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ON KNESER - TYPE SOLUTIONS OF SUBLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Summary. A nontrivial solution $u: [a, +\infty[\to R$ of an ordinary differential equation of n-th order is called a Kneser-type solution (KS) if $(-1)^i u^{(i)}(t) \geqq 0$ for $t \geqq a$ $(i = 0, \ldots, n-1)$. A KS is called degenerate (singular) if it is constant (zero) in some neighbourhood of $+\infty$, and nondegenerate otherwise. In the paper a class of equations admitting sufficiently many singular KS-s is introduced and studied. For the equations from this class a sufficient condition for the existence of a nondegenerate KS with a prescribed limit at $+\infty$ is established. Two-sided a priori asymptotic estimates of such solutions are obtained.

Keywords: n-th order ordinary differential equations, monotone solutions, asymptotic estimates

AMS classification: 34C99.

INTRODUCTION

Let n be a natural number and $f \in K_{\mathrm{loc}}(R_+ \times R^n; R)$[1]. Consider the equation

$$(0.1) \qquad u^{(n)} = f(t, u, u', \ldots, u^{(n-1)}) .$$

We say that a nontrivial solution $u: [\alpha, +\infty[\to R$ of (0.1) is a Kneser-type solution if

$$(0.2) \qquad (-1)^i u^{(i)}(t) \geqq 0 \quad \text{for} \quad t \geqq a \quad (i = 0, \ldots, n-1) .$$

This definition is motivated by the fact that the problem of finding a solution of (0.1) satisfying (0.2) together with the additional condition $u(a) = u_0 > 0$ was for the first time considered by A. Kneser [1] in the case $n = 2$. Later this problem was studied in [2]–[4] for the case $n = 2$ and in [5]–[8] for the general case.

It is proved in [6] that if

$$(0.3) \qquad \begin{aligned} &f(t, 0, \ldots, 0) \equiv 0 , \quad (-1)^n f(t, x_1, \ldots, x_n) \geqq 0 \\ &\text{for} \quad t \geqq 0 , \quad (-1)^{i-1} x_i \geqq 0 \quad (i = 1, \ldots, n) \end{aligned}$$

then the equation (0.1) has a one-parameter family of Kneser-type solutions. In

[1]) For the notation see Section 1.

what follows we assume that the conditions (0.3) are fulfilled. At the end of the paper we shortly discuss the case when $f(t, 0, \ldots, 0) \not\equiv 0$.

We say that a Kneser-type solution is *degenerate* if it is constant in some neighbourhood of $+\infty$, and *nondegenerate* otherwise. We call a Kneser-type solution *singular* if it is identically zero in some neighbourhood of $+\infty$. Throughout the paper for the sake of brevity we denote a Kneser-type solution by KS, a nondegenerate Kneser-type solution by NKS and a singular Kneser-type solution by SKS. It is clear that any NKS u: $[a, +\infty[\to R$ of (0.1) satisfies

$$(-1)^i u^{(i)}(t) > 0 \quad \text{for} \quad t \geqq a \quad (i = 0, \ldots, n-1),$$

and for any SKS u: $[a, +\infty[\to R$ of (0.1) there exists a point $b > a$, which we call its singular point, such that

$$(-1)^i u^{(i)}(t) > 0 \quad \text{for} \quad a \leqq t < b \ (i = 0, \ldots, n-1),$$
$$u(t) = 0 \quad \text{for} \quad t \geqq b .$$

In this paper sufficient conditions are given for the equation (0.1) to have a SKS with a prescribed singular point. For a class of equations these conditions turn out to be necessary as well. The problems of existence and two-sided a priori estimates of NKS-s are also studied in the case when the equation (0.1) admits sufficiently many SKS-s (see Definition 2.1). In this sense (the solution of (0.1) with zero initial conditions is not unique) the case considered here may be treated as sublinear (see also (2.3) and (2.19)). Similar problems were considered in [9], [10] in the case when f is bounded with respect to $x_2, \ldots, x_n$.

1. NOTATION AND AUXILIARY STATEMENTS

Throughout the paper we use the following notation.

$$R =]-\infty, +\infty[, \quad R_+ = [0, +\infty[, \quad R^n = \underbrace{R \times \ldots \times R}_{n \text{ times}},$$

$$R^n_+ = \underbrace{R_+ \times \ldots \times R_+}_{n \text{ times}},$$

$$\tilde{R}^n = \{(x_1, \ldots, x_n) \in R^n: (-1)^{i-1} x_i \geqq 0 \quad (i = 1, \ldots, n)\} ;$$

if I, $J \subset R$ are intervals and $\Gamma \subset R^n$ then $L_{\text{loc}}(I; J)$ is the set of all functions $p: I \to J$ which are Lebesgue integrable on every compact subinterval of I; $C(\Gamma; J)$ is the set of all continuous functions $h: \Gamma \to J$; $K_{\text{loc}}(I \times \Gamma; J)$ is the set of all functions $f: J \times \Gamma \to J$ satisfying the local Carathéodory conditions, i.e. $f(t, \cdot) \in C(\Gamma; J)$ for almost all $t \in I$ and $\sup\{|f(\cdot, x)|: x \in \Gamma_0\} \in L_{\text{loc}}(I; J)$ for any compact $\Gamma_0 \subset \Gamma$.

By a solution of the equation (0.1) defined on I we mean a function $u: I \to R$ which is absolutely continuous on each compact subinterval of I along with its derivatives up to and including the order $n - 1$ and satisfies (0.1) almost everywhere in I.

Now we present some known results which we will use later.

Lemma 1.1. ([6; p. 1388]) *Let* $u\colon [a, +\infty[\to R$ *be a KS of* (0.1). *Then* $\lim_{t \to +\infty} t^i u^{(i)}(t) = 0$ $(i = 1, \ldots, n - 1)$.

Let $\beta > 0$ and $c_i \in R$ $(i = 0, \ldots, n - 1)$. Consider the initial value problem for (0.1)

$$u^{(i)}(\beta) = c_i \quad (i = 0, \ldots, n - 1)\,. \tag{1.1}$$

Lemma 1.2. *Let the function*

$$(t, x_1, x_2, \ldots, x_n) \mapsto (-1)^n f(t, x_1, -x_2, \ldots, (-1)^{n-1} x_n)$$

be nondecreasing in $x_1, \ldots, x_n$ *and let* $u\colon]\alpha, \beta] \to R$ *be any solution of* (0.1), (1.1). *Then for any* $n - 1$ *times continuously differentiable function* $v\colon]\alpha, \beta] \to R$ *satisfying the inequalities*

$$\begin{gathered}(-1)^i v^{(i)}(\beta) \geqq (-1)^i c_i \quad (i = 0, \ldots, n - 2)\,, \\ (-1)^{n-1} v^{(n-1)}(t) > (-1)^{n-1} [c_{n-1} - \textstyle\int_t^\beta f(\tau, v(\tau), -v'(\tau), \ldots \\ \ldots, (-1)^{n-1} v^{(n-1)}(\tau)\, d\tau \quad \text{for} \quad \alpha < t \leqq \beta\end{gathered}$$

we have

$$(-1)^i v^{(i)}(t) > (-1)^i u^{(i)}(t) \quad \text{for} \quad \alpha < t < \beta \quad (i = 0, \ldots, n - 1)\,.$$

This lemma is a special case of a general statement concerning systems of ordinary differential equations (see e.g. [7; Lemma 4.6]).

2. ON SINGULAR KNESER-TYPE SOLUTIONS

Theorem 2.1. *Let* $b > 0$ *and let there exist* $a \in [0, b[$, $m \in \{1, \ldots, n\}$ *and* $\varrho > 0$ *such that*

$$\begin{gathered}(-1)^n f(t, x_1, \ldots, x_n) \geqq p(t)\, h(|x_m|, \ldots, |x_n|) \quad \text{for} \quad a \leqq t \leqq b\,, \\ 0 \leqq (-1)^{i-1} x_i \leqq \varrho \quad (i = 1, \ldots, n)\end{gathered} \tag{2.1}$$

where $p \in L([a, b]; R_+)$,

$$\textstyle\int_t^b p(\tau)\, d\tau > 0 \quad \text{for} \quad a \leqq t < b\,, \tag{2.2}$$

$h \in C(R_+^{n-m+1}; R_+)$ *is nondecreasing in each of its variables,* $h(x, \ldots, x) > 0$ *for* $x > 0$ *and*

$$\textstyle\int_0^x [y^{n-m} h(y, \ldots, y)]^{-1/(n-m+1)}\, dy < +\infty \quad \text{for} \quad x > 0\,. \tag{2.3}$$

Then the equation (0.1) *has a SKS with* b *as the singular point.*

Proof. Let u_ε be an arbitrary solution of (0.1) with initial conditions

$$u^{(i)}(b) = 0 \ (i = 0, \dots, n-2), \quad u^{(n-1)}(b) = (-1)^{n-1} \varepsilon . \tag{2.4}$$

It is not difficult to see that there exists $a_0 \in [a, b[$ such that for any $\varepsilon \in]0, \varrho/2]$ the maximal left-hand interval of existence of u_ε contains $[a_0, b]$ and

$$0 < (-1)^i u_\varepsilon^{(i)}(t) \leqq \varrho \quad \text{for} \quad a_0 \leqq t < b \quad (i = 0, \dots, n-1) . \tag{2.5}$$

By (2.5) the solutions $(u_\varepsilon)_{0<\varepsilon\leqq\varrho/2}$ of (0.1) are uniformly bounded and equicontinuous on $[a_0, b]$ along with their derivatives up to and including the order $n-1$. Therefore, one can choose such a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ that $\lim_{k\to\infty} \varepsilon_k = 0$ and

$$\lim_{k\to\infty} u_{\varepsilon_k}^{(i)}(t) = u_0^{(i)}(t) \quad \text{for} \quad a_0 \leqq t \leqq b \quad (i = 0, \dots, n-1) \tag{2.6}$$

where u_0 is a solution of (0.1). It obviously satisfies

$$u_0^{(i)}(b) = 0 , \quad 0 \leqq (-1)^i u_0^{(i)}(t) \leqq \varrho \quad \text{for} \quad a_0 \leqq t \leqq b \quad (i = 0, \dots, n-1) . \tag{2.7}$$

Let $\varepsilon \in]0, \varrho/2]$ be fixed. From the obvious inequalities

$$|u_\varepsilon^{(m-1)}(t)| = \frac{1}{(i-1)!} \int_t^b (\tau - t)^{i-1} |u^{(m+i-1)}(\tau)| \, d\tau \leqq$$
$$\leqq \frac{(b-t)^i}{i!} |u^{(m+i-1)}(t)| \ (i = 1, \dots, n-m)$$

we get

$$|u_\varepsilon^{(m+i-1)}(t)| \geqq i! \, b^{-i} |u_\varepsilon^{(m-1)}(t)| \quad \text{for} \quad a_0 \leqq t \leqq b \quad (i = 0, \dots, n-m) . \tag{2.8}$$

So (2.1), (2.5) and the monotonicity of h imply

$$(-1)^n u_\varepsilon^{(n)}(t) \geqq p(t) \, \tilde{h}(|u_\varepsilon^{(m-1)}(t)|) \quad \text{for} \quad a_0 \leqq t \leqq b$$

where

$$\tilde{h}(x) = h(x, b^{-1}x, \dots, (l-1)! \, b^{1-l}x) \quad \text{for} \quad x \geqq 0 , \quad l = n - m + 1 .$$

According to Lemma 1.2 this inequality together with (2.4) implies

$$(-1)^{m-1} u_\varepsilon^{(m-1)}(t) > v_\varepsilon(t) \quad \text{for} \quad a_0 \leqq t < b \tag{2.9}$$

where v_ε is any solution of the problem

$$v^{(l)} = (-1)^l p(t) \, \tilde{h}(v) , \tag{2.10}$$

$$v^{(i)}(b) = 0 \quad (i = 0, \dots, l-2) , \quad v^{(l-1)}(b) = (-1)^{l-1} \varepsilon/2 \tag{2.11}$$

with $l = n - m + 1$. Clearly

$$(-1)^i v_\varepsilon^{(i)}(t) > 0 \quad \text{for} \quad a_0 \leqq t < b \quad (i = 0, \dots, l-1) . \tag{2.12}$$

Without loss of generality we can assume that $p(t) \leqq 1$ for $a \leqq t \leqq b$. Using this along with (2.11), (2.12) and the monotonicity of $\tilde{h}$ we have from (2.10) (supposing first that $l > 1$).

$$(2.13)\qquad [v_\varepsilon^{(l-1)}(t)]^2 \leqq (\varepsilon/2)^2 + 2\int_t^b \tilde{h}(v_\varepsilon(\tau))\,|v_\varepsilon^{(l-1)}(\tau)|\,\mathrm{d}\tau \leqq$$
$$\leqq (\varepsilon/2)^2 + 2\tilde{h}(v_\varepsilon(t))\,|v_\varepsilon^{(l-2)}(t)| \quad \text{for} \quad a_0 \leqq t < b\,.$$

Due to Lemma 9.2′ in [7] the inequality

$$|v_\varepsilon^{(l-2)}(t)| \leqq [(l-1)!]^{1/(l-1)}\,[v_\varepsilon(t)]^{1/(l-1)}\,|v_\varepsilon^{(l-1)}(t)|^{(l-2)/(l-1)}$$

sholds for $a_0 \leqq t \leqq b$, which together with (2.13) implies (since $(\varepsilon/2)^2 \leqq (\varepsilon/2)^{l/(l-1)} \cdot [v_\varepsilon(t)]^{(l-2)/(l-1)}$)

$$(2.14)\qquad |v_\varepsilon^{(l-1)}(t)| < \varepsilon/2 + \omega(v_\varepsilon(t)) \quad \text{for} \quad a_0 \leqq t < b$$

where

$$(2.15)\qquad \omega(x) = 2lx^{1/l}[\tilde{h}(x)]^{(l-1)/l} \quad \text{for} \quad x \geqq 0\,.$$

(2.14) is obviously true for $l = 1$.

Now take an arbitrary $t_0 \in]a_0, b[$. Noting that $v_\varepsilon(t) \geqq [2(l-1)!]^{-1}\,\varepsilon(b-t)^{l-1}$ for $a_0 \leqq t \leqq b$ we get from (2.10) and (2.11)

$$[v_\varepsilon^{(l-1)}(t)] \geqq \varepsilon/2 + \gamma + \int_t^{t_0} p(\tau)\,\tilde{h}(v_\varepsilon(\tau))\,\mathrm{d}\tau \quad \text{for} \quad a_0 \leqq t \leqq t_0$$

where

$$\gamma = \int_{t_0}^b p(\tau)\,\tilde{h}([2(l-1)!]^{-1}\,\varepsilon(b-\tau)^{l-1})\,\mathrm{d}\tau > 0\,.$$

This inequality together with (2.14) implies

$$(2.16)\qquad \omega(v_\varepsilon(t)) > \gamma + \int_t^{t_0} p(\tau)\,h^*(\omega(v_\varepsilon(\tau)))\,\mathrm{d}\tau \quad \text{for} \quad a_0 \leqq t \leqq t_0$$

where

$$(2.17)\qquad h^*(x) = \tilde{h}(\omega^{-1}(x)) \quad \text{for} \quad x \geqq 0$$

and ω^{-1} is the function inverse to ω.

By the (2.15), (2.17) and the well-known properties of the Riemann-Stieltjes integral we have

$$(2.18)\qquad \int_y^x \frac{\mathrm{d}\xi}{h^*(\xi)} = \int_{\omega^{-1}(y)}^{\omega^{-1}(x)} \frac{\mathrm{d}\omega(z)}{\tilde{h}(z)} = 2l(l-1)\left[\frac{\omega^{-1}(y)}{\tilde{h}(\omega^{-1}(y))}\right]^{1/l} -$$
$$- 2l(l-1)\left[\frac{\omega^{-1}(x)}{\tilde{h}(\omega^{-1}(x))}\right]^{1/l} + 2l\int_{\omega^{-1}(y)}^{\omega^{-1}(x)} [z^{l-1}\,\tilde{h}(z)]^{-1/l}\,\mathrm{d}z$$
$$\text{for} \quad x > y > 0\,.$$

On the other hand, using the monotonicity of $\tilde{h}$ we conclude

$$\int_{y/2}^y [z^{l-1}\,\tilde{h}(z)]^{-1/l}\,\mathrm{d}z = \int_{y/2}^y \left[\frac{z}{\tilde{h}(z)}\right]^{1/l} \frac{\mathrm{d}z}{z} \geqq \frac{\ln 2}{2}\left[\frac{y}{\tilde{h}(y)}\right]^{1/l},$$

hence (2.3) yields

$$\text{(2.19)} \qquad \lim_{y \to 0+} \frac{y}{\tilde{h}(y)} = 0 .$$

Therefore, (2.18) implies

$$H(x) \equiv \int_0^x \frac{\mathrm{d}\xi}{h^*(\xi)} < +\infty \quad \text{for} \quad x \geqq 0 .$$

The function

$$x(t) = H^{-1}(H(\gamma) + \textstyle\int_t^{t_0} p(\tau)\,\mathrm{d}\tau) \quad \text{for} \quad a \leqq t \leqq t_0 ,$$

where H^{-1} is the function inverse to H, is the solution of the problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -p(t)\,h^*(x) , \quad x(t_0) = \gamma .$$

Therefore, according to Lemma 1.2, (2.16) implies

$$\omega(v_\varepsilon(t)) > x(t) > H^{-1}(\textstyle\int_t^{t_0} p(\tau)\,\mathrm{d}\tau) \quad \text{for} \quad a_0 \leqq t \leqq t_0 .$$

Hence for t_0 tending to b we get

$$\text{(2.20)} \qquad v_\varepsilon(t) \geqq \omega^{-1}(H^{-1}(\textstyle\int_t^b p(\tau)\,\mathrm{d}\tau)) \quad \text{for} \quad a_0 \leqq t \leqq b .$$

(2.2), (2.6), (2.9) and (2.20) imply that the function

$$u(t) = \begin{cases} u_0(t) & \text{for} \quad a_0 \leqq t < b \\ 0 & \text{for} \quad t \geqq b \end{cases}$$

is the SKS of (0.1) with b as the singular point. This completes the proof.

Remark. One can formally consider h in (2.1) to depend on more than $n - m + 1$ variables, so in (2.3) one can take m smaller. However, taking into consideration (2.9) it is easy to see that (2.3) implies its own validity with m larger and the example of the function $h(x_1, x_2) = x_2 |\ln x_2|^{3/2}$ shows that the inverse is not true. In some cases, however, as in the corollary below, the value of m is not important.

Corollary. *Let $b > 0$ and let there exist $a \in [0, b[$, $m \in \{1, \ldots, n\}$ and $\varrho > 0$ such that*

$$\text{(2.21)} \qquad \begin{aligned} &(-1)^n f(t, x_1, \ldots, x_n) \geqq p(t)\,|x_m|^{\lambda_m} \ldots |x_n|^{\lambda_n} \quad \textit{for} \quad a \leqq t \leqq b , \\ &0 \leqq (-1)^{i-1} x_i \leqq \varrho \quad (i = 1, \ldots, n) \end{aligned}$$

with $\lambda_i \geqq 0$ $(i = m, \ldots, n)$, $0 < \sum_{i=m}^{n} \lambda_i < 1$ and $p \in L([a, b]; R_+)$ satisfying (2.2). Then the conclusion of Theorem 2.1 is valid.

Theorem 2.2. *Let $l \geqq 1$, $b > a \geqq 0$, $p \in L_{\mathrm{loc}}([a, +\infty[; R_+)$, let $\tilde{h} \in C(R_+; R_+)$ be nondecreasing, $\tilde{h}(0) = 0$ and $\tilde{h}(x) > 0$ for $x > 0$. Then the condition* (2.2) *is necessary for the equation* (2.10) *to have a SKS with b as the singular point. If, in addition,* (2.22) $\operatorname{vraimax}\{p(t)\colon a_0 \leqq t \leqq b\} = c < +\infty$ *for some $a_0 \in [a, b[$ then the condition*

(2.23) $$\int_0^x [y^{l-1}\,\tilde{h}(y)]^{-1/l}\,\mathrm{d}y < +\infty \quad \text{for} \quad x > 0$$

is necessary as well.

Proof. The necessity of (2.2) is obvious. Let now (2.22) hold and let $v\colon [o_0, +\infty[\to R$ be a SKS with b as the singular point. Without loss of generality we may assume that $c = 1$, for otherwise we can consider $v_0 = c^{-1}v$ instead of v. Quite analogously to (2.15) we can get (notation is from the proof of Theorem 2.1)

$$\omega(v(t)) > \int_t^b p(\tau)\,h^*(\omega(v(\tau)))\,\mathrm{d}\tau \equiv x(t) \quad \text{for} \quad a_0 \leqq t < b\,.$$

We have

$$x'(t) > -p(t)\,h^*(x(t)) \quad \text{for} \quad a_0 \leqq t < b\,,$$

so

$$\int_0^{x(a_0)} \frac{d\xi}{h^*(\xi)} = -\int_{a_0}^b \frac{x'(t)\,\mathrm{d}t}{h^*(x(t))} \leqq \int_{a_0}^b p(t)\,\mathrm{d}t < +\infty\,.$$

Hence by (2.18) we get (2.23). The theorem is thus proved.

Definition 2.1. We say that the equation (0.1) has the *property S* provided for any $t_0 \geqq 0$ there exists a SKS of this equation with the singular point $b > t_0$.

Theorem 2.1 and its corollary immediately imply the following results.

Theorem 2.3. *Let there exist $a \geqq 0$, $m \in \{1, \ldots, n\}$ and a nonincreasing function $\varrho\colon [a, +\infty[\to]0, +\infty[$ such that*

(2.24) $$(-1)^n f(t, x_1, \ldots, x_n) \geqq \varphi(t, |x_m|, \ldots, |x_n|) \quad \text{for} \quad t \geqq a\,,$$
$$0 \leqq (-1)^{i-1}\,x_i \leqq \varrho(t) \quad (i = 1, \ldots, n)$$

with $\varphi \in K_{\mathrm{loc}}([a, +\infty[\times R^{n-m+1}; R_+)$ satisfying one of the following two conditions:

1) *φ does not increase in the first variable, does not decrease in the last $n - m + 1$ variables*

(2.25) $$\varphi(t, x, \ldots, x) > 0\,, \quad \int_0^x [y^{n-m}\varphi(t, y, \ldots, y)]^{-1(n-m+1)}\,\mathrm{d}y + \infty$$
$$\text{for} \quad t \geqq a\,, \quad x > 0\,;$$

2) *$\varphi(t, x_m, \ldots, x_n) \equiv p(t)\,h(x_m, \ldots, x_n)$ with h satisfying the conditions of Theorem 2.1 and $p \in L_{\mathrm{loc}}([a, +\infty[\to R_+)$ satisfying*

(2.26) $$\operatorname{mes}\{\tau \geqq t\colon p(\tau) > 0\} > 0 \quad \text{for} \quad t \geqq a\,.$$

Then the equation (0.1) *has the property S.*

Corollary. *Let there exist* $a \geqq 0$, $m \in \{1, \ldots, n\}$ *and a nonincreasing function* $\varrho: [a, +\infty[\to]0, +\infty[$ *such that the inequality* (2.21) *holds for* $t \geqq a$, $0 \leqq (-1)^{i-1} x_i \leqq \varrho(t)$ *with* $\lambda_i \geqq 0$ $(i = m, \ldots, n)$, $0 < \sum_{i=m}^{n} \lambda_i < 1$, *and* $p \in L_{\mathrm{loc}}([a, +\infty[; R_+)$ *satisfying* (2.26). *Then the equation* (0.1) *has the property S.*

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NONDEGENERATE KNESER-TYPE SOLUTIONS

For any $a > 0$, $r > 0$, $r_0 > 0$ put

$$D_n(a, r, r_0) = \{(t, x_1, \ldots, x_n) \in [a, +\infty[\times \tilde{R}^n: |x_1| \leqq r, \\ |x_i| \leqq r_0 t^{1-i} \quad (i = 2, \ldots, n)\}.$$

Theorem 3.1. *Let there exist* $a > 0$, $r > 0$, $r_0 > 0$ *such that*

$$\varphi(t, |x_1|, \ldots, |x_n|) \leqq (-1)^n f(t, x_1, \ldots, x_n) \leqq g(t) \tag{3.1}$$

on $D_n(a, r, r_0)$ *where* $\varphi \in K_{\mathrm{loc}}([a, +\infty[\times R^n_+; R_+)$ *does not decrease in the last n variables, the equation*

$$v^{(n)} = (-1)^n \varphi(t, |v|, \ldots, |v^{(n-1)}|) \tag{3.2}$$

has the property S, $g \in L_{\mathrm{loc}}([a, +\infty[; R_+)$ *and*

$$\int_a^{+\infty} \tau^{n-1} g(\tau)\, d\tau < +\infty. \tag{3.3}$$

Then for any $c \in [0, r[$ *the equation* (0.1) *has a NKS u such that*

$$\lim_{t \to +\infty} u(t) = c. \tag{3.4}$$

Proof. For any $\varrho > 0$ put

$$\pi(\varrho, x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x & \text{for} \quad 0 \leqq x \leqq \varrho \\ \varrho & \text{for} \quad x > \varrho \end{cases}$$

and let

$$\tilde{f}(t, x_1, \ldots, x_n) = f(t, \pi(r, x_1), -\pi(r_0 t^{-1}, -x_2), \ldots, (-1)^{n-1} \,.\\ .\, \pi(r_0 t^{1-n}, (-1)^{n-1} x_n)),$$

$$\tilde{\varphi}(t, x_1, \ldots, x_n) = \varphi(t, \pi(r, x_1), \pi(r_0 t^{-1}, x_2), \ldots, \pi(r_0 t^{1-n}, x_n)).$$

It is not difficult to see that $\tilde{f} \in K_{\mathrm{loc}}([a, +\infty[\times R^n; R)$, $\tilde{\varphi} \in K_{\mathrm{loc}}([a, +\infty[\times R^n_+; R_+)$, $\tilde{\varphi}$ does not decrease in the last n variables, the equation

$$v^{(n)} = (-1)^n \tilde{\varphi}(t, |v|, \ldots, |v^{(n-1)}|) \tag{3.5}$$

has the property S and by (3.1)

$$\tilde{\varphi}(t, |x_1|, \ldots, |x_n|) \leqq (-1)^n \tilde{f}(t, x_1, \ldots, x_n) \leqq g(t) \quad \text{for} \quad (t, x_1, \ldots, x_n) \in [a, +\infty[\times R^n . \tag{3.6}$$

Since the equation (3.5) has the property S, there exists a sequence $\{v_k\}_{k=1}^{\infty}$ of its SKS-s, the singular points t_k $(k = 1, 2, \ldots)$ of which satisfy

$$t_{k+1} > t_k > a \ (k = 1, 2, \ldots) . \ \lim_{k \to \infty} t_k = +\infty .$$

In view of (3.6) we may assume that all v_k-s are defined on $[a, +\infty[$.

Let k be fixed and let u_k be a solution of

$$u^{(n)} = \tilde{f}(t, u, \ldots, u^{(n-1)}) \tag{3.7}$$

with initial conditions

$$u(t_k) = c , \quad u^{(i)}(t_k) = 0 \quad (i = 1, \ldots, n-2) , \quad u^{(n-1)}(t_k) = (-1)^{n-1} \varepsilon_k \tag{3.8}$$

where

$$\varepsilon_k = \min_{0 \leqq i \leqq n-1} \{t_k^{-i} \textstyle\int_{t_k}^{+\infty} \tau^i g(\tau) \, d\tau\} . \tag{3.9}$$

By (3.6) u_k may be assumed to be defined on $[a, t_k]$. Clearly $\varepsilon_k > 0$ because the contrary would contradict the property S of the equation (3.2). Since, in addition, $(-1)^{n-1} v_{k+1}^{(n-1)}(t_k) > 0 = (-1)^{n-1} v_k^{(n-1)}(t_k)$, Lemma 1.2 yields

$$0 < (-1)^i v_k^{(i)}(t) < (-1)^i v_{k+1}^{(i)}(t) , \quad (-1)^i v_k^{(i)}(t) < (-1)^i u_k^{(i)}(t) \quad \text{for} \quad a \leqq t \leqq t_k \quad (i = 0, \ldots, n-1) . \tag{3.10}$$

Besides, we have by (3.6) and (3.9)

$$\begin{aligned} |u_k^{(i)}(t)| &= c_i + \frac{\varepsilon_k (t_k - t)^{n-i-1}}{(n-i-1)!} + \frac{1}{(n-i-1)!} \int_t^{t_k} (\tau - t)^{n-i-1} \, . \\ &\quad . \, |f(\tau, u_k(\tau), \ldots, u_k^{(n-1)}(\tau))| \, d\tau \leqq c_i + \int_{t_k}^{+\infty} \tau^{n-i-1} g(\tau) \, d\tau + \\ &\quad + \int_t^{t_k} \tau^{n-i-1} g(\tau) \, d\tau = c_i + \int_t^{+\infty} \tau^{n-i-1} g(\tau) \, d\tau \\ &\quad \text{for} \quad a \leqq t \leqq t_k \quad (i = 0, \ldots, n-1) \end{aligned} \tag{3.11}$$

with $c_0 = c$, $c_i = 0$ $(i = 1, \ldots, n-1)$.

It follows from (3.11) that for any compact interval $I \subset [a, +\infty[$ the sequence $\{u_k\}_{k=1}^{\infty}$ is eventually defined, uniformly bounded and equicontinuous on I along with its derivatives up to and including the order $n-1$. So without loss of generality we can assume that it converges to a solution u: $[a, +\infty[\to R$ of (3.7). By (3.10) and (3.11) u is a NKS and satisfies (3.4).

According to (3.4) and Lemma 1.1 there exists $a_0 \geqq a$ such that $(t, u(t), \ldots$ $\ldots, u^{(n-1)}(t)) \in D_n(a, r, r_0)$ for $t \geqq a_0$. Therefore, by definition of $\tilde{f}$, the restriction of u to $[a_0, +\infty[$ is a solution of (0.1). This completes the proof.

Remark. From the proof it is clear that if we suppose that instead of (3.1)

$$(3.12) \qquad (-1)^n f(t, x_1, \ldots, x_n) \leqq g(t)$$

holds on $D_n(a, r, r_0)$ where g satisfies (3.3), only the existence of a KS with the property (3.4) is guaranteed. This result generalizes Theorem 5 of the paper [5] of M. Švec. The example of the equation $u'' = -t^{-3}u'$ all the KS-s of which are constant shows that in general one cannot claim more, and the necessary conditions for the existence of a NKS of (0.1) given in Section 4 imply that (3.3) cannot be omitted, either.

According to Theorem 2.3 and its corollary, Theorem 3.1 implies

Corollary. *Let there exist $a > 0$, $r > 0$, $r_0 > 0$ and $m \in \{1, \ldots, n\}$ such that*

$$\varphi(t, |x_m|, \ldots, |x_n|) \leqq (-1)^n f(t, x_1, \ldots, x_n) \leqq g(t)$$

on $D_n(a, r, r_0)$ with φ satisfying one of the two conditions of Theorem 2.3 and $g \in L_{\text{loc}}([a, +\infty[; R_+)$ satisfying (3.3). Then the conclusion of Theorem 3.1 is valid. This is the case, in particular, if $\varphi(t, x_m, \ldots, x_n) \equiv p(t)\, x_m^{\lambda_m} \ldots x_n^{\lambda_n}$ with with $\lambda_i \geqq 0$ $(i = m, \ldots, n)$, $0 < \sum_{i=m}^{n} \lambda_i < 1$ and p satisfying (2.26).

4. NECESSARY CONDITIONS FOR THE EXISTENCE OF A NONDEGENERATE KNESER-TYPE SOLUTION. ESTIMATES

Let l be natural and let $I \subset]0, +\infty[$ be an interval. For any function $\varphi: I \times R_+^l \to$ $\to R$ which is continuous in the last l variables put

$$(4.1) \qquad \varphi_l(t, x) = \left[\frac{1}{(l-2)!}\int_0^x (x-y)^{l-2}\, \varphi\left(t, y, \frac{y}{t}, \ldots, \frac{(l-1)!\, y}{t^{l-1}}\right) \mathrm{d}y\right]^{1/l}$$

$$\text{if} \quad l > 1\,,$$

$$\varphi_1(t, x) = \varphi(t, x) \quad \text{for} \quad t \in I\,, \quad x \geqq 0\,.$$

Theorem 4.1. *Let there exist $a > 0$, $r > 0$, $r_0 > 0$ and $m \in \{1, \ldots, n\}$ such that the inequality (2.24) holds on $D_n(a, r, r_0)$ where $\varphi \in K_{\text{loc}}([a, +\infty[\times R_+^{n-m+1}; R_+)$ neither increases in the first variable nor decreases in the last $n - m + 1$ variables, $\varphi(t, x, \ldots, x) > 0$ for $x > 0$ and*

$$(4.2) \qquad \Phi(t, x) \equiv \int_0^x \frac{\mathrm{d}y}{\varphi_{n-m+1}(t, y)} < +\infty \quad \textit{for} \quad t \geqq a\,, \quad x \geqq 0$$

where φ_{n-m+1} is defined by (4.1). *Then the condition*

(4.3) $$\lim_{t\to+\infty} t^{m-1}\gamma(t) = 0$$

is necessary for existence of a NKS u of (0.1) *with* $\lim_{t\to+\infty} u(t) \in [0, r[$ *where*

(4.4) $$\gamma(t) = \sup\{\Phi^{-1}(s, s-t): s \geqq t\}$$

and $\Phi^{-1}(t, \cdot)$, $t \geqq a$, is the function inverse to $\Phi(t, \cdot)$. Moreover, any NKS u of (0.1) *satisfies*

(4.5) $$\left|u^{(m-1)}(t) - \lim_{t\to+\infty} u^{(m-1)}(t)\right| \geqq \gamma(t)$$

for large t.

Proof. Let $l = n - m + 1$, $v(t) \equiv (-1)^{m-1} u^{(m-1)}(t)$. By (4.1) and Lemma 1.1 a may be assumed to be so large that the inequality

(4.6) $$(-1)^l v^{(l)}(t) \geqq \varphi(t, |v(t)|, \ldots, |v^{(l-1)}(t)|) \quad \text{for} \quad t \geqq a$$

holds.

Let $s > a$ be fixed. Put

$$v_s(t) = v(t) - \sum_{j=0}^{l-1} \frac{|v^{(j)}(s)|(s-t)^j}{j!} \quad \text{for} \quad a \leqq t \leqq s\,.$$

Since u is a NKS,

(4.7) $$0 < (-1)^i v_s^{(i)}(t) < (-1)^i v^{(i)}(t) \quad \text{for} \quad a \leqq t \leqq s \quad (i = 0, \ldots, l-1)\,.$$

Besides, since $v_s^{(i)}(s) = 0$ $(i = 0, \ldots, l-1)$, quite analogously to (2.8) we obtain that

(4.8) $$|v_s^{(i)}(t)| \geqq i!\, s^{-1} v_s(t) \quad \text{for} \quad a \leqq t \leqq s \quad (i = 0, \ldots, l-1)\,.$$

(4.6)–(4.8) together with the monotonicity of φ imply

$$(-1)^l v_s^{(l)}(t) \geqq \varphi\left(s, v_s(t), \frac{v_s(t)}{s}, \ldots, \frac{(l-1)!\, v_s(t)}{s^{l-1}}\right) \quad \text{for} \quad a \leqq t \leqq s\,.$$

Multiply both sides of this inequality by $|v_s'(t)|$ and integrate from $t \in [a, s]$ to s. Integrating by parts and taking into consideration (4.7) we obtain

$$(-1)^l v_s^{(l-1)}(t)\, v_s'(t) \geqq \int_0^{v_s(t)} \varphi\left(s, y, \frac{y}{s}, \ldots, \frac{(l-1)!\, y}{s^{l-1}}\right) dy$$

for $a \leqq t \leqq s$.

Suppose first that $l > 1$ and apply this procedure for $l - 2$ more times. We get

$$-v_s'(t) \geqq \varphi_l(s, v_s(t)) \quad \text{for} \quad a \leqq t \leqq s\,.$$

This inequality is obviously true for $l = 1$. Hence we have

$$v_s(t) \geqq \Phi^{-1}(s, s - t) \quad \text{for} \quad a \leqq t \leqq s\,.$$

Since $s > a$ was fixed quite arbitrarily, from this inequality we easily get (4.5) which together with Lemma 1.1 implies the necessity of (4.3). The theorem is thus proved.

Remark 1. It is easy to see that if $m > 1$ then the estimates

$$\left|u^{(i)}(t) - \lim_{t \to +\infty} u^{(i)}(t)\right| \geqq \frac{1}{(m - i - 2)!} \int_t^{+\infty} (\tau - t)^{m-i-2}\, \gamma(\tau)\, \mathrm{d}\tau$$
$$(i = 0, \ldots, m - 2)$$

hold for large t. Hence we have the necessary condition

$$\int^{+\infty} t^{m-2}\, \gamma(t)\, \mathrm{d}t < +\infty$$

which, in general, is stronger than (4.3).

Corollary. *Let there exist $a > 0$, $r > 0$, $r_0 > 0$ and $m \in \{1, \ldots, n\}$ such that the inequality* (2.21) *holds on $D_n(a, r, r_0)$ with*

$$\lambda_i \geqq 0 \ (i = m, \ldots, n)\,, \quad 0 < \lambda = \sum_{i=m}^{n} \lambda_i < 1\,, \quad \textit{and}$$
$$p\colon [a, +\infty[\ \to\ ]0, +\infty[$$

nonincreasing. Then the condition

$$\text{(4.9)} \qquad \lim_{t \to +\infty} t^{n - \lambda_2 - \ldots - (n-1)\lambda_n}\, p(t) = 0 \quad \text{if} \quad m = 1\,,$$
$$\int^{+\infty} t^{m-2} [p_0(t)]^{1/(1-\lambda)}\, \mathrm{d}t < +\infty \quad \text{if} \quad m > 1$$

where $p_0(t) = \sup\{s^{-\lambda_{m+1} - \ldots - (n-m)\lambda_m}\, p(s)\, (s - t)^{n-m+1} : s \geqq t\}$ is necessary for the equation (0.1) *to have a NKS u with $\lim_{t \to +\infty} u(t) \in [0, r[$. Moreover, any NKS u of* (0.1) *satisfies*

$$\left|u^{(m-1)}(t) - \lim_{t \to +\infty} u^{(m-1)}(t)\right| \geqq \mu [p_0(t)]^{1/(1-\lambda)}$$

for large t where

$$\mu = [(n - m + 1)^{n-m+1} (1 - \lambda)^{-(n-m+1)} (\lambda + 1) \ldots$$
$$\ldots (\lambda + n - m)]^{1/(\lambda - 1)}\,.$$

For the proof it suffices to notice that if

$$\varphi(t, x_m, \ldots, x_n) \equiv p(t) x_m^{\lambda_m} \ldots x_n^{\lambda_n}$$

then

$$\Phi^{-1}(t, x) = \mu [t^{-\lambda_m - \ldots - (n-m)\lambda_m}\, p(t)\, x^{n-m+1}]^{1/(1-\lambda)}\,.$$

Remark 2. Since

$$\varphi_l(t, x) \leqq x^{(l-1)/l}\left[\varphi\left(t, x, \frac{x}{t}, \ldots, \frac{(l-1)!\,x}{t^{l-1}}\right)\right]^{1/l},$$

(4.2) implies (2.25). Besides, if φ is formally considered as depending on $l + 2$ variables then

$$\varphi_{l+1}(t, x) = \left[\int_0^x [\varphi_l(t, y)]^l \, \mathrm{d}y\right]^{1/(l+1)} \leqq \varphi_l(t, x)\left[\frac{x}{\varphi_l(t, x)}\right]^{1/(l+1)},$$

so (4.2) implies its own validity with m larger (see (2.19)). The example of the function $\varphi(t, x_1, x_2) \equiv x_2 |\ln x_2|^{3/2}$ shows that the inverse is not true. As to the role of m in the necessary conditions, consider the equation $u'' = t^{\lambda-2}(\ln \ln t)^{-1} |u'|^{\lambda}$ with $0 < \lambda < 1$. The condition (4.9) with $n = 2$, $m = 1$ is obviously fulfilled. Take now $m = 2$. It can be checked that then $[p_0(t)]^{1/(1-\lambda)} \sim ct^{-1}(\ln \ln t)^{-1}$ as $t \to +\infty$, so (4.9) does not hold and we are able to conclude that the equation under consideration has no NKS at all.

Remark 3. A necessary condition for the existence of NKS of $u^{(n)} = (-1)^n \,.\, p(t) |u|^{\lambda} \operatorname{sign} u$ where $0 < \lambda < 1$, $p(t) \geqq 0$ and p need not be monotone was obtained by N. A. Izobov in [11].

Theorem 4.2. *Let there exist $a > 0$, $r > 0$, $r_0 > 0$ and $m \in \{1, \ldots, n\}$ such that*

$$(4.10) \qquad (-1)^n f(t, x_1, \ldots, x_n) \leqq \psi(t, |x_m|, \ldots, |x_n|)$$

on $D_n(a, r, r_0)$ where $\psi \in K_{\mathrm{loc}}([a, +\infty[\times R_+^{n-m+1}; R_+)$ does not decrease in the last $n - m + 1$ variables and is such that the function $\psi_{n-m+1}: [a, +\infty[\to R_+$ is correctly defined by the equalities

$$(4.11) \qquad \psi_0(t, x_m, \ldots, x_n) \equiv \psi(t, x_m, \ldots, x_n),$$
$$\psi_i(t, x_m, \ldots, x_{n-i}) = \sup\{x \geqq 0: x \leqq \textstyle\int_t^{+\infty} \psi_{i-1}(\tau, x_1, \ldots, x_{n-i}, x)\, \mathrm{d}\tau\}$$
$$(i = 1, \ldots, n - m + 1).$$

Then any NKS u of (0.1) *with* $\lim\limits_{t \to +\infty} u(t) \in [0, r[$ *satisfies*

$$(4.12) \qquad \left|u^{(m-1)}(t) - \lim_{t \to +\infty} u^{(m-1)}(t)\right| \leqq \psi_{n-m+1}(t)$$

for large t.

Proof. According to (4.10) and Lemma 1.1 a may be assumed to be so large that the inequality

$$(-1)^n u^{(n)}(t) \leqq \psi(t, |u^{(m-1)}(t)|, \ldots, |u^{(n-1)}(t)|) \quad \text{for} \quad t \geqq a$$

holds. Integrating from t to $+\infty$ and using (4.11) and the monotonicity of ψ we get

$$(-1)^{n-1}\, u^{(n-1)}(t) \leqq \psi_1(t, |u^{(m-1)}(t)|, \ldots, |u^{(n-2)}(t)|) \quad \text{for} \quad t \geqq a\,.$$

Continuing in this way we get (4.12). The theorem is thus proved.

Corollary. *Let there exist $a > 0$, $r > 0$, $r_0 > 0$ and $m \in \{1, \ldots, n\}$ such that*

$$(4.13) \qquad (-1)^n f(t, x_1, \ldots, x_n) \leqq q(t)\, |x_m|^{\lambda_m} \ldots |x_n|^{\lambda_n}$$

on $D_n(a, r, r_0)$ where

$$\lambda_i \geqq 0 \; (i = m, \ldots, n)\,, \quad 0 < \sum_{i=m}^{n} \lambda_i < 1\,,$$

and $q \in L_{\mathrm{loc}}([a, +\infty[; R_+)$ is such that the function $q_{n-m+1}: [a, +\infty[\to R_+$ is correctly defined by the equalities

$$q_0(t) \equiv q(t)\,,$$
$$q_i(t) = \left(\int_t^{+\infty} q_{i-1}(\tau)\, d\tau\right)^{(1-\lambda_{n-i+2}-\ldots-\lambda_n)/(1-\lambda_{n-i+1}-\ldots-\lambda_n)}$$
$$(i = 1, \ldots, n-m+1)\,.$$

Then any NKS u of (0.1) *satisfies*

$$\left|u^{(m-1)}(t) - \lim_{t \to +\infty} u^{(m-1)}(t)\right| \leqq q_{n-m+1}(t)$$

for large t.

5. AN EXAMPLE. SOME REMARKS ON THE CASE WHEN $f(t, 0, \ldots, 0) \not\equiv 0$

Consider, as an example, the equation

$$(5.1) \qquad u^{(n)} = (-1)^n\, t^\sigma |u|^{\lambda_1}\, |u'|^{\lambda_2} \ldots |u^{(n-1)}|^{\lambda_n}$$

with

$$(5.2) \qquad \lambda_i \geqq 0 \; (i = 1, \ldots, n)\,, \quad 0 < \lambda = \sum_{i=1}^{n} \lambda_i < 1\,, \quad \sigma \in R\,.$$

The results obtained above imply

Theorem 5.1. *Let the conditions* (5.2) *hold. Then the equation* (5.1) *has the property S. The condition $\omega = n + \sigma - \lambda_2 - \ldots - (n-1)\lambda_n < 0$ is necessary and sufficient for a NKS of* (5.1) *to exist. Moreover, if it is fulfilled then any NKS u of* (0.1) *satisfies*

$$c_* t^{\omega/(1-\lambda)} \leqq u(t) - \lim_{t \to +\infty} u(t) \leqq c^* t^{\omega/(1-\lambda)}$$

for large t where c_* and c^* are positive numbers depending only on n, σ and λ_i $(i = 1, \ldots, n)$.

Note that under the condition $\omega < 0$ (5.1) has the exact solution $ct^{\omega/(1-\lambda)}$.

Finally, let us make some remarks on the condition $f(t, 0, \ldots, 0) \equiv 0$. If there is no neighbourhood of $+\infty$ where this condition holds then for every $\varepsilon > 0$ small enough

$$\operatorname{mes}\{\tau \geqq t : f_\varepsilon(\tau) > 0\} > 0 \quad \text{for} \quad t \geqq 0$$

where $f_\varepsilon(t) = \inf\{|f(t, x_1, \ldots, x_n)| : 0 \leqq (-1)^{i-1} x_i \leqq \varepsilon \ (i = 1, \ldots, n)\}$. It is easy to see that if

$$\int^{+\infty} t^{n-1} f_\varepsilon(t)\,\mathrm{d}t = +\infty$$

then the equation (0.1) has no KS u such that $\lim\limits_{t \to +\infty} u(t) \in [0, \varepsilon[$. If this integral converges and (3.12) holds on $D_n(a, \varepsilon, \varepsilon)$ with g satisfying (3.3) then for any $c \in [0, \varepsilon[$ (0.1) has a NKS u satisfying (3.4). The proof is quite analogous to that of Theorem 3.1. Indeed, the consequence of the property S which is crucial there – the existence of solutions v_k: $[a, t_k] \to R_+$ satisfying $t_k \uparrow +\infty$, $v_k^{(i)}(t_k) = 0$, $(-1)^i v_k^{(i)}(t) > 0$ for $a \leqq t < t_k$ $(i = 0, \ldots, n-1;\ k = 1, 2, \ldots)$ is trivially true for the equation $v^{(n)} = (-1)^n f_\varepsilon(t)$.

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Súhrn

O KNESEROVSKÝCH RIEŠENIACH SUBLINEÁRNYCH OBYČAJNÝCH DIFERENCIÁLNYCH ROVNÍC

GIORGI KVINIKADZE

Netriviálne riešenie u: $[a, +\infty[\rightarrow R$ obyčajnej diferenciálnej rovnice n-tého rádu sa nazýva kneserovským riešením (KR) ak $(-1)^i u^{(i)}(t) \geqq 0$ pre $t \geqq a$ $(i = 0, \ldots, n-1)$. KR sa nazýva degenerovaným (singulárnym), ak je konštantné (nulové) v nejakém okolí $+\infty$ a nedegenerovaným inak. V článku sa študuje trieda rovníc, ktoré majú dostatočný počet singulárnych KR. Pre rovnice z tejto triedy je dokázaná postačujúca podmienka pre existenciu nedegenerovaného KR s predpísanou limitou pre $t \rightarrow +\infty$. Sú odvodené dvojstranné apriorne odhady takých riešení.

Резюме

О КНЕЗЕРОВСКИХ РЕШЕНИЯХ СУБЛИНЕЙНЫХ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦАЬЛНЫХ УРАВНЕНИЙ

GIORGI KVINIKADZE

Нетривиальное решение u: $[a, +\infty[\rightarrow R$ обыкновенного дифференциального уравнения n-го порядка называется Кнезеровским решением (КР), если $(-1)^i u^{(i)}(t) \geqq 0$ при $t \geqq a$ $(i = 0, \ldots, n-1)$. КР называется вырожденным (сингулярным), если оно тождественно равно постоянной (нулю) в некоторой окрстности $+\infty$, и невырожденным в противном случае. В статье рассматривается один класс уравнений, имеющих достаточно много сингулярных КР. Для уравнения из этого класса установлено достаточное условие существования невырожденного КР с наперед заданным пределом при $t \rightarrow +\infty$. Приведены двусторонние априорные оценки таких решений.

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