivative respect to the k-th variable and i for the imaginary unit we use the usual notation

$$D_k = -i\partial_k , \quad D^\alpha = D_1^{\alpha_1} \ldots D_N^{\alpha_N}$$

and consider the differential operator (acting on functions or distributions in U) given by

$$P(D) = \sum_\alpha a_\alpha D^\alpha , \tag{2}$$

the sum being extended over all multiindices satisfying (1).

By a distinguished parallelepiped we mean an N-dimensional interval

$$K = \mathop{\times}_{k=1}^{N} I_k \tag{3}$$

arising as a Cartesian product of one-dimensional intervals I_k of the corresponding lengths r^{1/m_k} $(k = 1, \ldots, N)$; the value of the parameter $r > 0$ will be denoted by $|K|$ or, in case of need, more explicitly by $|K|_m$.

If u is a function integrable with respect to the Lebesgue measure λ over K, we denote by

$$u_K = \frac{1}{\lambda(K)} \int_K u \, d\lambda$$

its mean value on K. For any locally integrable function u in U and each compact set $Q \subset U$ with a non-empty interier Q^0 we put

$$\Omega_Q^m(\delta, u) = \sup_K \int_K |u - u_K| \, d\lambda \,, \tag{4}$$

K ranging over all distinguished parallelepipeds satisfying, for the given $\delta > 0$, the conditions

$$K \subset Q \,, \quad |K|_m \leqq \delta \,.$$

Writing

$$\|x\|_m = \sum_{k=1}^{N} |x_k|^{m_k}$$

for $x = [x_1, \ldots, x_N] \in \mathbb{R}^N$ we define for any locally bounded function v in U and any non-void compact set $Q \subset U$:

$$\omega_Q^m(\delta, v) = \sup \{ |v(x) - v(y)| ; \; x, y \in Q, \; \|x - y\|_m \leqq \delta \} \,, \quad \delta > 0 \,. \tag{5}$$

Finally, we employ the distinguished parallelepipeds for constructing measure of the Hausdorff type. By a measure function we mean any real-valued function f in $\mathbb{R}$ for which there is a $\delta_f > 0$ such that f is non-negative, non-decreasing and continuous on $(0, \delta_f)$. Given such an f we define for any nonvoid set $L \subset \mathbb{R}^N$ and any $\varepsilon \in (0, \delta_f)$

$$\mathscr{H}_{m,\varepsilon}^f(L) = \inf \sum_n f(|K_n|) \,,$$

where the infimum is taken over all sequences of distinguished parallelepipeds K_n satisfying the conditions

$$|K_n| \leqq \varepsilon \,, \quad L \subset \bigcup_n K_n \,.$$

Setting $\mathscr{H}_{m,\varepsilon}^f(\emptyset) = 0$ we define the (outer) Hausdorff measure of type m for any $L \subset \mathbb{R}^N$ by

$$\mathscr{H}_m^f(L) = \lim_{\varepsilon \downarrow 0} \mathscr{H}_{m,\varepsilon}^f(L) \,.$$

Using this notation we may formulate the following results proved in [12].

Theorem 1. *Let* $0 \leqq \gamma \leqq 1$ *and let* $g \geqq 0$ *be a function of the variable* $t > 0$ *satisfying*

$$\liminf_{t \downarrow 0} g(t) \, t^{-\gamma - b} > 0 \,.$$

Let u be a function in U such that each multiindex α fulfilling (1) can be split into a sum

$$\hat{\alpha} + \tilde{\alpha} = \alpha \tag{6}$$

in such a way that

$$|\tilde{\alpha} : m| \leqq \gamma \tag{7}$$

and $D^{\hat{\alpha}}u$ is locally integrable in U and satisfies

$$\Omega_Q^m(\delta, D^{\hat{\alpha}}u) = O(g(\delta)) \quad (\delta \downarrow 0) \tag{8}$$

on each compact set $Q \subset U$, $Q^0 \neq \emptyset$. Further, let $F \subset U$ be relatively closed in U and suppose that

$$P(D)\,u = 0 \quad in \quad U \setminus F \tag{9}$$

in the sense of distributions. If

$$f(t) = g(t)\,t^{-\gamma} \quad (t > 0)$$

defines a measure function and F has a locally finite $\mathscr{H}_m^f$-measure, then there exists a locally integrable (with respect to $\mathscr{H}_m^f$) function h in U vanishing on $U \setminus F$ such that

$$P(D)\,u = h\mathscr{H}_m^f \quad in \quad U \tag{10}$$

in the sense of distributions (which means that, for any infinitely differentiable test function ψ with support in U, the value of the distribution $P(D)\,u$ at ψ is given by the integral $\int_F \psi h \,\mathrm{d}\mathscr{H}_m^f$). In particular, if $\mathscr{H}_m^f(F) = 0$, then F is removable for u and $P(D)\,u = 0$ in the whole of U.

Corollary 1. *Let $0 \leqq \gamma \leqq 1$ and let $\omega \geqq 0$ be a function of the variable $t > 0$ satisfying*

$$\liminf_{t \downarrow 0} \omega(t)\,t^{-\gamma} > 0\,.$$

Let u be a function in U such that each multiindex α fulfilling (1) splits as in (6), (7) in such a way that $D^{\hat{\alpha}}u$ is continuous in U and satisfies

$$\omega_Q^m(\delta, D^{\hat{\alpha}}u) = O(\omega(\delta)) \quad (\delta \downarrow 0) \tag{11}$$

on each compact set $Q \subset U$. Let $F \subset U$ be relatively closed in U and suppose (9). If

$$f(t) = \omega(t)\,.\,t^{b-\gamma} \quad (t > 0)$$

defines a measure function and F has a locally finite $\mathscr{H}_m^f$-measure, then there exists a locally bounded Baire function h in U vanishing on $U \setminus F$ such that (10) holds in the sense of distributions. In particular, if $\mathscr{H}_m^f(F) = 0$, then $P(D)\,u = 0$ in the whole of U.

These results give, in terms of the measure $\mathscr{H}^f_m$, sufficient conditions for the removability of a singular set $F \subset U$ for functions which together with certain derivatives satisfy suitable conditions concerning their integral modulus of continuity (4) or their anisotropic modulus of continuity (5). In order to obtain necessary conditions one has, of course, to impose adequate restrictions on the type of the differential operator $P(D)$. We show in § 2 that for semielliptic operators it is possible to get, again with help of the measure $\mathscr{H}^f_m$, sharp estimates of the size of the set of singularities, which can be combined with the above results into necessary and sufficient conditions on the removability of singularities. These estimates are based on a preliminary investigation, included in § 1, of the behaviour in $\mathbb{R}^N$ of potentials derived from kernels in $\mathbb{R}^N \times \mathbb{R}^N$ with a specified singular behaviour near the diagonal.

§ 1. BEHAVIOUR OF POTENTIALS

Let $G(x, y)$ be a complex-valued Baire function on $\mathbb{R}^N \times \mathbb{R}^N$ $(x, y \in \mathbb{R}^N)$ which is bounded on every set of the form

$$\{[x, y];\ r \leqq |x - y| \leqq R,\ |x| \leqq R\},$$

where $0 < r < R < \infty$. Such a G will be called a kernel. We assume that there is a $\beta > 0$ such that the estimate

$$G(x, y) = O(\|x - y\|_m^{-\beta}) \quad \text{as} \quad |x - y| \downarrow 0 \tag{12}$$

holds locally uniformly with respect to $x \in \mathbb{R}^N$.

Given a complex-valued Borel measure μ with a compact support spt μ we denote by $|\mu|$ its variation and define

$$v^m(\delta, \mu) = \sup_K \{|\mu|(K),\ |K|_m \leqq \delta\} \tag{13}$$

for $\delta > 0$ with K ranging over all distinguished parallelepipeds with $|K|_m$ not exceeding δ. We shall see that, under adequate restrictions on $v^m(\cdot, \mu)$, the potential

$$G\,\mu(x) = \int_{\mathbb{R}^N} G(x, y)\, \mathrm{d}\mu(y) \tag{14}$$

is defined a.e. (or everywhere) in $\mathbb{R}^N$, and additional information concerning the behaviour of G near the diagonal permits to obtain estimates of the integral modulus of continuity $\Omega^m(\cdot, \cdot)$ (or, respectively, the ordinary modulus of continuity $\omega^m(\cdot, \cdot)$ of the potential (14)).

1.1. Lemma. *Let φ be a measure function such that*

$$\int_0^\delta \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t < \infty \tag{15}$$

for a suitable $\delta = \delta_\varphi > 0$. *Put*

$$B(x, r) = \{y \in \mathbb{R}^N;\ \|x - y\|_m < r\} \quad (x \in \mathbb{R}^N,\ r > 0)\,.$$

Then for any $s > 0$ *there is a* $c = c(s) > 0$ *such that, for each compact* $Q \subset \mathbb{R}^N$ *and each complex-valued compactly supported Borel measure* μ *satisfying*

$$v^m(r, \mu) = O(\varphi(r))\,, \quad r \downarrow 0\,, \tag{16}$$

the estimate

$$\int_{B(x,sr)} |G(x, y)|\, \mathrm{d}|\mu|\,(y) = O\left(\int_0^{cr} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t\right), \quad r \downarrow 0\,, \tag{17}$$

holds uniformly with respect to $x \in Q$.

Proof. In view of our assumptions on G there are positive constants c_1, r_0 such that, for all $x \in Q$ and $y \in B(x, sr)$ with $r \in (0, r_0)$, the following estimate holds:

$$|G(x, y)| \leqq c_1 \|x - y\|_m^{-\beta}\,.$$

Hence

$$\int_{B(x,sr)} |G(x, y)|\, \mathrm{d}|\mu|\,(y) \leqq c_1 \int_{B(x,sr)} \|x - y\|^{-\beta}\, \mathrm{d}|\mu|\,(y) =$$

$$= c_1 \int_0^\infty |\mu|\,(B(x, \min(sr, t^{-1/\beta})))\, \mathrm{d}t =$$

$$= c_1 (sr)^{-\beta}\, |\mu|\,(B(x, sr)) + c_1 \int_{(sr)^{-\beta}}^\infty |\mu|\,(B(x, t^{-1/\beta}))\, \mathrm{d}t\,.$$

Note that

$$B(x, \tau) \subset \bigtimes_{k=1}^N \langle x_k - \tau^{1/m_k},\ x_k + \tau^{1/m_k}\rangle$$

can be covered by 2^N distinguished parallelepipeds K with $|K|_m = \tau$. Assuming that r_0 has been fixed sufficiently small and c_1 sufficiently large we have for $r \in (0, r_0)$ in view of (16)

$$\int_{(sr)^{-\beta}}^\infty |\mu|\,(B(x, t^{-1/\beta})) \leqq 2^N c_1 \int_{(sr)^{-\beta}}^\infty \varphi(t^{-1/\beta})\, \mathrm{d}t = 2^N c_1 \beta \int_0^{sr} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t\,.$$

Consequently,

$$\int_{B(x,sr)} |G(x, y)|\, \mathrm{d}|\mu|\,(y) = O\left(r^{-\beta}\varphi(sr) + \int_0^{sr} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t\right), \quad r \downarrow 0\,,$$

uniformly with respect to $x \in Q$. Since φ is non-decreasing on $(0, \delta)$ we have the inequality

$$r^{-\beta}\varphi(sr) \leqq c_2 \int_{sr}^{2sr} \varphi(s\tau)\, \tau^{-\beta-1}\, \mathrm{d}\tau = c_2 s^\beta \int_{s^2 r}^{2s^2 r} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t$$

provided $(s + 2s^2)\, r_0 < \delta$ and $c_2 = \beta s^\beta (1 - 2^{-\beta})^{-1}$ which yields the estimate (17) with $c = s + 2s^2$.

1.2. Remark. It follows from the above lemma that the potential (14) is defined everywhere for any compactly supported Borel measure μ satisfying (16), where φ fulfils (15).

1.3. Lemma. *If $b > \beta$ (cf. (0), (12)) then the potential (14) is defined a.e. (λ) and is locally integrable in $\mathbb{R}^N$ (w.r. to λ) for any compactly supported Borel measure μ. If, in addition, $G(x, y)$ is continuously differentiable with respect to x off the diagonal and*

$$(19) \qquad \frac{\partial G(x, y)}{\partial x_k} = O(\|x - y\|_m^{-\beta - 1/m_k}), \quad |x - y| \downarrow 0 \quad (k = 1, \dots, N)$$

locally uniformly w.r. to x, then the integral modulus of continuity of $G\mu$ admits, for every compact set $Q \subset \mathbb{R}^N$ with $Q^0 \neq \emptyset$, the estimate

$$(20) \qquad \Omega_Q^m(\delta, G\mu) = O\left(\sum_{k=1}^N \delta^{b + 1/m_k} \int_\delta^\infty v^m(t, \mu)\, t^{-\beta - 1 - 1/m_k}\, \mathrm{d}t\right), \quad \delta \downarrow 0\,.$$

Proof. Note that, for any distinguished parallelepiped K, $\lambda(K) = |K|^b$. Applying **1.1** to the transposed kernel $G(y, x)$ and the measure function

$$\varphi(t) = t^b, \quad t > 0\,,$$

we get for any $s > 0$

$$\int_{B(y,sr)} |G(x, y)|\, \mathrm{d}\lambda(x) = O\left(\int_0^{cr} t^{b-\beta-1}\, \mathrm{d}t\right) = O(r^{b-\beta}), \quad r \downarrow 0\,,$$

which holds locally uniformly w.r. to $y \in \mathbb{R}^N$.

Now let μ be an arbitrary non-trivial compactly supported Borel measure in $\mathbb{R}^N$. Given $x^0 \in \mathbb{R}^N$ then

$$|\mu|\,(B(x^0, r)) \leqq 2^N v^m(r, \mu)$$

and, for $s \geqq \sum_{k=1}^N 2^{m_k}$,

$$(21) \qquad y \in B(x^0, r) \Rightarrow B(x^0, r) \subset B(y, sr)\,.$$

Hence

$$\int_{B(x^0,r)} \left(\int_{B(x^0,r)} |G(x, y)\, |\mathrm{d}|\mu|\,(y)\right) \mathrm{d}\lambda(x) \leqq$$

$$\leqq \int_{B(x^0,r)} \left(\int_{B(y,sr)} |G(x, y)|\, \mathrm{d}\lambda(x)\right) \mathrm{d}|\mu|\,(y) = O(r^{b-\beta} v^m(r, \mu))\,, \quad r \downarrow 0\,,$$

locally uniformly w.r.t. to $x^0 \in \mathbb{R}^N$. We see that the potential $u = G\mu$ is defined a.e. (λ) and is locally Lebesgue integrable. We now proceed to investigate its integral modulus of continuity on an arbitrarily fixed compact set $Q \subset \mathbb{R}^N$, $Q^0 \neq \emptyset$. Let $K \subset Q$ be a distinguished parallelepiped with $|K| = r$. Then

$$\int_K |u - u_K| \,\mathrm{d}\lambda \leqq r^{-b} \iint_{K \times K} |u(y) - u(z)| \,\mathrm{d}y \,\mathrm{d}z \leqq$$

$$\leqq r^{-b} \int_{\mathbb{R}^N} \left(\iint_{K \times K} |G(y, \xi) - G(z, \xi)| \,\mathrm{d}y \,\mathrm{d}z \right) \mathrm{d}|\mu|(\xi) .$$

If x is the center of K and $s \geqq \sum_{j=1}^{N} 2^{-m_j}$, then $K \subset B(x, sr)$. Note that $v^m(2r, \mu) \leqq 2^N v^m(r, \mu)\,(r > 0)$, so that $v^m(sr, \mu) = O(v^m(r, \mu))$ as $r \downarrow 0$ for any $s > 0$.

Hence

$$\int_{B(x,sr)} \left(\iint_{K \times K} |G(y, \xi) - G(z, \xi)| \,\mathrm{d}y \,\mathrm{d}z \right) \mathrm{d}|\mu|\,(\xi) \leqq$$

$$\leqq 2r^b \int_{B(x,sr)} \left(\int_K |G(y, \xi)| \,\mathrm{d}y \,\mathrm{d}|\mu|\,(\xi) = O(r^{2b-\beta}\, v^m(r, \mu) \right), \quad r \downarrow 0 ,$$

uniformly w.r.t. to $K \subset Q$, $|K| = r$. Now we shall make use of the estimates (19) of the derivatives $\partial_j G = \partial G / \partial x_j \,(j = 1, \ldots, N)$. If $\xi \in \operatorname{spt} \mu$ is not situated on the segment with end-points $y, z \in K$, then for a suitable constant k and $\beta_j = \beta + 1/m_j$ we have

$$|G(y, \xi) - G(z, \xi)| = \left| \int_0^1 \frac{\partial}{\partial \theta} G(y + \theta(z - y), \xi) \,\mathrm{d}\theta \right| \leqq$$

$$\leqq \sum_{j=1}^{N} \int_0^1 |y_j - z_j| \cdot |\partial_j G(y + \theta(z - y), \xi)| \,\mathrm{d}\theta \leqq$$

$$\leqq k \sum_{j=1}^{N} r^{1/m_j} \int_0^1 \|y + \theta(z - y) - \xi\|_m^{-\beta_j} \,\mathrm{d}\theta .$$

Writing

$$\bar{m} = \max \{m_j;\ 1 \leqq j \leqq N\} \tag{22}$$

we have

$$|y_j + \theta(z_j - y_j) - \xi_j|^{m_j} \geqq 2^{-m_j} |\xi_j - x_j|^{m_j} - |y_j + \theta(z_j - y_j) - x_j|^{m_j} \geqq$$

$$\geqq 2^{-\bar{m}} |\xi_j - x_j|^{m_j} - r ;$$

consequently, if s has been fixed large enough to guarantee that

$$h = 2^{-\bar{m}} - \frac{N}{s} > 0 ,$$

we have for $\theta \in \langle 0, 1 \rangle$ and

$$\xi \in CB(x, sr) = \{\xi \in \mathbb{R}^N;\ \|x - \xi\| \geqq sr\}$$

the inequality

$$\|\xi - [y + (z - y)]\|_m \geqq 2^{-\bar{m}}\|\xi - x\|_m - Nr \geqq h\|\xi - x\|_m,$$

which implies that ξ is not on the segment with end-points y, z and

$$|G(y, \xi) - G(z, \xi)| \leqq k \sum_{j=1}^{N} r^{1/m_j}(h\|\xi - x\|_m)^{-\beta_j},$$

$$\int_{CB(x,sr)} \left(\iint_{K \times K} |G(y, \xi) - G(z, \xi)| \, dy \, dz \right) d|\mu| (\xi) \leqq$$

$$\leqq k_1 r^{2b} \sum_{j=1}^{N} r^{1/m_j} \int_{CB(x,sr)} \|\xi - x\|_m^{-\beta_j} d|\mu| (\xi),$$

where $k_1 = k \sum_{j=1}^{N} h^{-\beta_j}$. We get

$$\int_{CB(x,sr)} \|\xi - x\|_m^{-\beta_j} \, d|\mu| (\xi) = \int_0^{\infty} |\mu| (Q_\tau) \, d\tau,$$

where

$$Q_\tau = \{y \in \mathbb{R}^N;\ sr \leqq \|x - y\|_m < \tau^{-1/\beta_j}.$$

Clearly, $Q_\tau = \emptyset$ for $\tau > (sr)^{-\beta_j}$, while for the other τ

$$|\mu| (Q_\tau) \leqq 2^N \, v^m(\tau^{-1/\beta_j}, \mu),$$

whence

$$\int_0^{\infty} |\mu| (Q_\tau) \, d\tau \leqq 2^N \int_0^{(sr)^{-\beta_j}} v^m(\tau^{-1/\beta_j}, \mu) \, d\tau = 2^N \beta_j \int_{sr}^{\infty} v^m(t, \mu) \, t^{-\beta_j - 1} \, dt.$$

We see that

$$\int_{CB(x,sr)} \left(\int_{K \times K} |G(y, \xi) - G(z, \xi) \, dy \, dz \right) d|\mu| (\xi) =$$

$$= O\left(\sum_{j=1}^{N} r^{2b + 1/m_j} \int_{sr}^{\infty} v^m(t, \mu) \, t^{-\beta_j - 1} \, dt \right) \quad \text{as} \quad |K| = r \downarrow 0.$$

Summarizing we obtain for $K \subset Q$, $|K| = r$,

$$\int_K |u - u_K| \, d\lambda = O\left(r^{b-\beta} v^m(r, \mu) + \sum_{j=1}^{N} r^{b + 1/m_j} \int_{sr}^{\infty} v^m(t, \mu) \, t^{-\beta_j - 1} \, dt \right), \quad r \downarrow 0.$$

Note that, for $r < 1$,

$$r^{b+1/m_j} \int_r^\infty v^m(t, \mu)\, t^{-\beta_j - 1}\, dt \geqq \beta_j^{-1} r^{b-\beta} v^m(r, \mu)\,,$$

so that

$$r^{b-\beta} v^m(r, \mu) = O\left(\sum_{j=1}^N r^{b+1/m_j} \int_r^\infty v^m(t, \mu)\, t^{-\beta_j - 1}\, dt \right), \quad r \downarrow 0\,.$$

Since $s \geqq 1$, we arrive at (20).

1.4. Lemma. *Let φ be a measure function such that the integral* (15) *converges for a suitable $\delta = \delta_\varphi > 0$. Then, for each compactly supported Borel measure satisfying* (16), *the potential* (14) *is everywhere defined and its modulus of continuity satisfies, on each compact set $Q \in \mathbb{R}^N$, the estimate*

$$\omega_Q^m(r, G\mu) = O\left(\int_0^r \varphi(t)\, t^{-\beta-1}\, dt + \sum_{j=1}^N r^{1/m_j} \int_r^\delta \varphi(t)\, t^{-\beta-1-1/m_j}\, dt \right), \quad r \downarrow 0\,.$$

Proof. Let μ be a non-trivial compactly supported Borel measure satisfying (16); according to **1.2**, the potential $u = G\mu$ is everywhere defined. Fix a compact set $Q \subset \mathbb{R}^N$, $r > 0$ and consider $y, z \in Q$ with $\|y - z\|_m \leqq r$. Writing $x = \frac{1}{2}(y + z)$ we have

$$|u(y) - u(z)| \leqq \int_{B(x,sr)} |G(y, \xi)|\, d|\mu|\,(\xi) + \int_{B(x,sr)} |G(z, \xi)|\, d|\mu|\,(\xi) + \\ + \int_{CB(x,sr)} |G(y, \xi) - G(z, \xi)|\, d|\mu|\,(\xi)$$

for $s > 0$, where $CB(x, sr)$ has the meaning described in the proof of **1.3**. As we have seen in the course of that proof, for s large enough and $\xi \in CB(x, sr)$ we have

$$|G(y, \xi) - G(z, \xi)| \leqq k_1 \sum_{j=1}^N r^{1/m_j} \|\xi - x\|_m^{-\beta_j}\,,$$

$$\int_{CB(x,sr)} \|\xi - x\|_m^{-\beta_j}\, d|\mu|\,(\xi) = O\left(\int_{sr}^\infty v^m(t, \mu)\, t^{-\beta_j - 1}\, dt \right), \quad r \downarrow 0\,.$$

In view of the implication (21) (which is valid for $s \geqq \sum_{j=1}^N 2^{m_j}$) we have the inclusions $B(x, sr) \subset B(y, s^2 r)$, $B(x, sr) \subset B(z, s^2 r)$, which yields by **1.1** for a suitable $c > 0$

$$\int_{B(x,sr)} |G(y, \xi)|\, d|\mu|\,(\xi) + \int_{B(x,sr)} |G(z, \xi)|\, d|\mu|\,(\xi) = O\left(\int_0^{cr} \varphi(t)\, t^{-\beta-1}\, dt \right), \quad r \downarrow 0\,.$$

Since

$$r^{1/m_j} \int_{sr}^\infty v^m(t, \mu)\, t^{-\beta_j - 1}\, dt = O\left(r^{1/m_j} \int_{sr}^\delta v^m(t, \mu)\, t^{-\beta_j - 1}\, dt \right), \quad r \downarrow 0\,,$$

we obtain

$$|u(y) - u(z)| = O\left(\int_0^{cr} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t + \sum_{j=1}^{N} r^{1/m_j} \int_{sr}^{\delta} v^m(t, \mu)\, t^{-\beta_j-1}\, \mathrm{d}t\right), \quad r \downarrow 0 .$$

If $c > 1$, then for small $r > 0$,

$$\int_r^{cr} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t \leqq c^{1/m_j} \int_r^{cr} \varphi(t)\, t^{-\beta-1} \left(\frac{t}{r}\right)^{-1/m_j} \mathrm{d}t \leqq c^{1/m_j} r^{1/m_j} \int_r^{\delta} \varphi(t)\, t^{-\beta_j-1}\, \mathrm{d}t .$$

Consequently,

$$\omega_Q^m(r, u) = O\left(\int_0^{r} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t + \sum_{j=1}^{N} r^{1/m_j} \int_r^{\delta} \varphi(t)\, t^{-\beta_j-1}\, \mathrm{d}t\right), \quad r \downarrow 0 .$$

The above estimates can be conveniently combined with the following version of Frostman's lemma which immediately follows from Lemma 7 proved in [8].

1.5. Lemma. *If φ is a measure function on $(0, \delta)$ and $F \subset \mathbb{R}^N$ is a compact set with $\mathscr{H}_m^{\varphi}(F) > 0$, then there exists a nontrivial Borel measure $\mu \geqq 0$ with its support contained in F such that*

$$\mu(K) \leqq \varphi(|K|_m)$$

for all distinguished parallelepipeds K with $|K|_m \leqq \delta$.

As a consequence of **1.5**, **1.3** and **1.4** we obtain

1.6. Proposition. *Let $G(x, y)$ be a kernel which is continuously differentiable w.r. to x off the diagonal and satisfies (12), (19) locally uniformly w.r. to x. If φ is a measure function on $(0, \delta_\varphi)$ and $F \subset \mathbb{R}^N$ is a compact set with $\mathscr{H}_\varphi^m(F) > 0$, then there is a nontrivial Borel measure $\mu \geqq 0$ with its support in F such that*

$$v^m(\delta, \mu) \leqq \varphi(\delta)\,, \quad \delta \in (0, \delta_\varphi)\,;$$

the potential $u = G\mu$ of any such μ is locally integrable and satisfies, for each compact $Q \subset \mathbb{R}^N$ with $Q^0 = \emptyset$, the estimate

$$\Omega_Q^m(\delta, u) = O\left(\sum_{j=1}^{N} \delta^{b+1/m_j} \int_\delta^{\delta_\varphi} \varphi(t)\, t^{-\beta-1-1/m_j}\, \mathrm{d}t\right), \quad \delta \downarrow 0 . \tag{23}$$

If

$$\int_0^{\delta_\varphi} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t < \infty ,$$

then u is everywhere defined and fulfils the estimates

$$\omega_Q^m(\delta, u) = O\left(\int_0^{\delta} \varphi(t)\, t^{-\beta-1}\, \mathrm{d}t + \sum_{j=1}^{N} \delta^{1/m_j} \int_\delta^{\delta_\varphi} \varphi(t)\, t^{-\beta-1-1/m_j}\, \mathrm{d}t\right), \quad \delta \downarrow 0 . \tag{23ω}$$

1.7. Remark. Proposition **1.6** provides estimates of moduli of smoothness which can be achieved by potentials of nontrivial measures whose supports are contained in a compact set of a positive $\mathscr{H}_m^\varphi$-measure for a given measure function φ. It is often useful to have, conversely, suitable conditions on the Hausdorff measure of a compact set F guaranteeing that F can support a nontrivial measure whose potential has a prescribed modulus of continuity. Such conditions are given in the following theorems.

1.8. Theorem. *Let $G(x, y)$ be a kernel which is differentiable w.r. to x off the diagonal and satisfies* (12), (19) *with $\beta < b$* (cf. (0)). *Let $g \geqq 0$ be a continuously differentiable function on an interval $(0, \delta_g)$ such that, for suitable $c > 0$,*

$$\varphi(t) = -t^{c+1+\beta-b}\left(\frac{g(t)}{t^c}\right)'$$

defines a positive measure function on a certain interval $(0, \delta_\varphi)$, $0 < \delta_\varphi \leqq \delta_g$; if $c > b + 1/\overline{m}$ (recall (22)) *then we suppose, in addition, that for a suitable $a > 0$,*

$$t \mapsto g(t)\, t^{a-b-1/\overline{m}} \tag{24}$$

is nonincreasing on $(0, \delta_\varphi)$. Then any compact set $F \subset \mathbb{R}^N$ with $\mathscr{H}_m^\varphi(F) > 0$ supports a nontrivial Borel measure μ with

$$v^m(\delta, \mu) \leqq \varphi(\delta)\,, \quad \delta \in (0, \delta_\varphi)\,; \tag{25}$$

its potential $u = G\mu$ is locally integrable and satisfies, for each compact set $Q \subset \mathbb{R}^N$ with $Q^0 \neq \emptyset$,

$$\Omega_Q^m(\delta, u) = O(g(\delta))\,, \quad \delta \downarrow 0\,. \tag{26}$$

Proof. If $\mathscr{H}_m^\varphi(F) > 0$ then, according to **1.6**, F supports a nontrivial Borel measure μ whose potential $u = G\mu$ fulfils (23). Taking into account the definition of φ we have for any j

$$\delta^{b+1/m_j}\int_\delta^{\delta_\varphi} \varphi(t)\, t^{-\beta-1-1/m_j}\,\mathrm{d}t = -\delta^{b+1/m_j}\int_\delta^{\delta_\varphi} t^{c-b-1/m_j}\left(\frac{g(t)}{t^c}\right)'\mathrm{d}t \leqq$$

$$\leqq g(\delta) + (c - b - 1/m_j)\,\delta^{b+1/m_j}\int_\delta^{\delta_\varphi} g(t)\, t^{-b-1-1/m_j}\,\mathrm{d}t\,.$$

If $c \leqq b + 1/m_j$, then this expression does not exceed $g(\delta)$. If $c > b + 1/m_j$ then, making use of our additional assumption on g, we get for a suitable $a > 0$

$$\delta^{b+1/m_j}\int_\delta^{\delta_\varphi} g(t)\, t^{-b-1-1/m_j}\,\mathrm{d}t = \delta^{b+1/m_j}\int_\delta^{\delta_\varphi} g(t)\, t^{a-b-1/m_j} \,.\, t^{-a-1}\,\mathrm{d}t \leqq$$

$$\leqq \delta^{b+1/m_j}\, g(\delta) \,.\, \delta^{a-b-1/m_j}\int_\delta^{\delta_\varphi} t^{-a-1}\,\mathrm{d}t \leqq \frac{1}{a}\, g(\delta)\,.$$

Thus (26) is true in any case.

1.9. Theorem. *Let $G(x, y)$ be a kernel which is differentiable w.r. to x off the diagonal and satisfies (12), (19) with $\beta < b$. Let $g \geqq 0$ be a continuously differentiable function on an interval $(0, \delta_g)$ such that, for suitable $c \in (0, b + 1/\overline{m})$,*

$$\varphi(t) = t^{c+1+\beta-b} \left(\frac{g(t)}{t^c}\right)' \tag{27}$$

defines a positive measure function on $(0, \delta_\varphi)$ $(0 < \delta_\varphi \leqq \delta_g)$ and, in addition, there is an $a > 0$ such that the function (24) is nonincreasing on $(0, \delta_\varphi)$. Then each compact set $F \subset \mathbb{R}^N$ with $\mathscr{H}^\varphi_m(F) > 0$ supports a nontrivial Borel measure μ fulfilling (25); its potential $u = G\mu$ satisfies (26) for each compact set $Q \subset \mathbb{R}^N$, $Q^0 \neq \emptyset$.

Proof. This again follows from **1.6**. Now we have for each j

$$\delta^{b+1/m_j} \int_\delta^{\delta_\varphi} \varphi(t)\, t^{-\beta-1-1/m_j}\, \mathrm{d}t = \delta^{b+1/m_j} \int_\delta^{\delta_\varphi} t^{c-b-1/m_j} \left(\frac{g(t)}{t^c}\right)' \mathrm{d}t \leqq$$

$$\leqq \delta^{b+1/m_j}\, g(\delta_\varphi)\, \delta_\varphi^{-b-1/m_j} + (b + 1/m_j - c)\, \delta^{b+1/m_j} \int_\delta^{\delta_\varphi} g(t)\, t^{-b-1-1/m_j}\, \mathrm{d}t =$$

$$= O\left(\delta^{b+1/m_j} \int_\delta^{\delta_\varphi} g(t)\, t^{-b-1-1/m_j}\, \mathrm{d}t\right),$$

because g cannot vanish identically on $(0, \delta_\varphi)$ in view of the assumption $\mathscr{H}^\varphi_m(F) > 0$. Since (24) is nonincreasing we conclude as in the final part of the proof in **1.8** that

$$\delta^{b+1/m_j} \int_\delta^{\delta_\varphi} g(t)\, t^{-b-1-1/m_j}\, \mathrm{d}t = O(g(\delta)), \quad \delta \downarrow 0.$$

1.10. Remark. If $g(t)$ tends to zero more quickly than t^b then it is natural to replace $g(t)/t^b$ by $\gamma(t)$ and to estimate the ordinary modulus of continuity $\omega^m(\cdot, \cdot)$ by means of γ. The following theorem deals with $\omega^m(\cdot, \cdot)$.

1.11. Theorem. *Let G be a kernel fulfilling the assumptions of* **1.9**. *Suppose that $\gamma \geqq 0$ is a continuously differentiable function on $(0, \delta_\gamma)$ such that*

$$\varphi(t) = t^{\beta+1}\, \gamma'(t)$$

defines a positive measure function on $(0, \delta_\varphi)$ $(0 < \delta_\varphi \leqq \delta_\gamma)$ and, for suitable $a > 0$,

$$t \mapsto \gamma(t)\, t^{a-1/\overline{m}} \tag{28}$$

is nonincreasing on $(0, \delta_\varphi)$. Then each compact set $F \subset \mathbb{R}^N$ with $\mathscr{H}^\varphi_m(F) > 0$ supports a nontrivial Borel measure μ fulfilling (25); its potential $u = G\mu$ is everywhere defined and satisfies

$$\omega^m_Q(\delta, u) = O(\gamma(\delta)) \quad \text{as} \quad \delta \downarrow 0$$

for every compact set $Q \subset \mathbb{R}^N$.

Proof. According to 1.6, F supports a nontrivial Borel measure μ with (25); its potential $u = G\mu$ satisfies (23ω). It follows from the definition of φ that

$$\int_0^\delta \varphi(t)\, t^{-\beta-1}\,\mathrm{d}t = \int_0^\delta \gamma'(t)\,\mathrm{d}t \leqq \gamma(\delta),$$

$$\delta^{1/m_j}\int_\delta^{\delta_\varphi} \varphi(t)\, t^{-\beta-1-1/m_j}\,\mathrm{d}t = \delta^{1/m_j}\int_\delta^{\delta_\varphi} \gamma'(t)\, t^{-1/m_j}\,\mathrm{d}t \leqq$$

$$\leqq \delta^{1/m_j}\gamma(\delta_\varphi)\,\delta_\varphi^{-1/m_j} + \frac{1}{m_j}\,\delta^{1/m_j}\int_\delta^{\delta_\varphi}\gamma(t)\,t^{-1-1/m_j}\,\mathrm{d}t = O\left(\delta^{1/m_j}\int_\delta^{\delta_\varphi}\gamma(t)\,t^{-1-1/m_j}\,\mathrm{d}t\right),$$

because $\mathscr{H}^\varphi_m(F) > 0$ guarantees that γ does not vanish identically on $(0, \delta_\varphi)$. Using the assumption that (28) is nonincreasing we conclude

$$\delta^{1/m_j}\int_\delta^{\delta_\varphi}\gamma(t)\,t^{-1-1/m_j}\,\mathrm{d}t \leqq \gamma(\delta)\,\delta^a\int_\delta^{\delta_\varphi} t^{-1-a}\,\mathrm{d}t = O(\gamma(\delta)), \quad \delta \downarrow 0.$$

1.12. Remark. Estimates of potentials derived from kernels G satisfying (19), (12) included in § 1 slightly extend (to general moduli of continuity) those presented in [10]; anisotropic Hölder continuity of such potentials was earlier proved in [8]. These results are related to the conditions on Hölder continuity of Riesz potentials obtained by H. Wallin [15]; estimates of Riesz potentials in Morrey's spaces were also investigated by D. R. Adams [1].

§ 2. SINGULARITIES OF SOLUTIONS OF SEMIELLIPTIC EQUATIONS

We shall consider a differential operator of the form (2), where now a_α are complex constants and α runs over the multiindices satisfying (1). Let us recall that $P(D)$ is called semielliptic if the associated polynomial

$$P_m(\xi) = \sum_{|\alpha:m|=1} a_\alpha \xi^\alpha \tag{29}$$

(where we put, as usual, $\xi^\alpha = \xi_1^{\alpha_1}\ldots\xi_N^{\alpha_N}$ for $= [\xi_1, \ldots, \xi_N] \in \mathbb{R}^N$) has no nontrivial real zeros in $\mathbb{R}^N$, i.e.

$$(\xi \in \mathbb{R}^N,\ P_m(\xi) = 0) \Rightarrow \xi = 0\ (\in\mathbb{R}^N).$$

It is well known that in this case $P(D)$ is precisely of order m_j in the j-th variable, $j = 1, \ldots. N$ (cf. [14]). We shall suppose, for simplicity, that

$$b = \sum_{j=1}^N \frac{1}{m_j} > 1. \tag{30}$$

Elementary examples of such operators include the Laplacian $\Delta = -\sum_{j=1}^N D_j^2$ in $\mathbb{R}^N$

with $N > 2$, the Cauchy-Riemann operator $D_2 - iD_1$ (here i is the imaginary unit) in $\mathbb{R}^2$ and the heat conduction operator $iD_N - \sum_{k=1}^{N-1} D_k^2$ in $\mathbb{R}^N$ with $N > 1$.

If $U \subset \mathbb{R}^N$ is an open set and $\mathscr{K}(U)$ is a class of locally integrable functions (or distributions) in U, then a relatively closed set $F \subset U$ is termed removable for $\mathscr{K}(U)$ with respect to $P(D)$, if every $u \in \mathscr{K}(U)$ satisfying (9) (in the sense of distributions) satisfies $P(D)\, u = 0$ in the whole of U. The results of § 1 give necessary conditions concerning the removability of relatively closed sets for various classes of functions.

2.1. Theorem. *Let* (2), (1) *be a semielliptic operator with constant complex coefficients satisfying* (30). *Suppose that* $0 \leqq \varepsilon < 1$ *and let* $g \geqq 0$ *be a continuously differentiable function on an interval* $(0, \delta_g)$ *such that, for a suitable* $c > 0$,

$$\varphi(t) = -t^{c+\varepsilon}\left(\frac{g(t)}{t^c}\right)'$$

defines a positive measure function on an interval $(0, \delta_\varphi)$, $0 < \delta_\varphi \leqq \delta_g$; *if* $c > b + 1/\bar{m}$ *(cf.* (22)) *then we suppose, in addition, that for a suitable* $a > 0$ *the function* (24) *is nonincreasing on* $(0, \delta_\varphi)$. *Then the condition*

$$\mathscr{H}_m^\varphi(F) = 0 \tag{31}$$

is necessary for a relatively closed set $F \subset U$ *to be removable* (*w.r. to* $P(D)$) *for the class of all functions* u *in* U *which, together with their derivatives* $D^\alpha u$ *corresponding to the multiindices* α *with*

$$|\alpha : m| \leqq \varepsilon\,, \tag{32}$$

are locally integrable in U *and satisfy*

$$\Omega_Q^m(\delta, D^\alpha u) = O(g(\delta))\,, \quad \delta \downarrow 0\,, \tag{33}$$

on each compact set $Q \subset U$ *with* $Q^0 \neq \emptyset$.

Proof. If (31) does not hold then F contains a compact subset of a positive $\mathscr{H}_m^\varphi$-measure; we may thus suppose that F itself is compact. Let us fix a fundamental solution E corresponding to $P(D)$. It is known that E coincides in $\mathbb{R}^N \setminus \{0\}$ with an infinitely differentiable function whose derivatives admit the estimates

$$|D^\alpha E(x)| = O(\|x\|_m^{1-b-|\alpha:m|})\,, \quad |x| \downarrow 0$$

(cf. [4]). Let now μ be a nontrivial measure supported by F with (25) and consider an arbitrary multiindex α satisfying (32). We then have $D^\alpha u = G\mu$, where the kernel

$$G(x, y) = \begin{cases} D^\alpha E(x - y)\,, & x \neq y \\ 0\,, & x = y \end{cases}$$

satisfies the assumptions of **1.8** with $\beta = b - (1 - \varepsilon)$. Consequently, $D^\alpha u$ is a locally integrable function whose integral modulus of continuity satisfies (33) on each compact set $Q \subset U$, $Q^0 \neq \emptyset$. Since $P(D)\,u = \mu$ in the sense of distributions, F is not removable for the class of functions described in our theorem.

2.2. Remark. Let $P(D)$ be the same as in **2.1** and suppose that $g \geqq 0$ is continuously differentiable on $(0, \delta_g)$, $0 \leqq \varepsilon < 1$ and that

$$\varphi(t) = t^{c+\varepsilon}\left(\frac{g(t)}{t^c}\right)'$$

defines a positive measure function on $(0, \delta_\varphi)$ $(0 < \delta_\varphi \leqq \delta_g)$ for a suitable $c \in \in (0, b + 1/\bar{m})$. Further suppose that (24) is nonincreasing on $(0, \delta_\varphi)$ for a suitable $a > 0$. Then (31) is necessary for a relatively closed set $F \subset U$ to be removable (w.r. to $P(D)$) for the class of all functions u in U which, together with their derivatives $D^\alpha u$ corresponding to the multiindices α with (32), are locally integrable in U and satisfy (33) on each compact set $Q \subset U$ with $Q^0 \neq \emptyset$.

This follows from **1.9** by the reasoning described in the proof of **2.1**. In a similar way **1.11** implies the following result which was presented in [9] for the case $\varepsilon = 0$.

2.3. Theorem. *Let $P(D)$ be a semielliptic operator satisfying the assumptions of* **2.1**. *Let $0 \leqq \varepsilon < 1$ and suppose that $\gamma \geqq 0$ is a continuously differentiable function on $(0, \delta_\gamma)$ $(\delta_\gamma > 0)$ such that*

$$\varphi(t) = t^{b+\varepsilon}\,\gamma'(t)$$

defines a positive measure function on $(0, \delta_\varphi)$ $(0 < \delta_\varphi \leqq \delta_\gamma)$ and the function (28) is nonincreasing on $(0, \delta_\varphi)$ for a suitable $a > 0$. Then (31) is necessary for a relatively closed set $F \subset U$ to be removable (w.r. to $P(D)$) for the class of all functions u in U which, together with their derivatives $D^\alpha u$ corresponding to the multiindices α satisfying (32), are representable by locally bounded functions satisfying

$$\omega_Q^m(\delta, D^\alpha u) = O(\gamma(\delta))\,, \quad \delta \downarrow 0\,,$$

on each compact set $Q \subset U$.

Proof follows from **1.11** where we put $\beta = \varepsilon + b - 1$ and apply the reasoning of the proof of **2.1**.

2.4. Corollary. *Let $P(D)$ be a semielliptic operator satisfying the assumptions of* **2.1**. *If*

(34) $$1 \leqq d \leqq b\,,$$

then

(35) $$\mathscr{H}_m^{d-1}(F) = 0$$

is necessary and sufficient for a relatively closed set $F \subset U$ to be removable (w.r. to $P(D)$) for the class of all locally integrable functions u in U satisfying the condition

$$\Omega_Q^m(\delta, u) = O(\delta^d), \quad \delta \downarrow 0, \tag{36}$$

on each compact set $Q \subset U$, $Q^0 \neq \emptyset$. If

$$b < d < b + 1/\bar{m}, \tag{37}$$

then the same condition (36) is necessary and sufficient for F to be removable for the class of all functions u satisfying

$$\omega_Q^m(\delta, u) = O(\delta^{d-b}), \quad \delta \downarrow 0, \tag{38}$$

on each compact set $Q \subset U$.

Proof. Let us first assume (34). Letting $\gamma = 1$ and $g(t) = t^d$ in Theorem 1 stated in Introduction we obtain that (35) is sufficient for F to be removable for the class of all locally integrable functions fulfilling (36). Conversely, if F is removable for this class then, letting $\varepsilon = 0$, $g(t) = t^d$ and $c = d + 1$ in Theorem **2.1**, we obtain (35). Next, consider the case (37). If (35) holds, then we let $\gamma = 1$ and $\omega(t) = t^{d-b}$ in Corollary 1 stated in Introduction and conclude that F is removable for the class of all functions u fulfilling (38). Conversely, if F is removable for this class then, employing **2.3** with $\varepsilon = 0$ and $\gamma(t) = t^{d-b}$, we arrive at (35).

2.5. Remark. Functions satisfying the Hölder condition (38) are known to be merely a special case of functions fulfilling the condition (36) on the integral modulus of continuity; accordingly, the second part of **2.4** may be obtained as a consequence of its first part dealing with $d > b$.

The result contained in **2.4** was presented in [10], Th. 3. When applied to the Cauchy-Riemann operator, it made it possible to extend to dimensions $d \leqq 1$ and to Campanato spaces characterization (due to Carleson and Dolženko) with help of the d-dimensional Hausdorff measure $(1 < d < 2)$ of removable singularities of holomorphic functions in Hölder classes with exponent $d - 1$; in particular, for $d = 1$, the relatively closed sets of linear measure zero were exhibited as removable singularities (with respect to holomorphy) of functions in BMO classes (cf. also [11]; MR 80i : 30037). In 1982 R. Kaufman (MR 84b : 30050) treated a related result in the Besicovitch setting admitting sets of singularities which need not be relatively closed.

Similarly, for the Laplacian ($P(D) = \Delta$) in $\mathbb{R}^N$ ($N > 2$), Th. 3 in [10] extended to dimensions $d \leqq N - 2$ and to Campanato spaces the result of L. Carleson (1963) characterizing in terms of the d-dimensional Hausdorff measure removable singularities of harmonic functions in Hölder classes with exponent $d - N + 2$($N - 2 < d < N - 1$); in particular, relatively closed sets with vanishing $(N - 2)$-dimensional Hausdorff measure were identified as removable singularities (with respect to harmonicity) of functions in BMO classes. For dimensions $d > N - 1$ the picture is naturally completed by considering functions whose derivatives belong to the corresponding spaces. In 1978 V. L. Shapiro (MR 58 # 11466) treated related topics for subharmonic functions.

The following Corollary dealing with elliptic operators shows that Theorems 1 and **2.1** permit to characterize, with help of ordinary Hausdorff measures in the full scale of dimensions between 0 and N, the removable singularities for functions which together with certain derivatives belong to adequate Campanato spaces. We write simply

$$\mathscr{H}_1^d(\cdot) = \mathscr{H}^d(\cdot),\ \Omega_Q^1(\cdot, \cdot) = \Omega_Q(\cdot, \cdot),\ \omega_Q^1(\cdot, \cdot) = \omega_Q(\cdot, \cdot).$$

2.6. Corollary. *Let $P(D)$ be an elliptic differential operator of order $\bar{m} < N$ with constant complex coefficients and suppose that k is an integer satisfying $0 \leqq k < \bar{m}$. If*

$$1 - k/\bar{m} \leqq d < (N + 1)/\bar{m} \tag{39}$$

then

$$\mathscr{H}^{k+\bar{m}(d-1)}(F) = 0 \tag{40}$$

is necessary and sufficient for a relatively closed set $F \subset U$ to be removable (w.r. to $P(D)$) for the class of all functions u which, together with their derivatives $D^\alpha u$ of order

$$|\alpha| = \alpha_1 + \ldots + \alpha_N \leqq k, \tag{41}$$

are locally integrable in U and satisfy the condition

$$\Omega_Q(\delta, D^\alpha u) = O(\delta^{\bar{m}d}), \quad \delta \downarrow 0,$$

on each compact set $Q \subset U$.

If $0 < d < 1$ then

$$\mathscr{H}^{k+N+d-\bar{m}}(F) = 0 \tag{42}$$

is necessary and sufficient for any relatively closed set $F \subset U$ to be removable for the class of all functions u in U which, together with their derivatives $D^\alpha u$ of order (41), *satisfy the condition*

$$\omega_Q(\delta, D^\alpha u) = O(\delta^d), \quad \delta \downarrow 0, \tag{43}$$

on each compact set $Q \subset U$.

Proof. Employing Theorem **2.1** with $\varepsilon = k/\bar{m}$, $g(t) = t^d$ and $c = d + 1$ we get, under the assumption (39), the measure function $\varphi(t) = t^{d-1+k/\bar{m}}$ such that (31) is necessary for F to be removable. Observing that, for this φ and $m = \bar{m} \,.\, \mathbf{1}$,

$$\mathscr{H}_m^{\varphi}(F) = 0 \Leftrightarrow \mathscr{H}^{k+\bar{m}(d-1)}(F) = 0\,,$$

we obtain the necessity of (40). Conversely, using Theorem 1 from Introduction with $\gamma = (\bar{m} - k)/\bar{m}$, $g(t) = t^d$ we get that the same condition (40) is also sufficient for F to be removable. If $0 < d < 1$ then Corollary 1 from Introduction with the same value of γ and $\omega(t) = t^{d/\bar{m}}$ provides the measure function $f(t) = t^{(d+N+k-\bar{m})/\bar{m}}$ such that

$$\mathscr{H}_m^f(F) = 0 \tag{44}$$

is sufficient for the removability of F for the functions u which together with the derivatives $D^\alpha u$ of order (41) satisfy on each compact set $Q \subset U$ the condition

$$\omega_Q^m(\delta, D^\alpha u) = O(\delta^{d/\bar{m}})\,, \quad \delta \downarrow 0\,,$$

which means just the same as (43). Conversely, Theorem **2.3** with $\varepsilon = k/\bar{m}$ and $\gamma(t) = t^{d/\bar{m}}$ shows that (44) is necessary for F to be removable for the same class of functions. It remains to note that (44) is equivalent to (42).

2.7. Remark. Hasudorff measures also permit to get necessary conditions for the removability of singularities for classes of functions admitting a specified growth near the singular set; on the other hand, sufficient conditions in known results of Bochner's type dealing with these classes usually employ Minkowskian content or comparable set functions (cf. [5], [6], [13]). Some natural questions in this direction remain open. We do not touch the vast field of various capacities (cf. [7]) and refer the reader to [9] for further references concerning the removability of singularities. The main results of this paper were presented in a lecture held in December 1982 in the Mathematical Institute of the University of Copenhagen.

References

[1] *D. R. Adams:* A note on Riesz potentials. Duke Math. J. *42* (1975), 765—778.

[2] *S. Campanato:* Appunti dell Corso di Analisi Superiore. Quaderno 1°, Anno accademico 1965—66, Università di Pisa.

[3] *L. Carleson:* Selected problems on exceptional sets. Van Nostrand 1967.

[4] *В. В. Грушин:* Связь между локальными и глобальными свойствами решений гипоэллиптических уравнений с постоянными коефициентами. Matem. sbornik *66* (108 (1965), 525 — 550.

[5] *U. Hamann:* Eigenschaften von Potentialen bezüglich elliptischer Differentialoperatoren. Math. Nachr. *96* (1980), 7—15.

[6] *R. Harvey, J. Polking:* Removable singularities of solutions of linear partial differential equations. Acta Mathematica (Uppsala) *125* (1970), 39—56.

[7] *R. Harvey, J. Polking:* A notion of capacity which characterizes removable singularities. Trans. Amer. Math. Soc. *169* (1972), 183—195.

[8] *J. Král:* Removable singularities of solutions of semielliptic equations. Rendiconti di Matematica (4) *6* (1973), 1—21.
[9] *J. Král:* Potentials and removability of singularities, in "Nonlinear evolution equations and potential theory". Proc. of a Summer School held in September 1973 at Podhradí, 95—106.
[10] *J. Král:* Singularités non essentielles des solutions des équations aux dérivées partielles. Séminaire de Théorie du Potentiel Paris 1972—1974, Lecture Notes in Math. vol. 518, 95—106.
[11] *J. Král:* Analytic capacity, in "Elliptische Differentialgleichungen". Proc. of a Conference held in October 1977 in Rostock.
[12] *J. Král:* Об особенностях решений уравнений в частных производных. Trudy Seminara S. L. Soboleva (Novosibirsk) 1983, No. 1, 78—89.
[13] *M. Marcus:* On removable singular sets for solutions of elliptic equations. American Journal of Mathematics *90* (1968), 197—213.
[14] *F. Trèves:* Lectures on linear partial differential equations with constant coefficients. Rio de Janeiro 1961.
[15] *H. Wallin:* Existence and properties of Riesz potentials satisfying Lipschitz conditions. Math. Scand. *19* (1966), 151—160.

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