Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 108 (1983), No. 2, 122--132

Persistent URL: `http://dml.cz/dmlcz/108418`

Časopis pro pěstování matematiky, roč. 108 (1983), Praha

OSCILLATION AND ASYMPTOTIC PROPERTIES OF STRONGLY SUBLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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(Received Juny 4, 1981)

This paper deals with the oscillatory and asymptotic behavior of the solutions of the n-th order ($n > 1$) differential equation with deviating arguments

$$\text{(E)} \qquad [r(t)\, x^{(n-1)}(t)]' + a(t)\, \Phi(x[g_1(t)], \ldots, x[g_m(t)]) = 0\,, \quad t \geqq t_0\,,$$

where the real-valued functions involved are subject to the following assumptions:

(i) *r is a positive continuous function on the interval $[t_0, \infty)$ such that*

$$\int^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty\,;$$

(ii) *a is a continuous function on $[t_0, \infty)$ which is of constant sign;*

(iii) *Φ is a continuous function which is defined at least on $\mathbb{R}^m_+ \cup \mathbb{R}^m_-$, where $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$, and has the following sign property:*

$$\Phi(y) > 0 \quad \text{for all} \quad y \in \mathbb{R}^m_+\,, \qquad \Phi(y) < 0 \quad \text{for all} \quad y \in \mathbb{R}^m_-\,;$$

(iv) *g_j ($j = 1, \ldots, m$) are continuous functions on $[t_0, \infty)$ with*

$$\lim_{t \to \infty} g_j(t) = \infty \quad (j = 1, \ldots, m)\,.$$

Smoothness sufficient for the existence of such solutions $x(t)$ of the equation (E) which are defined for all large t will be assumed. In what follows, we consider only such solutions $x(t)$ which are defined for all large t. The oscillatory character is considered in the usual sense, i.e., a continuous real-valued function defined on an interval $[T, \infty)$ is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*.

The oscillatory character and the asymptotic behavior of the bounded solutions of the differential equation (E) are well described by the following theorem, which is a particular case of a result due to the author [8].

Theorem 0. *Let* (i)–(iv) *be satisfied and suppose that*

$$(\mathrm{C}_0) \qquad \text{either} \quad \int^{\infty} |a(t)| \, \mathrm{d}t = \infty \quad \text{or} \quad \int^{\infty} \frac{t^{n-2}}{r(t)} \int_t^{\infty} |a(s)| \, \mathrm{d}s \, \mathrm{d}t = \infty \, .$$

If a is nonnegative (nonpositive), then for n even (odd) all bounded solutions of the differential equation (E) *are oscillatory, while for n odd, (even) respectively every bounded solution x of the equation* (E) *is either oscillatory or such that $x^{(i)}$ $(i = 0, 1, \ldots, n-2)$ and $rx^{(n-1)}$ tend monotonically to zero at ∞.*

Our aim is to study the oscillatory and asymptotic behavior of all solutions of the differential equation (E). Our interest is concentrated on the case when the equation (E) is *strongly sublinear* in the sense that *the function Φ is increasing and such that*

$$\int_{+0} \frac{\mathrm{d}y}{\Phi(y, \ldots, y)} < \infty \quad \text{and} \quad \int_{-0} \frac{\mathrm{d}y}{\Phi(y, \ldots, y)} < \infty \, .$$

Note that the increasing character of Φ is considered with respect to the usual order in $\mathbb{R}^m$ defined by the positive cone $\{y = (y_1, \ldots, y_m) \in \mathbb{R}^m : (\forall j = 1, \ldots, m) \; y_j \geqq 0\}$, i.e., $y \leqq z \Leftrightarrow (\forall j = 1, \ldots, m) \; y_j \leqq z_j$. Moreover, it is noteworthy (cf. Staikos [9, 10]) that, if the equation (E) is strongly sublinear, then

$$\lim_{y \to 0} \frac{\Phi(y, \ldots, y)}{y} = \infty \, .$$

For our purposes, we need the following lemma which has originated in two well-known lemmas due to Kiguradze [2, 3] (cf. also [1], [9, 10] and [7]).

Lemma. *Suppose that* (i) *holds and let h be a positive and $(n-1)$-times differentiable function on an interval $[\tau, \infty)$, $\tau \geqq t_0$, such that the function $rh^{(n-1)}$ is differentiable with its derivative of constant sign on $[\tau, \infty)$ and not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$.*

Then there exist a $T \geqq \tau$ and an integer l, $0 \leqq l \leqq n$, with $n + l$ odd for $[rh^{(n-1)}]'$ nonpositive or $n + l$ even for $[rh^{(n-1)}]'$ nonnegative so that

$$\begin{cases} l \leqq n - 1 \Rightarrow (-1)^{l+j} h^{(j)} > 0 \quad \text{on} \quad [T, \infty) \quad (j = l, \ldots, n-1) \\ l > 1 \Rightarrow h^{(i)} > 0 \quad \text{on} \quad [T, \infty) \quad (i = 1, \ldots, l-1) \, . \end{cases}$$

If, in addition, h is eventually increasing and $h^{(n-1)}[rh^{(n-1)}]'$ is eventually nonpositive, then for every ϑ, $0 < \vartheta < 1$, we have:

$$h(t) \geqq \frac{\vartheta}{(n-2)!} \left| r(t) \, h^{(n-1)}(t) \right| \int_{t_0}^{t} \frac{(s - t_0)^{n-2}}{r(s)} \, \mathrm{d}s \quad \text{for all large } t \, ,$$

if $l < n - 1$;

$$h(t) \geqq \frac{\vartheta}{(n-2)!} \left| r(t)\, h^{(n-1)}(t) \right| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{r(s)}\, ds \quad \textit{for all large } t\,,$$

if $l = n - 1$.

Proof. The first part of the lemma is a special case of a lemma given by the same author in [7]. Furthermoie, let us suppose that h is eventually increasing and $h^{(n-1)}[rh^{(n-1)}]'$ is eventually nonpositive. Then we must have $1 \leqq l \leqq n - 1$. Next, we consider the following two cases.

Case 1. $l < n - 1$. By using the Taylor formula with integral remainder, for every $t \geqq T$ we obtain

$$h(t) = \sum_{i=0}^{l-1} \frac{(t-T)^i}{i!} h^{(i)}(T) + \frac{1}{(l-1)!} \int_T^t (t-s)^{l-1} h^{(l)}(s)\, ds \geqq$$

$$\geqq \frac{1}{(l-1)!} \int_T^t (t-s)^{l-1} h^{(l)}(s)\, ds\,.$$

Furthermore, by applying again the Taylor formula with integral remainder, for every s, t with $T \leqq s \leqq t$ we get

$$h^{(l)}(s) = \sum_{j=l}^{n-2} \frac{(s-t)^{j-l}}{(j-l)!} h^{(j)}(t) + \frac{1}{(n-2-l)!} \int_t^s (s-u)^{n-2-l} h^{(n-1)}(u)\, du =$$

$$= \sum_{j=l}^{n-2} \frac{(t-s)^{j-l}}{(j-l)!} \left[(-1)^{l+j} h^{(j)}(t)\right] +$$

$$+ \frac{1}{(n-2-l)!} \int_s^t \frac{(u-s)^{n-2-l}}{r(u)} \left[(-1)^{l+n-1} r(u)\, h^{(n-1)}(u)\right] du \geqq$$

$$\geqq \frac{1}{(n-2-l)!} \left| r(t)\, h^{(n-1)}(t) \right| \int_s^t \frac{(u-s)^{n-2-l}}{r(u)}\, du\,.$$

Thus, for every $t \geqq T$ we have

$$h(t) \geqq \frac{1}{(l-1)!\,(n-2-l)!} \left| r(t)\, h^{(n-1)}(t) \right| \int_T^t (t-s)^{l-1} \int_s^t \frac{(u-s)^{n-2-l}}{r(u)}\, du\, ds \geqq$$

$$\geqq \frac{1}{(n-3)!} \left| r(t)\, h^{(n-1)}(t) \right| \int_T^t \int_s^t \frac{(u-s)^{n-3}}{r(u)}\, du\, ds =$$

$$= \frac{1}{(n-3)!} \left| r(t)\, h^{(n-1)}(t) \right| \int_T^t \frac{1}{r(u)} \left[\int_T^u (u-s)^{n-3}\, ds \right] du =$$

$$= \frac{1}{(n-2)!} \left| r(t)\, h^{(n-1)}(t) \right| \int_T^t \frac{(u-T)^{n-2}}{r(u)}\, du\,.$$

Next, by the L'Hospital rule, we derive that

$$\lim_{t\to\infty}\left[\int_T^t \frac{(s-T)^{n-2}}{r(s)}\,ds\Big/\int_{t_0}^t \frac{(s-t_0)^{n-2}}{r(s)}\,ds\right] = 1$$

and hence for every ϑ, $0 < \vartheta < 1$, we have

$$\int_T^t \frac{(s-T)^{n-2}}{r(s)}\,ds \geqq \vartheta \int_{t_0}^t \frac{(s-t_0)^{n-2}}{r(s)}\,ds$$

for t sufficiently large. Thus,

$$h(t) \geqq \frac{\vartheta}{(n-2)!}\left|r(t)\,h^{(n-1)}(t)\right| \int_{t_0}^t \frac{(s-t_0)^{n-2}}{r(s)}\,ds \quad \text{for all large } t\,.$$

Case 2. $l = n - 1$. By the Taylor formula with integral remainder, for $t \geqq T$ we obtain

$$\begin{aligned} h(t) &= \sum_{i=0}^{n-2} \frac{(t-T)^i}{i!}\,h^{(i)}(T) + \frac{1}{(n-2)!}\int_T^t (t-s)^{n-2}\,h^{(n-1)}(s)\,ds \geqq \\ &\geqq \frac{1}{(n-2)!}\int_T^t \frac{(t-s)^{n-2}}{r(s)}\,[r(s)\,h^{(n-1)}(s)]\,ds \geqq \\ &\geqq \frac{1}{(n-2)!}\left|r(t)\,h^{(n-1)}(t)\right| \int_T^t \frac{(t-s)^{n-2}}{r(s)}\,ds\,. \end{aligned}$$

But, by applying the L'Hospital rule, it is easy to see that

$$\lim_{t\to\infty}\left[\int_T^t \frac{(t-s)^{n-2}}{r(s)}\,ds\Big/\int_{t_0}^t \frac{(t-s)^{n-2}}{r(s)}\,ds\right] = 1$$

and hence for every ϑ, $0 < \vartheta < 1$, we get

$$h(t) \geqq \frac{\vartheta}{(n-2)!}\left|r(t)\,h^{(n-1)}(t)\right| \int_{t_0}^t \frac{(t-s)^{n-2}}{r(s)}\,ds \quad \text{for all large } t\,.$$

Now, in order to present our main result we introduce the functions σ_j $(j = 1, \ldots, m)$ and R_1, R_2 defined on $[t_0, \infty)$ as follows:

$$\sigma_j(t) = \min\{t, g_j(t)\} \quad (j = 1, \ldots, m)\,,$$

$$R_1(t) = \int_{t_0}^t \frac{(s-t_0)^{n-2}}{r(s)}\,ds \quad \text{and} \quad R_2(t) = \int_{t_0}^t \frac{(t-s)^{n-2}}{r(s)}\,ds\,.$$

Moreover, for two vectors $y = (y_1, \ldots, y_m)$, $z = (z_1, \ldots, z_m)$ in $\mathbb{R}^m$ we define

$$yz = (y_1 z_1, \ldots, y_m z_m)\,.$$

Theorem. *Suppose that* (i)–(iv) *hold. Moreover, let the differential equation* (E) *be strongly sublinear and suppose that the function* Φ *has the exponential property*

$$(\mathrm{P}) \qquad \begin{cases} \Phi(yz) \geqq K\,\Phi(y)\,\Phi(z) & \text{for all } y, z \text{ in } \mathbb{R}^m_+ , \\ -\Phi(-yz) \geqq K\,\Phi(y)\,\Phi(z) & \text{for all } y, z \text{ in } \mathbb{R}^m_- , \end{cases}$$

where K *is a positive constant.*

Then, under the condition

$$(\mathrm{C}) \qquad \int^{\infty} |a(t)|\,|\Phi(\delta R_k[\sigma_1(t)], \ldots, \delta R_k[\sigma_m(t)])|\,\mathrm{d}t = \infty \quad (\delta = \pm 1;\ k = 1, 2),$$

we have the following results:

(I) *For* a *nonnegative and* n *even, every solution of* (E) *is oscillatory.*

(II) *For* a *nonnegative and* n *odd, every solution* x *of* (E) *is oscillatory or satisfies*

$$(X_0) \qquad \begin{cases} \lim\limits_{t\to\infty} x^{(i)}(t) = 0 \quad \text{monotonically} \quad (i = 0, 1, \ldots, n-2), \\ \lim\limits_{t\to\infty} r(t)\,x^{(n-1)}(t) = 0 \quad \text{monotonically}. \end{cases}$$

(III) *For* a *nonpositive and* n *even, every solution* x *of* (E) *is oscillatory or satisfies one of* (X_0),

$$(X_\infty) \qquad \lim_{t\to\infty} x^{(i)}(t) = \infty \quad (i = 0, 1, \ldots, n-2) \quad \text{and} \quad \lim_{t\to\infty} r(t)\,x^{(n-1)}(t) = \infty,$$

$$(X_{-\infty}) \qquad \lim_{t\to\infty} x^{(i)}(t) = -\infty \quad (i = 0, 1, \ldots, n-2) \quad \text{and} \quad \lim_{t\to\infty} r(t)\,x^{(n-1)}(t) = -\infty.$$

(IV) *For* a *nonpositive and* n *odd, every solution* x *of* (E) *is oscillatory or satisfies one of* (X_∞), $(X_{-\infty})$.

Proof. By (P), for all $y > 0$ we have

$$K\Phi(y, \ldots, y)\,\Phi\left(\frac{1}{y}, \ldots, \frac{1}{y}\right) \leqq \Phi(1, \ldots, 1).$$

But, because of the sublinearity of the equation (E), the function Φ is such that

$$\lim_{y\to\infty} y\Phi\left(\frac{1}{y}, \ldots, \frac{1}{y}\right) = \lim_{z\to 0} \frac{\Phi(z, \ldots, z)}{z} = \infty$$

and consequently for all large y,

$$K\Phi(y, \ldots, y) \leqq K\Phi(y, \ldots, y)\,y\Phi\left(\frac{1}{y}, \ldots, \frac{1}{y}\right) \leqq \Phi(1, \ldots, 1)\,y.$$

Therefore, since Φ is increasing and $\lim\limits_{t\to\infty} R_1(t) = \infty$, we have that for all large t

$$K\Phi(R_1[\sigma_1(t)], \ldots, R_1[\sigma_m(t)]) \leqq K\Phi(R_1(t), \ldots, R_1(t)) \leqq \Phi(1, \ldots, 1)\,R_1(t).$$

Thus, by (C), we obtain

$$\int^{\infty} |a(t)|\, R_1(t)\, \mathrm{d}t = \infty\,,$$

that is

$$\int^{\infty} |a(t)| \int_{t_0}^{t} \frac{(s - t_0)^{n-2}}{r(s)}\, \mathrm{d}s\, \mathrm{d}t = \infty\,.$$

Now it is a matter of elementary calculus to obtain

$$\int^{\infty} \frac{(t - t_0)^{n-2}}{r(t)} \int_{t}^{\infty} |a(s)|\, \mathrm{d}s\, \mathrm{d}t = \infty$$

provided that $\int^{\infty} |a(t)|\, \mathrm{d}t < \infty$. Thus, we have proved that the condition (C) implies (C_0). Hence, by Theorem 0, it remains to study the unbounded solutions of the equation (E).

The substitution $w = -x$ transforms (E) into the equation

$$[r(t)\, w^{(n-1)}(t)]' + a(t)\, \hat{\Phi}(w[g_1(t)], \ldots, w[g_m(t)]) = 0\,,$$

where $\hat{\Phi}(y) = -\Phi(-y)$ for all y in the domain of Φ. This equation is subject to the assumptions posed for the equation (E). Thus, with respect to the nonoscillatory solutions of (E) we can restrict our attention only to the positive ones.

Now, let x be a positive unbounded solution on an interval $[\tau_0, \infty)$, $\tau_0 \geqq t_0$, of the equation (E). Moreover, by (iv), let $\tau \geqq \tau_0$ be chosen so that

$$g_j(t) \geqq \tau_0 \quad \text{for every} \quad t \geqq \tau \quad (j = 1, \ldots, m)\,.$$

Then, in virtue of (iii), (E) implies that

$$[r(t)\, x^{(n-1)}(t)]'\, I(a) \leqq 0 \quad \text{for every} \quad t \geqq \tau\,,$$

where

$$I(a) = \begin{cases} +1 & \text{if} \quad a \geqq 0 \\ -1 & \text{if} \quad a \leqq 0 \end{cases}$$

is the so called sign index of the function a. Moreover, the function $[rx^{(n-1)}]'$ is not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$, since, because of (C), the same holds for the function a. Thus, by Lemma, there exist a $T \geqq \tau$ and an integer l, $0 \leqq l \leqq n$, with $n + l$ odd for a nonnegative or $n + l$ even for a nonpositive so that

$$\begin{cases} l \leqq n - 1 \Rightarrow (-1)^{l+j}\, x^{(j)}(t) > 0 \quad \text{for every} \quad t \geqq T \quad (j = l, \ldots, n-1)\,, \\ l > 1 \Rightarrow x^{(i)}(t) > 0 \quad \text{for every} \quad t \geqq T \quad (i = 1, \ldots, l-1)\,. \end{cases}$$

Because of the unboundedness of x, we always have $l > 0$.

Next, let us suppose that $\lim_{t \to \infty} r(t)\, x^{(n-1)}(t) = 0$. Then we must have

$$x^{(n-1)}(t)\, I(a) > 0 \quad \text{for every} \quad t \geqq T\,.$$

Furthermore, by Lemma, for some $T_1 \geqq T$ and every $t \geqq T_1$ we have

$$x(t) \geqq M\, R(t)\, r(t)\, x^{(n-1)}(t)\, I(a),$$

where M is a positive constant, and $R = R_1$ for $l < n - 1$ or $R = R_2$ for $l = n - 1$. Thus, since the function $rx^{(n-1)} I(a)$ is decreasing on $[\tau, \infty)$, we get

$$x[\sigma_j(t)] \geqq MR[\sigma_j(t)]\, r(t)\, x^{(n-1)}(t)\, I(a) \quad (j = 1, \ldots, m)$$

for all $t \geqq T_2$, where T_2, $T_2 \geqq T_1$, is chosen so that

$$\sigma_j(t) \geqq T_1 \quad \text{for every} \quad t \geqq T_2 \quad (j = 1, \ldots, m).$$

Hence, by taking into account the increasing character of Φ and its exponential property (P), for every $t \geqq T_2$ we obtain

$$\begin{aligned}
[r(t)\, x^{(n-1)}(t)]'\, I(a) &= -|a(t)|\, \Phi(x[g_1(t)], \ldots, x[g_m(t)]) \leqq \\
&\leqq -|a(t)|\, \Phi(x[\sigma_1(t)], \ldots, x[\sigma_m(t)]) \leqq \\
&\leqq -|a(t)|\, \Phi(MR[\sigma_1(t)]\, r(t)\, x^{(n-1)}(t)\, I(a), \ldots, MR[\sigma_m(t)]\, r(t)\, x^{(n-1)}(t)\, I(a)) \leqq \\
&\leqq -K^2 |a(t)|\, \Phi(R[\sigma_1(t)], \ldots, R[\sigma_m(t)])\, \Phi(M, \ldots, M)\,. \\
&\quad .\, \Phi(r(t)\, x^{(n-1)}(t)\, I(a), \ldots, r(t)\, x^{(n-1)}(t)\, I(a)),
\end{aligned}$$

that is

$$\begin{aligned}
&|a(t)|\, \Phi(R[\sigma_1(t)], \ldots, R[\sigma_m(t)]) \leqq \\
&\leqq S \frac{-[r(t)\, x^{(n-1)}(t)]'\, I(a)}{\Phi(r(t)\, x^{(n-1)}(t)\, I(a), \ldots, r(t)\, x^{(n-1)}(t)\, I(a))},
\end{aligned}$$

where $S = 1/K^2 \Phi(M, \ldots, M)$. Therefore, by integration, we derive that

$$\int_{T_2}^{\infty} |a(t)|\, \Phi(R[\sigma_1(t)], \ldots, R[\sigma_m(t)])\, dt \leqq S \int_{+0}^{\delta} \frac{dy}{\Phi(y, \ldots, y)},$$

where $\delta = r(T_2)\, x^{(n-1)}(T_2)\, I(a) > 0$. Because of (C) and the fact that (E) is strongly sublinear, the last inequality leads to a contradiction.

We have thus proved that $\lim_{t\to\infty} r(t)\, x^{(n-1)}(t)$ must be nonzero. If $\lim_{t\to\infty} r(t)\, x^{(n-1)}(t) < 0$, then, by using (i), it is easy to derive that $\lim_{t\to\infty} x(t) = -\infty$ which contradicts the positivity of x. Hence, we always have that $\lim_{t\to\infty} r(t)\, x^{(n-1)}(t) > 0$. This by (i) gives $\lim_{t\to\infty} x^{(i)}(t) = \infty$ $(i = 0, 1, \ldots, n - 2)$. Thus, by using the L'Hospital rule, it is easy to obtain that

$$(n-2)! \lim_{t\to\infty} x(t) \left[\int_{t_0}^{t} \frac{(t-s)^{n-2}}{r(s)}\, ds\right]^{-1} = \lim_{t\to\infty} r(t)\, x^{(n-1)}(t) > 0\,.$$

Therefore, we have for some $\tau_1 > T$ and every $t \geqq \tau_1$ that

$$x(t) \geqq L \int_{t_0}^{t} \frac{(t-s)^{n-2}}{r(s)} \, ds = L R_2(t),$$

where L is a positive constant. Hence,

$$x[\sigma_j(t)] \geqq L R_2[\sigma_j(t)] \quad (j = 1, \ldots, m)$$

for all $t \geqq \tau_2$, where τ_2, $\tau_2 \geqq \tau_1$, is chosen so that

$$\sigma_j(t) \geqq \tau_1 \quad \text{for every} \quad t \geqq \tau_2 \quad (j = 1, \ldots, m).$$

Next, by the increasing chaiacter of Φ and its property (P), for $t \geqq \tau_2$ we get

$$[r(t)\, x^{(n-1)}(t)]'\, I(a) = -|a(t)|\, \Phi(x[g_1(t)], \ldots, x[g_m(t)]) \leqq$$
$$\leqq -|a(t)|\, \Phi(x[\sigma_1(t)], \ldots, x[\sigma_m(t)]) \leqq -|a(t)|\, \Phi(L R_2[\sigma_1(t)], \ldots, L R_2[\sigma_m(t)]) \leqq$$
$$\leqq -K \Phi(L, \ldots, L)\, |a(t)|\, \Phi(R_2[\sigma_1(t)], \ldots, R_2[\sigma_m(t)]).$$

Therefore, by integration,

$$I(a) \lim_{t \to \infty} r(t)\, x^{(n-1)}(t) - I(a)\, r(\tau_2)\, x^{(n-1)}(\tau_2) \leqq$$
$$\leqq -K \Phi(L, \ldots, L) \int_{\tau_2}^{\infty} |a(t)|\, \Phi(R_2[\sigma_1(t)], \ldots, R_2[\sigma_m(t)])\, dt$$

and hence, by the condition (C),

$$\lim_{t \to \infty} r(t)\, x^{(n-1)}(t) = -I(a)\, \infty .$$

However, this is case where a is nonpositive and therefore (X_∞) can easily be derived.

Next, let us consider the special case where $r = 1$, i.e. the differential equation

$$(\tilde{\mathrm{E}}) \qquad x^{(n)}(t) + a(t)\, \Phi(x[g_1(t)], \ldots, x[g_m(t)]) = 0 .$$

Then we obviously have for $k = 1, 2$

$$R_k(t) = \frac{1}{n-1} (t - t_0)^{n-1} \geqq \mu t^{n-1} \quad \text{for all large } t,$$

where μ is a positive constant. Thus, if (ii)–(iv) hold and the function Φ is increasing and has the exponential property (P), then the condition (C) follows from the following one:

$$(\tilde{\mathrm{C}}) \qquad \int^{\infty} |a(t)|\, |\Phi(\delta[\sigma_1(t)]^{n-1}, \ldots, \delta[\sigma_m(t)]^{n-1})|\, dt = \infty, \quad \delta = \pm 1 .$$

So, in the special case of the equation $(\tilde{E})$ our theorem leads to some recent results of Staikos [9, 10]. Note that the method used here patterns after that of Staikos [9, 10]. For earlier oscillation results concerning sublinear retarded differential equations we refer to Kusano [4], and Kusano and Onose [5, 6].

Now, we remark that in the case of ordinary or retarded differential equations of the form (E) the condition (C) becomes

$$\text{(C*)} \quad \int^{\infty} |a(t)|\, |\Phi(\delta R_k[g_1(t)], \ldots, \delta R_k[g_m(t)])|\, dt = \infty \quad (\delta = \pm 1;\ k = 1, 2).$$

For differential equations of the form (E) which are of advanced or mixed type our theorem ceases to hold with the condition (C*) in place of (C). This is illustrated by the following four examples of advanced differential equations. These equations fail to satisfy the condition (C). However, they satisfy the rest of the assumptions of Theorem and the condition (C*).

Example 1. The equation

$$[t^{1/3} x'(t)]' + (1/9)\, t^{-5/3} x^{1/3}(t^3) = 0\,, \quad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{1/3}$, a contradiction to conclusion (I) of Theorem.

Example 2. The equation

$$[t^{1/2} x''(t)]' + 5t^{-3/2}(1 + t^6)^{-1/3} x^{1/3}(t^6) = 0\,, \quad t \geqq 1$$

has the solution $x(t) = 1 + 1/t$ for which $\lim_{t\to\infty} x(t) = 1$, a contradiction to conclusion (II) of Theorem.

Example 3. The equation

$$[t^{1/2} x'''(t)]' - (3/8)\, t^{-7/2} x^{1/3}(t^3) = 0\,, \quad t \geqq 1$$

has the solution $x(t) = t^{3/2}$ for which $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$ while $\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} t^{1/2} x'''(t) = \infty$, a contradiction to conclusion (III) of Theorem.

Example 4. The equation

$$[t^{1/2} x''(t)]' - (1/4)\, t^{-5/2} x^{1/3}(t^3) = 0\,, \quad t \geqq 1$$

has the solution $x(t) = t^{1/2}$ for which we have $\lim_{t\to\infty} x(t) = \infty$ while $\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} t^{1/2} x''(t) = 0$, a contradiction to conclusion (IV) of Theorem.

We now turn our attention to a particular class of differential equations of the form (E), which includes the ordinary, retarded equations and some others of advanced or mixed type. This class is characterized by the condition

$$\text{(H)} \qquad \limsup_{t\to\infty} \frac{R_k[g_j(t)]}{R_k(t)} < \infty \quad (j = 1, \ldots, m;\ k = 1, 2).$$

For sublinear equations of the class considered the condition (C) can be replaced by (C*) in our theorem. That is, we have the following corollary.

Corollary. *Suppose that* (i)–(iv) *hold. Moreover, let the differential equation* (E) *be strongly sublinear and suppose that the function Φ has the exponential property* (P).

If (H) *holds, then, under the condition* (C*), *we have the conclusions* (I)–(IV) *of Theorem.*

Proof. For any $j \in \{1, \ldots, m\}$ and for all large t we have

$$\frac{R_k[\sigma_j(t)]}{R_k[g_j(t)]} = \begin{cases} 1, & \text{if} \quad g_j(t) \leqq t \\ \dfrac{R_k(t)}{R_k[g_j(t)]}, & \text{if} \quad g_j(t) > t. \end{cases} \qquad (k = 1, 2)$$

Thus, by virtue of (H), there exists a positive constant M so that for $k = 1, 2$

$$R_k[\sigma_j(t)] \geqq MR_k[g_j(t)] \quad \text{for all large } t \quad (j = 1, \ldots, m)$$

and consequently, by taking into account (iii) and the increasing character of Φ and its exponential property (P), we obtain that for all large t

$$\left|\Phi(\delta R_k[\sigma_1(t)], \ldots, \delta R_k[\sigma_m(t)])\right| \geqq$$
$$\geqq \left|\Phi(\delta M R_k[g_1(t)], \ldots, \delta M R_k[g_m(t)])\right| \geqq KL\left|\Phi(\delta R_k[g_1(t)], \ldots, \delta R_k[g_m(t)])\right|,$$

where $L = \min\{\Phi(M, \ldots, M), |\Phi(-M, \ldots, -M)|\} > 0$. Thus, the condition (C*) implies (C) and hence the corollary follows from our theorem.

Note that the condition (H) cannot be omitted from the above corollary. This is demonstrated by Examples 1–4 of advanced differential equations which fail to satisfy (H). Also, we notice that in the special case where $r = 1$ the condition (H) is satisfied if

$$(\tilde{\text{H}}) \qquad \limsup_{t\to\infty} \frac{g_j(t)}{t} < \infty \quad (j = 1, \ldots, m).$$

In the conclusion we remark that it would be desirable to study the oscillatory and asymptotic behavior of the solutions of sublinear differential equations of the form (E) without the condition of the exponential property (P). From the arguments presented here it is apparent that the role of this condition is essential to our method. Perhaps another method would be of significant importânce. This is said in view of the fact that in the superlinear case no such condition is imposed. We hasten to add that as far as we known the sublinear equations that appeared in the bibliography satisfy the exponential property mentioned above. So, one usually encounters sublinear differential equations of particular forms of (E) with the continuous function Φ defined by

$$\Phi(y_1, \ldots, y_m) = |y_1|^{\alpha_1} \ldots |y_m|^{\alpha_m} \operatorname{sgn} y_1$$

at least on $\mathbb{R}^m_+ \cup \mathbb{R}^m_-$. The simplest case where $m = 1$, i.e. $\Phi(y) = |y|^\alpha \operatorname{sgn} y$, drew much attention in the literature.

It remains an open question to the author whether our theorem can be extended to more general strongly sublinear differential equations of the form

$$(\hat{\mathrm{E}}) \qquad [r_{n-1}(t)\,[r_{n-2}(t)\,[\ldots[r_2(t)\,[r_1(t)\,x'(t)]']' \ldots]']]' + \\ + a(t)\,\Phi(x[g_1(t)], \ldots, x[g_m(t)]) = 0\,,$$

where r_i $(i = 1, \ldots, n - 1)$ are positive continuous functions on the interval $[t_0, \infty)$ such that $\int^\infty [1/r_i(t)]\,\mathrm{d}t = \infty$ $(i = 1, \ldots, n - 1)$.

References

[1] *M. K. Grammatikopoulos, Y. G. Sficas* and *V. A. Staikos:* Oscillatory properties of strongly superlinear differential equations with deviating arguments. J. Math. Anal. Appl. *67* (1979), 171—187.

[2] *I. T. Kiguradze:* On the oscillation of solutions of the equation $\mathrm{d}^m u/\mathrm{d}t^m + a(t)\,|u|^n \operatorname{sgn} u = 0$ (Russian). Mat. Sb. *65* (1964), 172—187.

[3] *I. T. Kiguradze:* The problem of oscillation of solutions of nonlinear differential equations (Russian). Differencial'nye Uravnenija *1* (1965), 995—1006.

[4] *T. Kusano:* Oscillatory behavior of solutions of higher order retarded differential equations. Proceedings of Carathéodory International Symposium (Athens, September 3—7, 1973), The Greek Mathematical Society, pp. 370—389.

[5] *T. Kusano* and *H. Onose:* Nonlinear oscillation of a sublinear delay equation of arbitrary order. Proc. Amer. Math. *40* (1973), 219—224.

[6] *T. Kusano* and *H. Onose:* Oscillatory and asymptotic behavior of sublinear retarded differential equations. Hiroshima Math. J. *4* (1974), 343—355.

[7] *Ch. G. Philos:* Oscillatory and asymptotic behavior of all solutions of differential equations with deviating arguments. Proc. Roy. Soc. Edinburgh Sect. A *81* (1978), 195—210.

[8] *Ch. G. Philos:* On the oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments. Ann. Mat. Pura Appl. *119* (1979), 125—140.

[9] *V. A. Staikos:* Differential Equations with Deviating Arguments-Oscillation Theory. Unpublished manuscripts.

[10] *V. A. Staikos:* Basic results on oscillation for differential equations with deviating arguments. Hiroshima Math. J. *10* (1980), 495—516.

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