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REMARK IN CONNECTION WITH AN ARTICLE OF HAMEDANI

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1. G. G. Hamedani [1] proved under suitable assumptions that the equation

$$y'(t) = f(t, y(h_1(t)), \ldots, y(h_n(t)), y'(h_{n+1}(t)), \ldots, y'(h_{n+m}(t)), \lambda)$$

has exactly one solution defined in an interval $J = (-\alpha, \alpha)$ and fulfilling an initial condition $y(0) = \eta$, and this solution depends continuously on the parameter λ. In this note we use a contraction principle (given in Sec. 2 as Proposition) to establish the well-posedness of the Cauchy problem for the above type functional-differential equation with f, h_i, λ and η in certain $\mathfrak{L}^*$-spaces (see e.g. [2]) which arise in a natural way. We shall treat the case $n = m = 1$ and $J = [0, \infty)$, since for $n, m > 1$ and $J = (-\alpha, \alpha)$ with $0 < \alpha \leqq \infty$ $0 \leqq t_i\, h(t)$, $|h_i(t)| \leqq t$, $t \in J$, the proof is similar and the reader can repeat it himself.

2. Let E be a Fréchet space with a saturated sequence $p_1, p_2, \ldots$ of seminorms which generates the topology of E (see e.g. [5]). Let A be a nonempty subset of E and let T be a one-to-one transformation of A into E foı which $T[A]$ is a closed set. Suppose that F_n $(n = 1, 2, \ldots)$ and F_0 are mappings from A into E satisfying the following conditions:

(1) $F_n[A] \subset T[A]$ for all $n \geqq 1$, (2) $\lim_{n\to\infty} F_n x = F_0 x$ for all x in A, and (3) $p_i(F_n x - F_n y) \leqq k \,.\, p_i(Tx - Ty)$ for all $i \geqq 1$, $n \geqq 1$ and x, y in A, where $0 \leqq k < 1$.

Now, we give the following result of the type of Banach contraction principle:

Proposition ([3], [4]). *Under the above assumptions there exists a unique point* x_m *(*$m = 0, 1, \ldots$*) in* A *such that* $F_m x_m = Tx_m$, *and* $Tx_n \to Tx_0$ *as* $n \to \infty$.

3. Throughout this part, $J = [0, \infty)$, R is the Euclidean space, and $C(J)$ denotes the set of all continuous real functions defined on J.

The set $C(J)$ let be considered as a vector space with the topology of almost uniform convergence (i.e., uniform convergence on compact subsets of J). This topology is determined by the sequence (p_n) of seminorms given as $p_n(x) = \sup_{0 \leqq t \leqq n} |x(t)|$ for x in $C(J)$, and therefore $C(J)$ is a Fréchet space.

Let K and $L < 1$ be nonnegative constants, and let G be a locally bounded function of J into itself. Next, we use the following notation:

$\mathfrak{F}$ – the set of all continuous real functions f defined on $J \times R \times R \times R$ such that $|f(t, x_1, y_1, \lambda) - f(t, x_2, y_2, \lambda)| \leqq K|x_1 - x_2| + L|y_1 - y_2|$ for $t \geqq 0$ and $x_1, x_2, y_1, y_2, \lambda$ in R;

$\mathfrak{F}_0$ – the set of all f in $\mathfrak{F}$ such that $|f(t, x, y, \lambda_1) - f(t, x, y, \lambda_2)| \leqq G(t)\,|\lambda_1 - \lambda_2|$ for $t \geqq 0$ and $x, y, \lambda_1, \lambda_2$ in R;

$\mathfrak{U}$ – the set of all continuous functions φ of J into itself with $\varphi(t) \leqq t$ for $t \geqq 0$.

By (PC) we shall denote the problem of finding the solution on the half-line $t \geqq 0$ of the differential equation

$$y'(t) = f(t, y(g(t)), y'(h(t)), \lambda)$$

satisfying the initial condition

$$y(0) = \eta\,;$$

here $f \in \mathfrak{F}$, g and h in $\mathfrak{U}$, and λ, η in R are given. Obviously, our (PC) problem is equivalent to the equation

$$x(t) = f\left(t, \eta + \int_0^{g(t)} x(s)\,\mathrm{d}s,\ x(h(t)), \lambda\right)$$

in the space $C(J)$.

Theorem. *For an arbitrary $f \in \mathfrak{F}$, $g \in \mathfrak{U}$, $h \in \mathfrak{U}$, $\lambda \in R$ and $\eta \in R$ there exists a unique function $y_{(f,g,h,\lambda,\eta)}$ satisfying the (PC) problem on J.*

Assume, moreover, that the sets $\mathfrak{F}_0$, $\mathfrak{U}$ are given the $\mathfrak{L}^*$-space structures ([2]) by the almost uniform convergence on $J \times R \times R \times R$ and J, respectively. Then the transformation

$$(f, g, h, \lambda, \eta) \mapsto y_{(f,g,h,\lambda,\eta)}$$

maps continuously the $\mathfrak{L}^*$-product ([2]) $\mathfrak{F}_0 \times \mathfrak{U} \times \mathfrak{U} \times R \times R$ into $C(J)$.

Proof. Let $r > 0$ be a constant such that $r^{-1}K + L < 1$. Let $\psi = (f, g, h, \lambda, \eta) \in \mathfrak{F} \times \mathfrak{U} \times \mathfrak{U} \times R \times R$. Define:

$$(Tx)(t) = \exp(-rt)\,x(t)\,,$$

$$(Fx)(t) = \exp(-rt) f\left(t, \eta + \int_0^{g(t)} x(s)\,\mathrm{d}s, x(h(t)), \lambda\right)$$

for x in $C(J)$. Then $F[C(J)] \subset C(J) = T[C(J)]$. For a positive integer n and u, v in $C(J)$ and $0 \leqq t \leqq n$, we have

$$\left|f\left(t, \eta + \int_0^{g(t)} v(s)\,\mathrm{d}s, v(h(t)), \lambda\right) - f\left(t, \eta + \int_0^{g(t)} u(s)\,\mathrm{d}s, u(h(t)), \lambda\right)\right| \leqq$$

$$\leq K \int_0^{g(t)} |u(s) - v(s)| \, ds + L|u(h(t)) - v(h(t))| \leq$$

$$\leq K \int_0^t \exp(rs) |(Tu)(s) - (Tv)(s)| \, ds + L \exp(rh(t)) |(Tu)(h(t)) - (Tv)(h(t))| \leq$$

$$\leq \left(\int_0^t \exp(rs) \, ds + L \exp(rt) \right) p_n(Tu - Tv) \leq (r^{-1}K + L) \exp(rt) \, p_n(Tu - Tv),$$

and it follows that $p_n(Fu - Fv) \leqq (r^{-1}K + L) \, p_n(Tu - Tv)$. Consequently, Proposition is applicable to the mappings T, F and the space $C(J)$. We conclude that there exists a unique x_ψ in $C(J)$ and

$$x_\psi(t) = f\left(t, \eta + \int_0^{g(t)} x_\psi(s) \, ds, x_\psi(h(t)), \lambda\right) \quad \text{for} \quad t \geqq 0,$$

which proves the first part of our result.

Let $\psi_m = (f_m, g_m, h_m, \lambda_m, \eta_m) \in \mathfrak{F}_0 \times \mathfrak{U} \times \mathfrak{U} \times R \times R$ for $m = 0, 1, \ldots$. Assume that $\lim_{n\to\infty} f_n = f_0$, $\lim_{n\to\infty} g_n = g_0$, $\lim_{n\to\infty} h_n = h_0$, and $|\lambda_n - \lambda_0| \to 0$ and $|\eta_n - \eta_0| \to 0$ as $n \to \infty$. Further, let I $(= [0, a])$ be a compact subset of J. We prove that $\sup_{t\in I} |y_{\psi_n}(t) - y_{\psi_0}(t)| \to 0$ as $n \to \infty$.

Denote by $C(I)$ the Banach space of all continuous real functions on I with the usual supremum norm $\|\cdot\|$. Now, let us denote by T, F_m $(m = 0, 1, \ldots)$ the mappings on $C(I)$ defined as above whenever $f = f_m$, $g = g_m$, $h = h_m$, $\lambda = \lambda_m$, $\eta = \eta_m$ and $x \in C(I)$. Obviously, $F_n[C(I)] \subset C(I) = T[C(I)]$ and $\|F_n u - F_n v\| \leqq (r^{-1}K + L)$. . $\|Tu - Tv\|$ for $n \geqq 1$ and u, v in $C(I)$. Moreover, for $n \geqq 1$ and x in $C(I)$ we obtain

$$|(F_n x)(t) - (F_0 x)(t)| \leqq K|\eta_n - \eta_0| + K \left| \int_0^{g_n(t)} x(s) \, ds - \int_0^{g_0(t)} x(s) \, ds \right| +$$

$$+ L|x(h_n(t)) - x(h_0(t))| + G(t) |\lambda_n - \lambda_0| +$$

$$+ \left| f_n\left(t, \eta_0 + \int_0^{g_0(t)} x(s) \, ds, x(h_0(t)), \lambda_0\right) - f_0\left(t, \eta_0 + \int_0^{g_0(t)} x(s) \, ds, x(h_0(t)), \lambda_0\right) \right|$$

for t in I. So we have $\|F_n x - F_0 x\| \to 0$ as $n \to \infty$.

Finally, by our Proposition there exists a unique $x_m \in C(I)$ $(m = 0, 1, \ldots)$ such that $x_{\psi_m|I} = x_m$ and $\|x_n - x_0\| \to 0$ as $n \to \infty$. This completes the proof of the theorem.

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