Mária Jakubíková
Hereditary radical classes of linearly ordered groups

Časopis pro pěstování matematiky, Vol. 108 (1983), No. 2, 199--207

Persistent URL: http://dml.cz/dmlcz/108408

Časopis pro pěstování matematiky, roč. 107 (1982), Praha

HEREDITARY RADICAL CLASSES OF LINEARLY ORDERED GROUPS

MÁRIA JAKUBÍKOVÁ, Košice

(Received December 7, 1981)

The study of radical classes and semisimple classes of linearly ordered groups was begun by Chehata and Wiegandt [1]. The basic properties of the lattice $\mathscr{R}$ of all radical classes of linearly ordered groups were described in [3]; for analogous questions concerning semisimple classes cf. [4]. In the papers [5], [7] and [8] radical classes and semisimple classes of abelian linearly ordered groups were dealt with.

In [3] and [4] it was proved that the lattice $\mathscr{R}$ has no atoms, no antiatoms and fails to be modular.

A radical class $X \in \mathscr{R}$ is said to be hereditary if, whenever $G \in X$ and H is a convex subgroup of G, then $H \in X$. The collection of all hereditary radical classes will be denoted by $\mathscr{R}_h$.

In this note it will be shown that $\mathscr{R}_h$ (partially ordered by inclusion) is a complete distributive lattice. In fact, $\mathscr{R}_h$ fulfils the infinite distributive law

$$A \wedge (\bigvee B_i) = \bigvee(A \wedge B_i),$$

hence $\mathscr{R}_h$ is a Brouwer lattice. The corresponding dual infinite distributive law does not hold in $\mathscr{R}_h$. Further, it will be proved that $\mathscr{R}_h$ has infinitely many atoms and that the collection $\mathscr{P}$ of all prime intervals of the lattice $\mathscr{R}_h$ is a proper collection. Thus some properties of the lattice $\mathscr{R}_h$ are analogous to those of the lattice of all radical classes of l-groups [2] or the lattice of all torsion classes of l-groups (cf. Martinez [6]).

The collection of all principal elements of $\mathscr{R}_h$ will be denoted by $\mathscr{R}_{hp}$. It will be shown that if $X \in \mathscr{R}_h$, $Y \in \mathscr{R}_{hp}$ and $X \leqq Y$, then $X \in \mathscr{R}_{hp}$. If $I \neq \emptyset$ is a set and $\{X_i\}_{i \in I} \subset \mathscr{R}_{hp}$, then $\bigvee_{i \in I} X_i$ belongs to $\mathscr{R}_{hp}$ as well. (Let us remark that analogous results do not hold for principal elements of the lattice of all radical classes of abelian linearly ordered groups; cf. [5].)

1. BASIC NOTIONS

A collection X will be said to be propre if there exists a one-to-one mapping of the class of all cardinals into X.

The group operation in a linearly ordered group will be denoted by $+$; the commutativity of this operation is not assumed. We recall some definitions; cf. [1].

Let $\mathscr{G}$ be the class of all linearly ordered groups. When considering a subclass X of $\mathscr{G}$ we always suppose that X is closed with respect to isomorphisms and that the zero linearly ordered group $\{0\}$ belongs to X.

A subclass X of $\mathscr{G}$ is said to be closed with respect to transfinite extensions if, whenever $G \in \mathscr{G}$ and

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \delta)$$

is an ascending chain of convex normal subgroups of G such that

$$G_\beta / \bigcup_{\gamma<\beta} G_\gamma \in X \quad \text{for each} \quad \beta < \delta\,,$$

then $\bigcup_{\alpha<\delta} G_\alpha$ belongs to X.

We also say that the linearly ordered group $\bigcup_{\alpha<\delta} G_\alpha$ is a transfinite extension of linearly ordered groups $G'_\beta(\mathrm{b} < \delta)$, where G'_β is isomorphic to $G_\beta / \bigcup_{\gamma<\beta} G_\gamma$ for each $\beta < \delta$.

1.1. Definition. *A class X of linearly ordered groups is called a radical class, if*

(a) *X is closed under homomorphisms, and*

(b) *X is closed with respect to transfinite extensions.*

We denote by $\mathscr{R}$ the collection of all radical classes. Further, let $\mathscr{R}_h$ be the collection of all hereditary radical classes. Both $\mathscr{R}$ and $\mathscr{R}_h$ are partially ordered by inclusion. Then $\mathscr{G}$ is the greatest element in both $\mathscr{R}$ and $\mathscr{R}_h$; the trivial variety R_0 containing all one-element l-groups is the least element in both $\mathscr{R}$ and $\mathscr{R}_h$.

If $\{A_i\}_{i\in I}$ is a non-empty collection of hereditary radical classes, then $\bigcap_{i\in I} A_i$ also is a hereditary radical class. Thus $\mathscr{R}_h$ is a complete lattice. The lattice operations in $\mathscr{R}_h$ will be denoted by $\wedge$ and $\vee$. The operation $\wedge$ in $\mathscr{R}_h$ coincides with the intersection of classes.

Let $Y \subseteq \mathscr{G}$ and $G \in \mathscr{G}$. The intersection of all hereditary radical classes X with $Y \subseteq X$ will be denoted by $T_h(X)$. Similarly, the intersection of all hereditary radical classes Z with $G \in Z$ is denoted by $T_h(G)$; the hereditary radical class $T_h(G)$ is said to be principal. We denote by $\mathscr{R}_{hp}$ the collection of all principal hereditary radical classes.

2. THE OPERATION $\vee$ IN THE LATTICE $\mathscr{R}_h$

Let X be a subclass of $\mathscr{G}$. We denote by

Hom X – the class of all homomorphic images of linearly ordered groups belonging to X;

Sub X – the class of all convex subgroups of linearly ordered groups belonging to X;

Ext X — the class of all transfinite extensions of linearly ordered groups belonging to X.

Now we define for each ordinal $\varkappa$ the class $\text{Ext}_\varkappa X$ by induction as follows. We put $\text{Ext}_1 X = \text{Ext}\, X$; if $\varkappa > 1$, then we set

$$\text{Ext}_\varkappa X = \text{Ext} \bigcup_{\tau<\varkappa} \text{Ext}_\tau X .$$

Next we denote

$$\text{ext}\, X = \bigcup_\varkappa \text{Ext}_\varkappa X ,$$

where $\varkappa$ runs over the class of all ordinals.

2.1. Theorem. *Let X be a subclass of $\mathscr{G}$. Then $T_h(X) = \text{ext Hom Sub}\, X$.*

Proof. Denote ext Hom Sub $X = Z$. Clearly $Z \subseteq T_h(X)$ and $X \subseteq Z$. Hence it suffices to prove that Z is a hereditary radical class. Thus we have to verify that Z fulfils the following conditions: (i) Ext $Z \subseteq Z$, (ii) Sub $Z \subseteq Z$; (iii) Hom $Z \subseteq Z$.

For each subclass Z_1 of $\mathscr{G}$ we have Ext ext $Z_1 = \text{ext}\, Z_1$, hence (i) is valid. In [3] (Lemma 2.1) it was proved that for each subclass Z_2 of $\mathscr{G}$ the relation

$$\text{Hom ext Hom}\, Z_2 = \text{ext Hom}\, Z_2$$

holds; therefore (iii) holds as well.

Let $G \in Z$ and let H be a convex subgroup of G with $H \subset G$. Hence there is an ordinal $\varkappa$ such that $G \in \text{Ext}_\varkappa \text{Hom Sub}\, X$. Thus it suffices to verify that for each ordinal $\varkappa$ we have

$$\text{Sub Ext}_\varkappa \text{Hom Sub}\, X \subseteq \text{Ext}_\varkappa \text{Hom Sub}\, X . \tag{1}$$

a) Let $\varkappa = 1$. There is an ascending chain of convex normal subgroups

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (x < \delta) \tag{2}$$

of G such that

$$\bigcup_{\alpha<\delta} G_\alpha = G \tag{3}$$

and for each $\beta < \delta$, $G_\beta / \bigcup_{\gamma<\beta} G_\gamma \in \text{Hom Sub}\, X$. Let λ be the first ordinal with $\lambda < \delta$ and $G_\lambda \supseteq H$. Denote $H_\alpha = H \cap G_\alpha$ for each $\alpha < \delta$. Then $\{H_\alpha\}$ $(\alpha < \delta)$ is an ascending chain of convex normal subgroups of H and $\bigcup_{\alpha<\delta} H_\alpha = H$. If $\beta < \lambda$, then

$$G_\beta / \bigcup_{\gamma<\beta} G_\gamma = H_\beta / \bigcup_{\gamma<\beta} H_\gamma ;$$

if $\beta > \lambda$, then $H_\beta / \bigcup_{\gamma<\beta} H_\gamma = \{0\}$. In the case $\beta = \lambda$ we have

$$H_\beta / \bigcup_{\gamma<\beta} H_\gamma \in \text{Sub}\, \{G_\beta / \bigcup_{\gamma<\beta} G_\gamma\} \subseteq \text{Sub Hom Sub}\, X = \text{Hom Sub}\, X ,$$

thus for $\varkappa = 1$ the relation (1) holds. (We use the well-known relation Sub Hom $Y \subseteq$ $\subseteq$ Hom Sub Y which is valid for each $Y \subseteq \mathscr{G}$.)

b) Assume that $\varkappa > 1$ and that (1) holds for each ordinal less than $\varkappa$. Then there is an ascending chain of convex normal subgroups (2) of G such that (3) is valid and for each $\beta < \delta$ there is an ordinal $\tau(\beta) < \varkappa$ having the property

$$G_\beta / \bigcup_{\gamma<\beta} G_\gamma \in \mathrm{Ext}_{\tau(\beta)} \mathrm{Hom\ Sub}\, X \,.$$

Let λ and H_α $(\alpha < \gamma)$ be as in part a). The cases $\mathrm{b} < \lambda$ and $\mathrm{b} > \lambda$ are analogous as in a). Let $\mathrm{b} = \lambda$. Then

$$H_\beta / \bigcup_{\gamma<\beta} H_\gamma \in \mathrm{Sub}\,\{G_\beta / \bigcup_{\gamma<\beta} G_\gamma\} \subseteq \mathrm{Sub\ Ext}_{\tau(\beta)} \mathrm{Hom\ Sub}\, X = $$
$$= \mathrm{Ext}_{\tau(\beta)} \mathrm{Hom\ Sub}\, X \,,$$

hence (1) is valid for each ordinal $\varkappa$, which completes the proof.

2.2. Theorem. *Let I be a nonempty class and for each $i \in I$ let X_i be a hereditary radical class. Then $\bigvee_{i\in I} X_i = \mathrm{ext} \bigcup_{i\in I} X_i$.*

Proof. From 2.1 it follows immediately that the relation

$$\bigvee_{i\in I} X_i = \mathrm{ext\ Hom\ Sub} \bigcup_{i\in I} X_i$$

is valid. Since X_i are hereditary radical classes, we have $\mathrm{Hom\ Sub}\, X_i = X_i$, therefore $\bigvee_{i\in I} X_i = \mathrm{ext} \bigcup_{i\in I} X_i$.

From 2.2 and [3] (Thm. 2.3) we obtain:

2.2.1. Corollary. R_h *is a closed sublattice of the complete lattice $\mathscr{R}$.*

2.3. Theorem. *Let $A \in \mathscr{R}_h$, $\{B_i\}_{i\in I} \subseteq \mathscr{R}_h$. Then*

$$A \wedge (\bigvee_{i\in I} B_i) = \bigvee_{i\in I} (A \wedge B_i)\,.$$

Proof. It suffices to verify that $A \wedge (\bigvee_{i\in I} B_i) \leqq \bigvee_{i\in I} (A \wedge B_i)$. Let $G \in A \wedge (\bigvee_{i\in I} B_i)$. Hence $G \in A$ and $G \in \bigvee_{i\in I} B_i$. In view of 2.2, $G \in \mathrm{ext} \bigcup_{i\in I} B_i$. Thus G is constructed by the operation ext from certain linearly ordered groups G_{ij} $(i \in I, j \in K_i)$ such that G_i belongs to B_i for each $i \in I$ and each $j \in K_i$.

According to the definition of ext, for each G_{ij} there exists a normal convex subgroup H_{ij} of G and a homomorphic image G'_{ij} of H_{ij} such that G'_{ij} is somorphic to G_{ij}. Because A is hereditary the linearly ordered group H_{ij} belongs to A and hence $G_{ij} \in A$. Thus $G_{ij} \in A \wedge B_i$ for each $i \in I$ and each $j \in K_i$. Therefore $G \in \mathrm{ext} \bigcup_{i\in I} . (A \wedge B_i) = \bigvee_{i\in I} (A \wedge B_i)$.

The following example shows that the relation

$$A \vee (\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} (A \vee B_i)$$

does not hold in general in the lattice $\mathscr{R}_n$. (The symbols $\Gamma_{j\in J}\, G_j$ and $G_1 \circ G_2$ denote lexicographic products of linearly ordered groups; cf., e.g., [5].)

2.4. Example. Let N be the set of all positive integers with the natural linear order. Let J be the linearly ordered set dual to N and for each $j \in J$ let G_j be an archimedean linearly ordered group, $G_j \neq \{0\}$, such that $G_{j(1)}$ and $G_{j(2)}$ fail to be isomorphic whenever $j(1)$ and $j(2)$ are distinct elements of J. For each $j \in J$ let $J_j = \{k \in J : k \leqq j\}$ (with the induced linear order). Put

$$
\begin{aligned}
G &= \Gamma_{j \in J} G_j , \\
G_{(j)} &= \Gamma_{k \in J_j} G_k \quad \text{for each} \quad j \in J , \\
A &= \bigvee_{j \in J} T_h(G_j) , \\
B_j &= T_h(G_{(j)}) \quad \text{for each} \quad j \in J .
\end{aligned}
$$

Then we have $G \notin A$, $\bigwedge_{j \in J} B_j = R_0$, hence

$$A \vee (\bigwedge_{j \in J} B_j) = A$$

and thus $G \notin A \vee (\bigwedge_{j \in J} B_j)$.

On the other hand, $G \in A \vee B_j$ for each $j \in J$, hence

$$G \in \bigwedge_{j \in J} (A \vee B_j)$$

and therefore $A \vee (\bigwedge_{j \in J} B_j) \neq \bigwedge_{j \in J} (A \vee B_j)$.

2.5. Lemma. *Let $X \subseteq \mathscr{G}$, $H \in T_h(X)$, $H \neq \{0\}$. Then there exists a convex subgroup H_1 of H with $H_1 \neq \{0\}$ such that $H_1 \in$ Hom Sub X.*

Proof. In view of 2.1 we have $H \in$ ext Hom Sub X, hence there is an ordinal τ such that $H \in \text{Ext}_\tau$ Hom Sub X. Thus there is an ordinal $\varkappa < \tau$ having the property that there exists a convex subgroup H' of H with $H' \neq \{0\}$ such that $H' \in \text{Ext}_\varkappa$ Hom Sub X.

Now let χ be the first ordinal having the property that there is a convex subgroup H'' of H with $H'' \neq \{0\}$ such that $H'' \in \text{Ext}_\chi$ Hom Sub X. Assume that $\chi > 1$. Then there is $\chi' < \chi$ such that there exists a convex subgroup $H^* \neq \{0\}$ of H'' with $H^* \in \text{Ext}_{\chi'}$ Hom Sub X. Since H^* is a convex subgroup of H, we have arrived at a contradiction. Hence $\chi = 1$. Therefore there is a convex subgroup $H_1 \neq \{0\}$ of H'' such that $H_1 \in$ Hom Sub X, which completes the proof.

3. ATOMS IN $\mathscr{R}_h$

3.1. Proposition. *Let $G \in \mathscr{G}$, $G \neq \{0\}$. Assume that G is archimedean. Then $T_h(G)$ is an atom in the lattice $\mathscr{R}_h$.*

Proof. We have $R_0 < T_h(G)$. Let $A \in \mathscr{R}_h$, $R_0 < A \leqq T_h(G)$. There exists $H \in A$ with $H \neq \{0\}$. In view of 2.1 we have $T_h(G) =$ ext Hom Sub $\{G\}$. Since G is archimedean, Hom Sub $\{G\}$ is the class of all linearly ordered groups G' such that either $G' = \{0\}$ or G' is isomorphic to G. Hence H can be constructed by the operation ext from a system of linearly ordered groups G_i $(i \in I)$ such that each G_i is isomorphic to G. Let $i \in I$ be fixed. There exists a normal convex subgroup H_i of G and a homomorphic image G'_i of H_i such that G'_i is isomorphic to G_i. Since A is hereditary, we have $H_i \in A$ and thus $G'_i \in A$. Therefore $G \in A$ and hence $A = T_h(G)$.

Because there is an infinite set of mutually nonisomorphic archimedean linearly ordered groups, 3.1 implies:

3.2. Corollary. *The class of all atoms of the lattice $\mathscr{R}_h$ is infinite.*

3.3. Proposition. *Let $X \in \mathscr{R}_h$, $X \neq R_0$. Then there exists an archimedean linearly ordered group $H \neq \{0\}$ such that $T_h(H) \leqq X$.*

Proof. There exists $G \in X$ such that $G \neq \{0\}$. Choose $g \in G$, $g > 0$ and let $\mathscr{H} = \{H_i\}_{i \in I}$ be the set of all convex subgroups of G not containing the element g. Let H_1 be the convex subgroup of G generated by g. Because the set $\mathscr{H}$ is linearly ordered, $\mathscr{H}$ has a unique maximal element H_2. Then H_2 is the largest proper convex subgroup of H_1. Hence H_2 is a normal subgroup in H_1. Therefore $H = H_1/H_2$ is o-simple and thus it is archimedean. Clearly $H \neq \{0\}$. Now we have $T_h(H) = T_h(H_1/H_2) \leqq T_h(G) \leqq T_h(X)$.

From 3.1 and 3.3 we infer:

3.4. Theorem. *Let $X \in \mathscr{R}_h$. Then the following conditions are equivalent:*

(i) *X covers R_0 in the lattice $\mathscr{R}_h$.*

(ii) *There is an archimedean linearly ordered group $H \neq \{0\}$ such that $X = T_h(G)$.*

Let A_0 be a set of non-zero archimedean linearly ordered groups such that (a) if G_1 and G_2 are distinct elements of A_0, then G_1 is not isomorphic to G_2, and (b) for each non-zero archimedean linearly ordered group G there is G' in A_0 such that G is isomorphic to G'. Put

$$X_0 = \bigvee_{G \in A_0} T_h(G)\,.$$

A collection X will be said to be small if there exists a set Y and a mapping of Y onto X.

3.5. Proposition. *Let $\mathscr{G}_1 = [R_0, X_0]$ (the interval taken in $\mathscr{R}_h$). Then*

(i) *$\mathscr{G}_1$ is a small collection;*

(ii) *$\mathscr{G}_1$ is a complete atomic Boolean algebra; the collection of atoms of $\mathscr{G}_1$ is $\{T_h(G)\}_{G \in A_0}$.*

Proof. $\mathscr{G}_1$ is obviously a complete lattice and in view of 2.3, $\mathscr{G}_1$ is distributive. From 3.4 it follows that $A'_0 = \{T_h(G)\}_{G \in A_0}$ is the collection of all atoms of $\mathscr{G}_1$. Let $R_0 \neq X \in \mathscr{G}_1$ and let $X' = \{T_h(G) : G \in A_0 \cap X\}$. Then

$$X = X \wedge X_0 = X \wedge (\bigvee_{G \in A_0} T_h(G)) = \bigvee_{G \in A_0} (X \wedge T_h(G)) = \bigvee_{G \in A_0 \cap X} (X \wedge T_h(G)) = \sup X'\,.$$

Moreover, if $X'' \subseteq A'_0$ and $\sup X'' = X$, then 2.3 implies that $X' = X''$. Hence $\mathscr{G}_1$ is isomorphic to the Boolean algebra of all subsets of the set A'_0.

3.6. Lemma. *Let $X \in \mathscr{G}_1$, $X \neq R_0$. Let I be a linearly ordered set isomorphic to the set of all negative integers (with the natural linear order). Let $G = \Gamma_{i \in I} G_i$, where each G_i belongs to $A_0 \cap X$. Assume that for each $G' \in A_0 \cap X$ and each $j \in I$ there is $i \in I$ with $i < j$ such that G' is isomorphic to G_i. Then*

(i) *$T_h(G)$ covers X,*

(ii) *$T_h(G)$ does not belong to $\mathscr{G}_1$,*

(iii) *$T_h(G) \wedge T_h(G') = R_0$ whenever $G' \in A_0$ and $G' \notin X$.*

Proof. We apply the same notations as in the proof of 3.5. For each $G' \in A_0 \cap X$ we have $T_h(G') \leqq T_h(G)$, hence $X = \bigvee_{G' \in A_0 \cap X} T_h(G') \leqq T_h(G)$. In view of 2.5, $T_h(G)$ does not belong to $\mathscr{G}_1$ and thus $X < T_h(G)$. Let $Y \in \mathscr{R}_h$, $X < Y \leqq T_h(G)$. There exists $H \in Y \setminus X$. Hence $H \in T_h(G)$. According to Thm. 2.1, H can be constructed from a subset S of the class Hom Sub $\{G\}$ by the operation ext. Because H does not belong to X, the set S must contain a linearly ordered group isomorphic to $\Gamma_{i \in I, i < j} G_i$ for some $j \in I$. Then we have $G \in Y$, whence $Y = T_h(G)$ and so (i) is valid. (iii) is a consequence of 2.1 and 2.3.

For each $X \in R_h$ we denote by $a(X)$ the collection of all $Y \in \mathscr{R}_h$ such that Y covers X in the lattice $\mathscr{R}_h$.

From 3.6 we immediately obtain:

3.7. Corollary. *Let $X \in \mathscr{G}_1$, $X \neq R_0$. Then there exists $Y \in a(X) \cap \mathscr{R}_{hp}$ such that $Y \notin \mathscr{G}_1$.*

The proof of the following proposition will be omitted (it can be established by using similar arguments as in the proof of 3.6).

3.8. Proposition. *Let $X \in \mathscr{G}_1$, $X \neq R_0$. Let I be as in 3.6 and let $G = \Gamma_{i \in I} G_i$, where each G_i belongs to $A_0 \cap X$. Then the following conditions are equivalent:*

(i) *$T_h(G)$ covers X;*

(ii) *for each $G' \in A_0 \cap X$ and each $j \in I$ there is $i \in I$ such that $i < j$ and G' is isomorphic to G_i.*

4. PRINCIPAL ELEMENTS OF $\mathscr{R}_h$

4.1. Proposition. *Let $X, Y \in \mathscr{R}_h$, $X \leqq Y$. Assume that Y is a principal element of $\mathscr{R}_h$. Then X is principal as well.*

Proof. Let $Y = T_h(G)$. In view of 2.1, $Y = \text{ext Hom Sub}\{G\}$. There exists a set $S = \{H_i\}_{i \in I}$ of linearly ordered groups such that $S \subset \text{Hom Sub}\{G\}$ and for each

$G_1 \in \mathrm{Hom\,Sub}\,\{G\}$ there is $i \in I$ such that G_1 is isomorphic to H_i. Hence $Y = \mathrm{ext}\,\{H_i\}_{i\in I}$ and $X \subseteq \mathrm{ext}\,\{H_i\}_{i\in I}$. Thus there is $\emptyset \neq J \subseteq I$ such that $X = \mathrm{ext}\,\{H_i\}_{i\in J}$. We can assume that J is well-ordered (by using the Axiom of Choice). Put $H = \Gamma_{i\in J} H_i$. Then $H_i \in T_h(H)$ holds for each $i \in J$, hence $X = \mathrm{ext}\,\{H_i\}_{i\in J} = \bigvee_{i\in J} T_h(H_i) \leqq T_h(H)$. On the other hand, $H \in \mathrm{Ext}\,\{H_i\}_{i\in J}$ and so $T_h(H) \leqq T_h(\{H_i\}_{i\in J}) = X$. Thus $X = T_h(H) \in \mathscr{R}_{hp}$.

4.2. Proposition. *Let I be a nonempty set and for each $i \in I$ let X_i be a principal element of $\mathscr{R}_h$. Then $X = \bigvee_{i\in I} X_i$ is a principal element of $\mathscr{R}_h$ as well.*

Proof. There are $G_i \in \mathscr{G}$ such that $X_i = T_h(G_i)$. We clearly have $X = T_h(\{G_i\}_{i\in I}) = \mathrm{ext\,Hom\,Sub}\,\{G_i\}_{i\in I}$. There is a set $S = \{H_j\}_{j\in J} \subset \mathscr{G}$ such that (i) $S \subset \mathrm{Hom\,Sub}\,\{G_i\}_{i\in I}$, and (ii) for each $G_1 \in \mathrm{Hom\,Sub}\,\{G_i\}_{i\in I}$ there is $j \in J$ having the property that G_1 is isomorphic to H_j. Again, we can assume that J is well-ordered. Put $H = \Gamma_{j\in J} H_j$. It is easy to verify that $X = T_h(H)$, hence X is principal.

Let α be a cardinal. We denote by $I(\alpha)$ the first ordinal having the property that the set of all ordinals less than $I(\alpha)$ has the cardinality α. Let $J(\alpha)$ be the linearly ordered set dual to $I(\alpha)$.

Let $G \in \mathscr{G}$, $G \neq \{0\}$. We put

$$G_{(\alpha)} = \Gamma_{j\in J(\alpha)} G_j\,,$$

where each G_j is isomorphic to G.

4.3. Lemma. *Let $G \in \mathscr{G}$, $G \neq \{0\}$, $\alpha >$* card G. *Then $T_h(G) < T_h(G_{(\alpha)})$.*

Proof. We have $G \in \mathrm{Hom}\,\{G_{(\alpha)}\}$, hence $T_h(G) \leqq T_h(G_{(\alpha)})$. In view of 2.5, $G_{(\alpha)} \notin T_h(G)$. Hence $T_h(G) < T_h(G_{(\alpha)})$.

4.4. Corollary. *The class $\mathscr{R}_{hp}$ has no maximal element. In particular, $\mathscr{G}$ does not belong to $\mathscr{R}_{hp}$.*

Let $G \in \mathscr{G}$, $G \neq \{0\}$. In view of 4.3 there is a least cardinal $\beta = \beta(G)$ such that $T_h(G) < T_h(G_{(\beta(G))})$.

The following proposition shows that there are many prime intervals in the lattice $\mathscr{R}_h$.

4.5. Proposition. *Let $G \in \mathscr{G}$, $G \neq \{0\}$. Then $T_h(G)$ is covered by $T_h(G_{\beta(G)})$ in the lattice $\mathscr{R}_h$.*

Proof. We have $T_h(G) < T_h(G_{(\beta(G))})$. Let $X \in \mathscr{R}_h$, $T_h(G) < X \leqq T_h(G_{(\beta(G))})$. There exists $G_1 \in X \setminus T_h(G)$. Then $G_1 \in \mathrm{ext\,Hom\,Sub}\,\{G_{(\beta(G))}\}$. Hence there exists a set $S \subset \mathrm{Hom\,Sub}\,\{G_{(\beta(G))}\}$ such that G_1 can be constructed by means of ext from the set S. In view of $G_1 \notin T_h(G)$ there is $H \in S$ such that H does not belong to Hom Sub . . $\{G\}$. Therefore, from the construction of $G_{(\beta(G))}$ it follows that there is a convex subgroup H_1 of H such that H_1 is isomorphic to $G_{(\beta(G))}$. Since $H_1 \in X$ we obtain $G_{(\beta(G))} \in X$, implying $X = T_h(G_{(\beta(G))})$.

From 4.5 and 3.1 we infer:

4.6. Corollary. *Let* $G \in \mathcal{R}_{hp}$. *Then* $a(T_h(G)) \cap \mathcal{R}_{hp} \neq \emptyset$.

Let $\mathcal{P}$ be the class of all prime intervals of the lattice R_h. From 4,5 and 4.2 we obtain:

4.7. Proposition. $\mathcal{P}$ *is a proper collection.*

References

[1] *C. G. Chehata, R. Wiegandt:* Radical theory for fully ordered groups. Rév. Anal. Numér. Théor. Approx. *20* (43), 1979, 143—157.

[2] *J. Jakubík:* Radical mappings and radical classes of lattice ordered groups. Symposia Mathematica, Vol. *21*, Academic Press, London—New York, 1977, 451—477.

[3] *J. Jakubík:* On the lattice of radical classes of linearly ordered groups. Studia Sci. Math. Hungar. (to appear).

[4] *J. Jakubík:* On the lattice of semisimple classes of linearly ordered groups. Čas. Pěst. Mat. (submitted).

[5] *J. Jakubík:* On radical classes of abelian linearly ordered groups (submitted).

[6] *G. Pringerová:* Covering condition in the lattice of radical classes of linearly ordered groups. Math. Slovaca (submitted).

[7] *G. Pringerová:* On semisimple classes of abelian linearly ordered groups. Časopis Pěst. Mat. (submitted).

[8] *J. Martinez:* Torsion theory for lattice ordered groups, I, Czech. Math. J. *25*, 1975, 284—299.

Author's address: 040 01 Košice, Švermova 9 (Vysoká škola technická).