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HYPERGRAPHS AND INTERVALS, II

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0. Similarly as in [2], by a hypergraph we mean an ordered pair $(V, \mathscr{E})$ with the property that V is a finite nonempty set and $\mathscr{E}$ is a set of nonempty subsets of V. Let $H = (V, \mathscr{E})$ be a hypergraph. Consider arbitrary $A, B \subseteq V$; if at least one of the sets $A \cap B$, $A - B$ and $B - A$ is empty, we shall write $A \sim B$; otherwise, we shall write $A \nsim B$. We shall say that a set $F \subseteq V$ is free in H if $F \sim E$ for every $E \in \mathscr{E}$. We denote by $\Sigma(H)$ the set of all $A \subseteq V$ with the property that at least one of the following conditions (i)–(iv) is fulfilled:

(i) $A = V$,

(ii) $|A| = 1$,

(iii) $A \in \mathscr{E}$,

(iv) there exist $A', A'' \in \Sigma(H)$ such that $A' \nsim A''$ and $A \in \{A' \cap A'', A' \cup A'', A' - A''\}$.

Denote $n = |V|$. By an arrangement on V we mean a sequence $(v_1, \ldots, v_n)$ of n distinct elements of V. Consider an arbitrary arrangement $\alpha = (v_1, \ldots, v_n)$ on V; a set $A \subseteq V$ is referred to as an interval set in α if there exist integers j and m such that $1 \leqq j \leqq m \leqq n$ and $A = \{v_k; j \leqq k \leqq m\}$; we denote by $\operatorname{Int}(\alpha)$ the set of all interval sets in α. We shall say that an arrangement α on V is a projectoidic arrangement on H if $\mathscr{E} \subseteq \operatorname{Int}(\alpha)$; note that the property "to be projectoidic" has a connexion with the property "to be projective" in the sense of mathematical linguistics (see for example [1]). We denote by $\Pi(H)$ the set of all projectoidic arrangements on H. We shall say that H is a projectoid if $\Pi(H) \neq \emptyset$.

The following lemma has been proved in [2]:

Lemma 0. *Let H be a projectoid, let α be a projectoidic arrangement on H, and let A be an interval set in α. If $A \notin \Sigma(H)$, then there exists an interval set F in α with the properties that $F \nsim A$ and F is free in H.*

In [2] the following theorem has been derived from Lemma 0:

Theorem 0. *If H is a projectoid, then*

$$\Sigma(H) = \bigcap_{\alpha \in \Pi(H)} \operatorname{Int}(\alpha).$$

In the present note we shall derive two more theorems from Lemma 0.

1. If $H = (V, \mathscr{E})$ is a hypergraph, then we shall write $V(H) = V$ and $\mathscr{E}(H) = \mathscr{E}$. If H_1 and H_2 are hypergraphs, then we denote by $H_1 \cup H_2$ the hypergraph $(V(H_1) \cup V(H_2), \mathscr{E}(H_1) \cup \mathscr{E}(H_2))$. We shall say that a hypergraph H is a classification if the following two conditions hold:

(a) if $E \in \mathscr{E}(H)$, then $1 < |E| < |V(H)|$, and

(b) if $E', E'' \in \mathscr{E}(H)$, then $E' \sim E''$.

It is clear that every classification is a projectoid.

Let H and H' be projectoids with $V(H) = V(H')$. It has been proved in [2] that $\Pi(H) = \Pi(H')$ if and only if $\Sigma(H) = \Sigma(H')$.

Theorem 1. *Let H be a projectoid. Then there exist classifications H_1 and H_2 such that $V(H_1) = V(H) = V(H_2)$, $\mathscr{E}(H_1) \cap \mathscr{E}(H_2) = \emptyset$, and $\Sigma(H) = \Sigma(H_1 \cup H_2)$.*

Proof. Denote $n = |V(H)|$ and $s = |\Sigma(H)|$. According to (i) and (ii), $s \geqq n + 1$. The case when $s \leqq 3$ is obvious. Let $s \geqq 4$. Assume that for every projectoid H' with $|\Sigma(H')| < s$, the statement of the theorem has been proved. We distinguish two cases:

Case 1. Assume that there exists no free set F of H with the property that $1 < |F| < n$. Consider a projectoidic arrangment $\alpha = (v_1, \ldots, v_n)$ of H. It follows from Lemma 0 that every interval set in α belongs to $\Sigma(H)$. This means that $\Sigma(H) = \operatorname{Int}(\alpha)$. Since $s \geqq 4$, $n \geqq 3$. We denote by $n^\#$ or n^b the maximum integer m such that $m \leqq n$ and m is even or odd, respectively. We define H_1 and H_2 as follows: $V(H_1) = V(H) = V(H_2)$,

$$\mathscr{E}(H_1) = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{n^\# - 1}, v_{n^\#}\}\} \quad \text{and}$$
$$\mathscr{E}(H_2) = \{\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{n^b - 1}, v_{n^b}\}\}.$$

It is clear that H_1 and H_2 are classifications, $\mathscr{E}(H_1) \cap \mathscr{E}(H_2) = \emptyset$ and $\Sigma(H) = \operatorname{Int}(\alpha) = \Sigma(H_1 \cup H_2)$.

Case 2. Assume that there exists a free set F of H with the property that $1 < |F| < n$. The case when $s = n + 1$ is obvious. Let $s \geqq n + 2$. Then there exists $A \in \Sigma(H)$ such that A is a free set of H and $1 < |A| < n$.

Subcase 2.1. Assume that there exists no subset E_0 of A such that $1 < |E_0| < |A|$ and $E_0 \in \mathscr{E}(H)$. Consider an arbitrary $u \in A$. Without loss of generality we shall assume that $\{u\} \notin \mathscr{E}(H)$. We denote by H^* the hypergraph with $V(H^*) = V(H) - \{u\}$

and $\mathscr{E}(H^*) = \{E - \{u\}; E \in \mathscr{E}(H)\}$. Denote $A^* = A - \{u\}$. Clearly, H^* is a projectoid, A^* is a free set in H^* and $A^* \in \Sigma(H^*)$. According to (ii), $\{u\} \in \Sigma(H)$. Hence, $|\Sigma(H^*)| < s$. According to the induction hypothesis there exist classifications H_1^* and H_2^* such that $V(H_1^*) = V(H^*) = V(H_2^*)$, $\mathscr{E}(H_1^*) \cap \mathscr{E}(H_2^*) = \emptyset$ and $\Sigma(H^*) = \Sigma(H_1^* \cup H_2^*)$. For $i = 1, 2$ we denote by H_i the hypergraph with $V(H_i) = V(H)$ and

$$\mathscr{E}(H_i) = \{E'; E' \in \mathscr{E}(H_i^*) \text{ and } A^* \cap E' = \emptyset\} \cup \cup \{E'' \cup \{u\}; E'' \in \mathscr{E}(H_i^*) \text{ and } A^* \subseteq E''\}.$$

It is clear that H_1 and H_2 have the desired properties.

Subcase 2.2. Assume that there exists a subset E_0 of A such that $1 < |E_0| < |A|$ and $E_0 \in \mathscr{E}(H)$. We denote by H^1 and H^2 the hypergraphs with $V(H^1) = V(H)$, $V(H^2) = A$,

$$\mathscr{E}(H^1) = \{A\} \cup \{E' \in \mathscr{E}(H); E' - A \neq \emptyset\} \quad \text{and} \quad \mathscr{E}(H^2) = \{E'' \in \mathscr{E}(H); E'' \subseteq A\}.$$

It is obvious that H^1 and H^2 are projectoids, $\Sigma(H) = \Sigma(H^1) \cup \Sigma(H^2)$, $|\Sigma(H^1)| < s$ and $|\Sigma(H^2)| < s$. This means that there exist classifications H_1^1, H_2^1, H_1^2 and H_2^2 such that for $i = 1, 2$, $V(H_1^i) = V(H^i) = V(H_2^i)$, $\mathscr{E}(H_1^i) \cap \mathscr{E}(H_2^i) = \emptyset$ and $\Sigma(H^i) = \Sigma(H_1^i \cup H_2^i)$. According to (a), $A \notin \mathscr{E}(H_1^2) \cup \mathscr{E}(H_2^2)$. For $j = 1, 2$ we denote $H_j = H_j^1 \cup H_j^2$. It is easy to see that H_1 and H_2 have the desired properties. Thus, the theorem is proved.

2. Let H be a hypergraph. We shall say that an arrangement on $V(H)$ is an antiprojectoidic arrangement on H if for no $E \in \mathscr{E}(H)$ such that $1 < |E| < |V(H)|$, E is an interval set in α.

Theorem 2. *Let H be a projectoid, and let H^* be a classification such that $V(H) = V(H^*)$ and $\Sigma(H) \cap \mathscr{E}(H^*) = \emptyset$. Then there exists a projectoidic arrangement on H which is an antiprojectoidic arrangement on H^*.*

Proof. Denote $n = |V(H)|$. For every $\alpha_0 \in \Pi(H)$, we denote by $d(\alpha_0)$ the number of $E \in \mathscr{E}(H^*)$ with the property that $E \in \mathrm{Int}(\alpha_0)$. Consider a projectoidic arrangement α on H such that for every $\alpha' \in \Pi(H)$, $d(\alpha') \geqq d(\alpha)$. We wish to prove that $d(\alpha) = 0$. On the contrary, we shall assume that $d(\alpha) \geqq 1$. Then there exists $E \in \mathscr{E}(H^*)$ such that $E \in \mathrm{Int}(\alpha)$. Since $E \notin \Sigma(H)$, it follows from Lemma 0 that there exists an interval set F in α with the properties that $F \nsim E$ and F is free in H. Obviously, there exist distinct $v_1, \ldots, v_n \in V(H)$ such that $\alpha = (v_1, \ldots, v_n)$. Since $F \nsim E$, without loss of generality we may assume that there exist integers h, i, j and k such that $1 \leqq h < i < j < k \leqq n$, $E = \{v_h, \ldots, v_j\}$ and $F = \{v_i, \ldots, v_k\}$. We denote by β the arrangement

$$(v_1, \ldots, v_{i-1}, v_k, \ldots, v_i, v_{k+1}, \ldots, v_n) \quad \text{if} \quad k < n \quad \text{or}$$
$$(v_1, \ldots, v_{i-1}, v_k, \ldots, v_i) \quad \text{if} \quad k = n.$$

Since F is a free set of H, β is a projectoidic arrangement on H. Obviously, $E \notin$ $\notin \text{Int}(\beta)$. Since $d(\beta) \geqq d(\alpha)$, there exists $E' \in \mathscr{E}(H^*)$ such that $E' \in \text{Int}(\beta) - \text{Int}(\alpha)$. This implies that either (a) v_{i-1}, $v_k \in E'$, or (b) $n > k$ and v_i, $v_{k+1} \in E'$. Since H^* is a classification, $E \subseteq E'$. Since $E' \in \text{Int}(\beta)$, $F \subseteq E'$. Hence, $E' \in \text{Int}(\alpha)$, which is a contradiction. This means that $d(\alpha) = 0$, and the proof is complete.

References

[1] *L. Nebeský:* Graph theory and linguistics. In: Applications of Graph Theory (R. J. Wilson and L. W. Beineke, eds.). Academic Press, London 1979, pp. 357—380.

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