Časopis pro pěstování matematiky

Jan Chvalina

Range closure extension of input-output relations of topological general systems

Časopis pro pěstování matematiky, Vol. 108 (1983), No. 2, 208--213

Persistent URL: http://dml.cz/dmlcz/108409

Časopis pro pěstování matematiky, roč. 108 (1983), Praha

RANGE CLOSURE EXTENSION OF INPUT-OUTPUT RELATIONS OF TOPOLOGICAL GENERAL SYSTEMS

JAN CHVALINA, Brno

(Received March 30, 1982)

A topological general system – briefly tgs – is a triad $\mathscr{S} = (X, R, Y)$, where X, Y are topological spaces and $R \subset X \times Y$ is a binary relation called the input-output relation of $\mathscr{S}$ (cf. [11] and [14]). A relation R between spaces X, Y is said to be *continuous in the sense of Grimeisen* (cf. [6], [7]) – or briefly G-*continuous*, and the corresponding system (X, R, Y) is said to be continuously functionally parametrizable ([14]) – if there exists a family of continuous mappings $F \subset Y^{\operatorname{dom} R}$ such that $R = \bigcup_{f \in F} \operatorname{gr}(f)$ (where $\operatorname{gr}(f)$ is the graph of the mapping f) – see also the definition of the functional system decomposition [11], chap. X, Definition 2.4. It is to be noted that the notion of the G-continuity of relations is necessary (as is shown in [14]) for the construction of continuous realizations of topological time systems. Some criteria of the G-continuity of binary relations between topological spaces have been obtained in [3]. The present contribution is devoted to certain connections of the above mentioned notion with the extension of relations by means of closures of relation images of points.

In what follows (if not said otherwise) we suppose all input-output relations R of systems (X, R, Y) to be domain full, i.e. $\operatorname{dom} R = X$. Separation axioms and compactness are considered in the sense of Kelley's monograph [9] and hence e.g. a regular or a normal space need not be T_1.

Definition 1. A tgs $\mathscr{S}_e = (X, R_e, Y)$ is said to be the *range closure extension* of a tgs $\mathscr{S} = (X, R, Y)$ if $R_e(x) = \overline{R(x)}$ (which is the Y-closure of the set $R(x) = \{y: xRy\}$) for each $x \in \operatorname{dom} R$. The relation R_e is also called the range closure extension of R.

Remark 1. It is not difficult to construct examples showing that a tgs which is not continuously functionally parametrizable (i.e. with the input-output relation being not G-continuous) possesses the parametrizable range closure extension. Consider an at least six-element set X and put $\mathscr{S} = (X, R, X)$, where

$$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5)\} \cup \{(x, x_6) : x \in X,\ x \neq x_i,\ i = 1, 2, 3\}$$

and $x_i \in X$ for $i = 1, 2, \ldots, 6$. The set X is endowed with the left quasi-discrete topology induced by the transitive cover R^τ of R. That means that the least neighbourhood of a point $x \in X$ is the set $(R^\tau)^{-1}(x) \cup \{x\}$. Since $R \cap [(X \setminus \{x_1\}) \times (X \setminus \{x_1\})]$ is a functional relation (i.e. a mapping), the only functional parametrization of R is $R = \mathrm{gr}(f) \cup \mathrm{gr}(g)$, where $f, g : X \to X$ are mappings defined by: $f(x_1) = x_3$, $g(x_1) = x_2, f(x_3) = g(x_3) = x_5, f(x_3) = g(x_2) = x_4$ and $f(x) = g(x) = x_6$ for each $x \in X \setminus \{x_1, x_2, x_3\}$. Since the set $M = X \setminus \{x_1, x_3, x_5\}$ is closed but $g^{-1}(M) = X \setminus \{x_2\}$ is not closed (this set is dense in the space X), $g : X \to X$ is not continuous and hence R is not G-continuous. On the other hand, it is easy to verify that the range closure extension $\mathscr{S}_e = (X, R_e, X)$ of the system $\mathscr{S}$ has the G-continuous input-output relation $R_e = R^\tau$.

Using results of S. P. Franklin and R. H. Sorgenfrey we get the below stated assertions giving certain sufficient but (cf. Remark 1) not necessary conditions for the G-continuity of the range closure extension of a binary relation.

Recall that a topological space is said to be feebly locally connected if each of its points has at least one connected neighbourhood or, which is equivalent, each of its components is open — [2] Theorem 21B.5. A relation R is called point $\mathscr{P}$ (where $\mathscr{P}$ is a topological set property) if each $R(x)$ has the property $\mathscr{P}$. By a continuous relation we mean a relation which is simultaneously lower semicontinuous (l.s.c.) and upper semicontinuous (u.s.c.).

Proposition 1. *A* tgs $\mathscr{S} = (X, R, Y)$ *with a feebly locally connected input space X and a continuous point open input-output relation R admits the continuously functionally parametrizable range closure extension $\mathscr{S}_e$.*

Proof. Let $\{K_i : i \in I\}$ be the collection of all components of the space X. Since R is continuous and point open we have according to [4] Proposition 2 the equality $R_e(x) = R_e(y)$ for any pair of points $x, y \in K_i$ and each $i \in I$. For $i \in I$, denote by Y_i the set $R_e(x)$, where $x \in K_i$ is an arbitrary point. For any pair we define the mapping $f_{a,b} : X \to Y$ as follows: If K_j (for $j \in I$) contains the point a, we put $f_{a,b}(x) = b$ for each $x \in K_j$, and further for each $i \in I$, $i \neq j$ we choose an arbitrary point $y_i \in Y_i$ and put $f_{a,b}(x) = y_i$ for any $x \in K_i$. Then $\mathrm{gr}(f_{a,b}) \subset R_e$ and since the collection $\{K_i : i \in I\}$ is a decomposition of X we have $\mathrm{dom} f_{a,b} = X$. Consider a point $x_0 \in X$ and a neighbourhood U of the point $f_{a,b}(x_0)$. The component K_i containing x_0 is an open neighbourhood of x_0 and $f_{a,b}(K_i) = \{f_{a,b}(x_0)\} \subset U$. Thus $f_{a,b} : X \to Y$ is continuous and since $R_e = \bigcup_{(a,b) \in R_e} \mathrm{gr}(f_{a,b})$, we conclude that R_e is G-continuous, q.e.d.

The relationship between systems $\mathscr{S}_e = (X, R_e, Y)$ and $\bar{\mathscr{S}} = (X, \bar{R}, Y)$ (where $\bar{R}$ means the topological closure of R in the product space $X \times Y$) establishes the assertion (1) from [5] which says: If Y is regular and R is u.s.c. then $R_e = \bar{R}$. Thus we have:

Corollary. *Let* $\mathscr{S} = (X, R, Y)$ *be a* tgs *with a feebly locally connected input space* X, *regular output space* Y *and a continuous point open input-output relation* R. *Then the extension* $(X, \bar{R}, Y)$ *of* $\mathscr{S}$ *admits a continuous functional parametrization.*

Remark 2. It is to be noted in this connection that using one result of L. J. Billera ([1] Theorem 4.1) we immediately get the below stated necessary and sufficient condition for the closedness (in the product topology) of point closed relations $R \subset \subset X \times Y$ provided either Y is a locally compact Hausdorff space or $X \times Y$ is a k-space (i.e. a Kelley space): The range closure extension $R_e \subset X \times Y$ of R is closed iff, given a compact subset $C \subset Y$, a subset $M \subset X$ such that $R_e(M) \cap C = \emptyset$ is open. Indeed, the relation $R_e \subset X \times Y$ satisfies the above condition iff the corresponding mapping $\hat{R}_e$ of X into the family 2^Y of all closed subsets of Y endowed with the compact open topology is continuous. Now the assertion follows from [1] Theorem 4.1.

Relations considered in the following proposition need not be domain full.

Proposition 2. *The range closure extension of any lower semicontinuous binary relation between arbitrary topological spaces is lower semicontinuous.*

Proof. Let $x_0 \in \operatorname{dom} R \subset X$ be an arbitrary point at which the corresponding multivalued mapping $\hat{R}$ is l.s.c. Suppose $y \in \overline{R(x_0)} = R_e(x_0)$, and U is an open subset of Y containing the point y. Then $U \cap R(x_0) \neq \emptyset$ and there exists an open subset V of X such that $x_0 \in V$ and $\emptyset \neq R(t) \cap U \subset R_e(t) \cap U$ for each $t \in V$. Hence R_e is l.s.c. (see e.g. [12] § 2), q.e.d.

Since the input-output relation of a continuously functionally parametrizable tgs is l.s.c. – see e.g. [6] Theorem 1 – we get:

Corollary 1. *The range closure extension of a continuously functionally parametrizable* tgs *has the* l.s.c. *input-output relation.*

In particular, the following assertion which is in fact a reformulation of Proposition 1 of R. S. Liničuk [10] is an immediate consequence:

Corollary 2. *Let* X, Y *be* T_1*-spaces and* $\{f_n : X \to Y \mid n \in \mathbf{N}\}$ *a sequence of continuous mappings. Then the relations* $R_n = \bigcup_{k=1}^{n} \operatorname{gr}(f_k)$, $n = 1, 2, \ldots$ *(which coincide with their range closure extensions) and the range closure extension of the relation* $R = \bigcup_{n \in \mathbf{N}} \operatorname{gr}(f_n)$ *are* l.s.c.

The following example shows that the range closure extension does not preserve the G-continuity of relations in general.

Example. Let X be an at least four-element set, $x_1, x_2, x_3, x_4 \in X$ different elements. Define a topology $\mathscr{T}$ on X by putting $\operatorname{cl}_{\mathscr{T}} M = M \cup \{x_3, x_4\}$ whenever $M \cap \{x_1, x_2\} \neq \emptyset$ and $\operatorname{cl}_{\mathscr{T}} M = M$ otherwise. It is easy to verify that the mappings

$f, g : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined by $f(x_1) = f(x_2) = x_1$, $g(x_1) = g(x_2) = x_2$ and $f(x) = g(x) = x$ for all $x \in X \setminus \{x_1, x_2\}$ are continuous. Then the range closure extension of the relation $R = \mathrm{gr}(f) \cup \mathrm{gr}(g)$ is

$$R_e = \Delta_X \cup \{(x_1, x_i) : i = 2, 3, 4\} \cup \{(x_2, x_i) : i = 1, 3, 4\},$$

where Δ_X is the diagonal of $X \times X$. If $F \subset X^X$ is any family of mappings such that $R_e = \bigcup_{h \in F} \mathrm{gr}(h)$, then for some $h_0 \in F$ we have $h_0(x_1) = x_3$, $h_0(x_2) \in \{x_1, x_2, x_3, x_4\}$ and $h_0(x) = x$ if $x_1 \neq x \neq x_2$. But this mapping is not continuous for $h_0(\mathrm{cl}_{\mathcal{T}} \{x_1\}) = h_0(\{x_1, x_3, x_4\}) = \{x_3, x_4\}$ and $\mathrm{cl}_{\mathcal{T}} h_0(\{x_1\}) = \{x_3\}$, therefore the relation $R_e \subset (X, \mathcal{T}) \times (X, \mathcal{T})$ is not G-continuous.

On the other hand, by virtue of [5] Proposition (5) and Proposition 2 above we immediately get:

Proposition 3. *If $R \subset X \times Y$ is continuous and Y normal, then the range closure extension R_e of R is continuous.*

We say that a tgs $\mathcal{S}$ is finitely parametrizable if there exists a continuous functional parametrization of $\mathcal{S}$ consisting of finitely many mappings. The following assertion is similar to that of Proposition 2 from [10] which says (in other words) that the input-output relation of a finitely parametrizable system with the output T_1-space is continuous.

Proposition 4. *Let $\mathcal{S} = (X, R, Y)$ be a finitely parametrizable* tgs *with Y normal. Then the range closure extension of R is continuous.*

Proof. Suppose $\{f_k : X \to Y \mid k = 1, 2, \ldots, n\}$ is a family of continuous mapping such that $R = \bigcup_{k=1}^{n} \mathrm{gr}(f_k)$. Let $x_0 \in X$ be an arbitrary point, U an open subset of the space Y such that $R_e(x_0) \subset U$. There exists an open subset W of Y with $R_e(x_0) \subset W \subset \overline{W} \subset U$. Put $V = \bigcap_{k=1}^{n} f_k^{-1}(W)$. Then V is an open neighbourhood of x_0 and since $f_k(x) \in W$ for any $x \in V$ and $k = 1, 2, \ldots, n$ we have

$$R_e(V) = \bigcup_{x \in V} \overline{R(x)} = \bigcup_{x \in V} \{f_k(x) : k = 1, 2, \ldots, n\}^- \subset \overline{W} \subset U.$$

Consequently R_e is u.s.c. Since R is l.s.c. the relation R_e is also l.s.c. in virtue of Proposition 2, q.e.d.

We get other sufficient conditions for the continuity and closedness of range closure extensions by using some results of R. E. Smithson and J. E. Joseph. In accordance with [13] § 2 and [8] § 3 we formulate the following definitions:

Definition 2. A relation $R \subset X \times Y$ (where X, Y are topological spaces) is said to be *subcontinuous* if whenever $\{x_\alpha : \alpha \in A\}$ is a convergent net in $\mathrm{dom}\, R$ and

$\{y_\alpha : \alpha \in A\}$ is a net in $R(X)$ with $(x_\alpha, y_\alpha) \in R$ for each $\alpha \in A$, then $\{y_\alpha : \alpha \in A\}$ has a convergent subnet.

Definition 3. A relation $R \subset X \times Y$ is said to be *subclosed* if for each $x \in \operatorname{dom} R$ and every net $\{x_\alpha : \alpha \in A\} \subset \operatorname{dom} R \setminus \{x\}$ which converges to x and every net $\{y_\alpha : \alpha \in A\}$ with $(x_\alpha, y_\alpha) \in R$ for each $\alpha \in A$ which converges to some $y \in Y$, we have $(x, y) \in R$.

Proposition 5. *Let $R \subset X \times Y$ be a subclosed* l.s.c. *relation. If R_e is subcontinuous then R_e is closed and continuous.*

Proof. First we show that the range closure extension preserves the subclosedness of relations. Suppose $x_0 \in X$, $\{x_\alpha : \alpha \in A\}$ is a net in $X \setminus \{x_0\}$ which converges to x_0 and $\{y_\alpha : \alpha \in A\} \subset Y$ is a net with $(x_\alpha, y_\alpha) \in R_e$ for each $\alpha \in A$ which converges to $y_0 \in Y$. Since $y_\alpha \in \overline{R(x_\alpha)}$ there exists a net $\{t_\beta^{(\alpha)} : \beta \in B_\alpha\} \subset R(x_\alpha)$ $(\alpha \in A)$ converging to y_α. Using the theorem on iterated limits – [2] Theorem 15B.13 or [19] chap. II, Theorem 4 – we get a net $\{p_\gamma : \gamma \in A \times \prod_{\alpha \in A} B_\alpha\} \subset \bigcup_{\alpha \in A} R(x_\alpha) \subset R(X)$ which converges to y_0. Let $\{p_\alpha : \alpha \in A\}$ be a subnet of $\{p_\gamma : \gamma \in A \times \prod_{\alpha \in A} B_\alpha\}$ which also converges to y_0 (cf. [2] Theorem 15B.20 and part (d) of Remark 15B.18). Then $(x_\alpha, p_\alpha) \in R$ for all $\alpha \in A$ and since R is supposed to be subclosed, we have $(x_0, y_0) \in R \subset R_e$, hence R_e is subclosed as well. Since R_e is subcontinuous by assumption, it is u.s.c. according to [8] Theorem 3.2. By Proposition 2, R_e is l.s.c., thus it is continuous and by [8] Theorem 2.1(g) (R_e is point closed and subclosed) it is also closed in the product space $X \times Y$.

Remark 3. The following example shows that the range closure extension does not preserve the subcontinuity of relations:

Consider the set $\mathbf{N}$ of all non-negative integers endowed with the left order topology $\mathscr{T}^-$, i.e. $\mathscr{T}^- = \{\emptyset, \mathbf{N}\} \cup \{\{0, 1, 2, \ldots, k\} : k \in \mathbf{N}\}$. The mapping $f : (\mathbf{N}, \mathscr{T}^-) \to$ $\to (\mathbf{N}, \mathscr{T}^-)$ defined by $f(n) = n + 1$ is continuous, hence subcontinuous (as a single-valued relation). The range closure extension f_e of f is the relation

$$f_e = \{(n, k) : n + 1 \leqq k,\ n \in \mathbf{N}\}\,.$$

Since it follows immediately from the definition of subcontinuity that point closed subcontinuous relations are point compact ([13] p. 284) and sets $f_e(n) = \{n + 1, n + 2, \ldots\}$ for $n \in \mathbf{N}$ are not compact in the space $(\mathbf{N}, \mathscr{T}^-)$, we conclude that the relation f_e is not subcontinuous.

Proposition 6. *Let $\mathscr{S} = (X, R, Y)$ be a finitely parametrizable* tgs *with X locally compact. Then R is subcontinuous.*

Proof. Suppose $R = \bigcup_{k=1}^{n} \operatorname{gr}(f_k)$ with $f_k : X \to Y$ continuous for $k = 1, 2, \ldots, n$. If K is a compact subset of the space X then $R(K) = \bigcup_{k=1}^{n} f_k(K)$ is also compact, thus by [8] Theorem 3.4 the relation R is subcontinuous.

Proposition 7. *Let Y be a* T_4*-space (i.e. normal and* T_1*). The range closure extension of any* u.s.c. *relation $R \subset X \times Y$ is closed.*

Proof. Suppose $R \subset X \times Y$ is u.s.c. The relation R_e is also u.s.c. by [5] (5). Since R_e is point closed, by [13] Theorem 3.3 we get that R_e is closed.

References

[1] *L. J. Billera:* Topologies for 2^X; set-valued functions and their graphs, Trans. Amer. Math Soc. *155* (1971), 137—147.

[2] *E. Čech:* Topological Spaces (revised by *Z. Frolík* and *M. Katětov*). Academia, Prague 1966.

[3] *J. Chvalina:* On continuous functional parametrizations of topological general systems. General Topology and its Relations to Modern Analysis and Algebra V, Proc. Fifth Prague Topol. Sym. 1981 J. Novak (ed.), Heldermann Verlag Berlin 1982, 79—85.

[4] *S. P. Franklin:* Open and image open relations, Colloq. Math. *12* (1964), 209—211.

[5] *S. P. Franklin, R. H. Sorgenfrey:* Closed and image closed relaions, Pacific J. Math. *19* (1966), 433—439.

[6] *G. Grimeisen:* Continuous relations, Math. Zeit. *127* (1972), 35—44.

[7] *G. Grimeisen:* Continuous incidence relations of topological planes. To appear.

[8] *J. E. Joseph:* Multifunctions and graphs, Pacific J. Math. *79* (1978), 509—529.

[9] *J. L. Kelley:* General Topology, D. van Nostrand Co., Princeton 1955.

[10] *P. C. Линичук:* О некоторых многозначных отображениях, Докл. АН УССР, Сер. А, *2* (1981), 15—18.

[11] *M. D. Mesarović, Y. Takahara:* General Systems Theory, Academic Press, New York 1975.

[12] *R. E. Smithson:* Multifunctions, Nieuw Arch. Wisk *20* (1972), 31—55.

[13] *R. E. Smithson:* Subcontinuity for multifunctions, Pacific J. Math. *61* (1975), 283—288.

[14] *R. Valk:* Realisierungen allgemeiner Systeme, GMD Bericht Nr. 7, Bonn 1976, 131 pp.

Author's address: 662 80 Brno, Poříčí 31 (Ped. fakulta UJEP).