Dagmar Medková

Invariance of the Fredholm radius of the Neumann operator

Časopis pro pěstování matematiky, Vol. 115 (1990), No. 2, 147--164

Persistent URL: `http://dml.cz/dmlcz/108370`

115 (1990) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 2, 147—164

INVARIANCE OF THE FREDHOLM RADIUS OF THE NEUMANN OPERATOR

DAGMAR MEDKOVÁ, Praha

(Received February 3, 1988)

Summary. One of the classical methods of solving the Dirichlet problem and the Neumann problem in R^m is the method of integral equations. If we wish to use the Fredholm-Radon theory to solve the problem, we need to know the Fredholm radius of the Neumann operator. It is shown in the paper that the Fredholm radius of the Neumann operator does not change under a deformation of the domain investigated by a diffeomorphism which is conformal (i.e. preserves angles) on a precisely specified part of the boundary.

Keywords: Neumann operator, interior normal in Federer's sense, reduced boundary, Fredholm radius, perimeter, Lipschitz mapping, diffeomorphism, Hausdorff measure.

AMS classification: 31B20.

The present paper follows the paper [Do], which proves that the Fredholm radius of the Neumann operator is invariant with respect to conformal deformations of the Jordan domain investigated. We have attempted to generalize this result, first to prove a similar theorem for dimensions of higher order.

If $H \subset R^m$ $(m \geqq 2)$ is an open set whith a compact boundary, we denote by $\mathscr{C}(\partial H)$ the space of all bounded continuous functions on ∂H and by $\mathscr{C}'(\partial H)$ the space of all finite signed measures on ∂H. For a given function h harmonic on H we define the weak normal derivative $N^H h$ as a distribution

$$\langle \varphi, N^H h \rangle = \int_H \operatorname{grad} \varphi \,.\, \operatorname{grad} h \, d\varkappa_m$$

for $\varphi \in \mathscr{D}$ (= the space of all compactly supported infinitely differentiable functions in R^m). We formulate the Neumann problem for the Laplace equation with a boundary condition $\mu \in \mathscr{C}'(\partial H)$ as follows: determine a harmonic function h on H for which $N^H h = \mu$. We wish to find the function h in the form of the single layer potential

$$\mathscr{U}\, \nu(x) = \int_{R^m} h_x(y) \, d\nu(y) \,,$$

where $\nu \in \mathscr{C}'(\partial H)$,

$$h_x(y) = (m-2)^{-1} A^{-1} |x-y|^{2-m} \quad \text{for} \quad m > 2 \,,$$
$$A^{-1} \log |x-y|^{-1} \quad \text{for} \quad m = 2 \,,$$

A is the area of the unit sphere in R^m. The operator $N^H \mathscr{U}$ is a bounded linear operator on $\mathscr{C}'(\partial H)$ if and only if $V^H < \infty$ (see the definition in § 2). Under the assumption $V^H < \infty$ we look for a solution of the Dirichlet problem for the Laplace equation on the set $R^m - \operatorname{cl} H$ with the boundary condition g (where $\operatorname{cl} H$ is the closure of the set H) in the form $u(x) = \langle f, N^H h_x \rangle$, where $f \in \mathscr{C}(\partial H)$. A solution f of the problem satisfies

$$Wf(x) \equiv \langle f, N^H h_x \rangle = g(x) .$$

Let ω be the distance of the Neumann operator $N = 2W - I$ (where I is the identity operator) from the space of all linear compact operators on $\mathscr{C}(\partial H)$. If ω ($=2V_0^H$ – see the definition in Section 2) is less then 1 then the Riesz-Schauder theory permits to apply the Fredholm theorems to the dual equations

$$[I + (2W - I)] f = 2g ,$$
$$[I + (2N^H \mathscr{U} - I)] \nu = 2\mu .$$

Notice that $1/\omega$ is the so-called Fredholm radius. As is proved in [KW], [AKK2], even in the case $V_0^H > 1/2$ we may in several cases utilize the Riesz-Schauder theory if we replace the maximum norm on $\mathscr{C}(\partial H)$ by the norm

$$\|f\|_v = \sup \{|f(x)|/v(x);\ x \in \partial H\} ,$$

where v is a positive lower semicontinuous function on ∂H, and then prove that $V_{0,v}^H < 1/2$ (see the definition in Section 2).

1. PERIMETER

Some auxiliary propositions concerning mappings of sets with a finite perimeter are proved in this part of the paper. It is possible to omit this section and to prove the propositions via the results of [DG1], [DG2].

We shall denote by $\mathscr{D}$ the class of all infinitely differentiable functions with compact support in R^m ($m \geqq 2$). Further, we denote by $\varkappa_k$ the outer k-dimensional Hausdorff measure, by $\mathrm{U}(y; r) = \{x; |x - y| < r\}$ the open ball of radius r and center y, by $A \div B = (A \cup B) - (A \cap B)$ the symmetric difference of sets A and B.

Definition 1. For any Borel set $H \subset R^m$ put

$$P(H) = \sup \{\textstyle\int_H \operatorname{div} w \, d\varkappa_m;\ w = (w_1, \ldots, w_m);\ w_j \in \mathscr{D}\ \forall j = 1, \ldots, m;$$
$$\sum_{j=1}^m w_i^2 = 1\} .$$

This quantity $P(H)$ is called the *perimeter* of H.

The aim of this section is to deduce the following result, which we shall use in the sequel.

Theorem 1. *Let $D \subset R^m$ be an open set, $\psi: D \to R^m$ a homeomorphism. Let H be a bounded Borel set whose closure cl H lies in D. If the mapping ψ is Lipschitz in a neighbourhood of ∂H with the Lipschitz constant L then*

$$P(\psi(H)) \leqq L^{m-1} P(H) .$$

Before proving this theorem we state several well-known auxiliary propositions.

Definition 2. Let $H \subset R^m$ be an open set. We call H *an open polyhedral set* if its boundary ∂H is locally a hypersurface (i.e. every point of ∂H has a neighbourhood in ∂H which is homeomorphic to R^{m-1}) and ∂H is formed by a finite number of $(m - 1)$- dimensional bounded polyhedrons.

Lemma 1. *Let H be a bounded open set. Then $P(H) \leqq \varkappa_{m-1}(\partial H)$. If H is an open polyhedral set then $P(H) = \varkappa_{m-1}(\partial H)$.*

Proof. See [0], Theorem 1.6, Theorem 2.5, Theorem 2.6, Theorem 1.3.

Lemma 2. *Let $H, H_1, H_2, \ldots$ be Borel sets such that $\varkappa_m(H \div H_k) \to 0$ for $k \to \infty$. Then*

$$\liminf_{k \to \infty} P(H_k) \geqq P(H) .$$

Proof. See [0], Theorem 1.5.

Lemma 3. *Let H be a nonempty Borel set with a finite perimeter. Then there exists a sequence $\{\Pi_k\}$ of open polyhedral sets such that $\partial \Pi_k \subset \{x;$ dist $(x, \partial H) < 1/k\}$ for each k, $\varkappa_m(H \div \Pi_k) \to 0$, $P(\Pi_k) \to P(H)$ for $k \to \infty$, where* dist $(x, \partial H) = \inf\{|x - y|;\ y \in \partial H\}$.

Proof. See [0], Theorem 1.7.

Proof of Theorem 1. It suffices to suppose that $P(H) < \infty$. According to Lemma 3 there exists a sequence $\{\Pi_k\}$ of open polyhedral sets such that $\partial \Pi_k \subset \{x;$ dist $(x, \partial H) < 1/k\}$ for each k, $P(\Pi_k) \to P(H)$ and $\varkappa_m(\Pi_k \div H) \to 0$ for $k \to \infty$. Further, there is k_0 such that ψ is a Lipschitz mapping on $\{x;$ dist $(x, \partial H) \leqq 1/k_0\} \subset D$ with a Lipschitz constant L. We prove that there is $k_1 > k_0$ such that $\Pi_k \subset \{x;$ dist $(x, \text{cl}\, H) < 1/k_0\}$ and $H - \Pi_k \subset \{x;$ dist $(x, \partial H) < 1/k_0\}$ for each $k \geqq k_1$. Denote $M = \{x;$ dist $(x, \text{cl}\, H) \leqq 1/(2k_0)\}$. Since cl H is compact, there is $R > 0$ such that cl $H \subset$ U$(0; R)$. Denote by $\{\Phi_j\}$ the components of U$(0; R + 2) - M$. Since U$(0; R + 1) - \{x;$ dist $(x, \text{cl}\, H) < 1/k_0\} \subset$ U$(0; R + 2) - M = \bigcup \{\Phi_j; j\}$, there exists a finite set $\Phi_1, \ldots, \Phi_n$ such that cl U$(0; R + 1) - \{x;$ dist $(x, \text{cl}\, H) < 1/k_0\} \subset \{\Phi_j; j = 1, \ldots, n\}$. Since $\varkappa_m(\Pi_k \div H) \to 0$, there is $k_1 > 2k_0$ such that

$$(1) \qquad \varkappa_m(\Pi_k \div H) < \min_{i=1,\ldots,n} \varkappa_m(\Phi_i)\,,$$

$$(2) \qquad \varkappa_m(\Pi_{k'} \div H) < (2k_0)^{-m}\, \varkappa_m(\mathrm{U}(0;1))$$

for each $k \geqq k_1$. If there were $k \geqq k_1$ such that $\Pi_k - \{x; \operatorname{dist}(x, \operatorname{cl} H) < 1/k_0\} \neq \emptyset$ then there would exist $y \in \Pi_k \cap (\operatorname{cl} \mathrm{U}(0; R+1) - \{x; \operatorname{dist}(x, \operatorname{cl} H) < 1/k_0\})$ because $\partial\Pi_k \subset \{x; \operatorname{dist}(x, \partial H) < 1/k\} \subset \mathrm{U}(0; R+1)$. Thus there would exist $j \in \{1, \ldots, n\}$ for which $\Pi_k \cap \Phi_j \neq \emptyset$. Since the set Φ_j is connected and $\partial\Pi_k \cap \Phi_j \subset M \cap \Phi_j = \emptyset$, we have $\Phi_j \subset \Pi_k$. Hence

$$\varkappa_m(\Pi_k \div H) \geqq \varkappa_m(\Phi_j)\,,$$

which contradicts (1).

Now suppose that there are $k \geqq k_1$ and $x \in H - \Pi_k$ such that $\operatorname{dist}(x, \partial H) \geqq 1/k_0$. If $y \in \partial\Pi_k$ then $\operatorname{dist}(y, \partial H) < 1/k_1$ and thus $\operatorname{dist}(x, y) > 1/(2k_0)$. Since $\operatorname{dist}(x, \partial\Pi_k) \geqq 1/(2k_0)$ we have $\mathrm{U}(x; 1/(2k_0)) \subset H - \Pi_k$. Hence

$$\varkappa_m(\Pi_k \div H) \geqq \varkappa_m(\mathrm{U}(x; 1/(2k_0)))\,,$$

which contradicts (2).

Let $k \geqq k_1$. Then $\psi(\Pi_k)$ is a Borel set and

$$\varkappa_{m-1}(\partial\psi(\Pi_k)) = \varkappa_{m-1}(\psi(\partial\Pi_k)) \leqq L^{m-1}\varkappa_{m-1}(\partial\Pi_k)\,,$$

because $|\psi(x) - \psi(y)| \leqq L|x - y|$ for each $x, y \in \partial\Pi_k$. According to Lemma 1

$$P(\psi(\Pi_k)) \leqq \varkappa_{m-1}(\partial\psi(\Pi_k)) \leqq L^{m-1}\varkappa_{m-1}(\partial\Pi_k) = L^{m-1}\, P(\Pi_k) < \infty\,.$$

Since $\Pi_k \subset \{x; \operatorname{dist}(x, \operatorname{cl} H) < 1/k_0\}$ we have $\Pi_k - H \subset \{x; \operatorname{dist}(x, \partial H) < 1/k_0\}$. Since $H - \Pi_k \subset \{x; \operatorname{dist}(x, \partial H) < 1/k_0\}$ we have $|\psi(x) - \psi(y)| \leqq L|x - y|$ for each $x, y \in \Pi_k \div H$. Hence

$$\varkappa_m(\psi(\Pi_k) \div \psi(H)) \leqq L^m\varkappa_m(\Pi_k \div H) \to 0 \quad \text{for} \quad k \to \infty\,.$$

According to Lemma 3

$$P(\psi(H)) \leqq \liminf_{k\to\infty} P(\psi(\Pi_k)) \leqq \liminf_{k\to\infty} L^{m-1}\, P(\Pi_k) = L^{m-1}\, P(H)\,.$$

2. INVARIANCE OF THE FREDHOLM RADIUS

Notation. Let $H \subset R^m$ be a Borel set with a compact boundary ∂H, let g be a lower semicontinuous function on ∂H such that $0 < \inf g \leqq \sup g < \infty$. For $x \in R^m$, $r > 0$ we denote

$$v^H_{r,g}(x) = \sup \{\textstyle\int_H \operatorname{grad} h_x(y) \,.\, \operatorname{grad} \varphi(y)\, \mathrm{d}\varkappa_m(y);\ \varphi \in \mathcal{D},\ |\varphi| \leqq \sup g\,,$$
$$|\varphi| \leqq g \text{ on } \partial H,\ \operatorname{spt} \varphi \subset \mathrm{U}(x; r) - \{x\}\}/g(x)\,,$$

where $g(x) = \sup g$ for $x \in R^m - \partial H$,

$$V^H_{r,g} = \sup \{v^H_{r,g}(x);\ x \in \partial H\}\ ,$$
$$V^H_{0,g} = \lim_{r \to 0+} V^H_{r,g}\ .$$

For $g \equiv 1$ we write $v^H_r(x)$, V^H_r, V^H_0. Instead of $v^H_\infty(x)$ and V^H_∞ we write $v^H(x)$ and V^H, respectively.

It is easily seen that $v^H_r(x)\, K^{-1} \leqq v^H_{r,g}(x) \leqq K\, v^H_r(x)$, where $K = \sup g/\inf g$ and thus $V^H_{\infty,g} < \infty \Leftrightarrow V^H < \infty$, $V^H_{0,g} = 0 \Leftrightarrow V^H_0 = 0$.

Lemma 4. *Let $H \subset R^m$ be a Borel set with a compact boundary. If $V^H_0 < \infty$ then $P(H) < \infty$.*

Proof. (See [K2] pp. 596–7 and the proof of Theorem 2.12 in [K1]). Since $V^H_0 = V^M_0$, $P(H) = P(M)$ for $M = R^m - H$ it suffices to suppose that H is a bounded set. Since $V^H_0 < \infty$ there is $r > 0$ for which $V^H_r < \infty$. Since $\mathrm{cl}\, H$ is compact, there are $x^1, \ldots, x^n \in \mathrm{cl}\, H$ such that $\mathrm{cl}\, H \subset \bigcup_{i=1}^{n} \mathrm{U}(x^i; r/2)$. Further, there exist $\alpha_1, \ldots, \alpha_n$ such that $0 \leqq \alpha_i \leqq 1$, $\mathrm{spt}\, \alpha_i \subset \mathrm{U}(x^i; r/2)$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$ in a neighbourhood of $\mathrm{cl}\, H$. Since

$$P(H) = \sup \{\sum_{j=1}^{m} \sum_{i=1}^{n} \int_H \partial_j(\alpha_i h_j)\, \mathrm{d}\varkappa_m;\ h_1, \ldots, h_m \in \mathscr{D},\ \sum_{j=1}^{m} h_j^2 \leqq 1\}$$

it suffices to prove

$$\sup \{\int_{H \cap \mathrm{U}(x^i; r/2)} \partial_j(\alpha_i \varphi)\, \mathrm{d}\varkappa_m;\ \varphi \in \mathscr{D},\ |\varphi| \leqq 1\} < \infty \quad \text{for} \quad i = 1, \ldots, n\ .$$

Let us fix i. If the points of $\mathrm{U}(x^i; r/2) \cap \partial H$ are situated in a single hyperplane then $P(H \cap \mathrm{U}(x^i; r/2)) < \infty$. Therefore

$$\sup \{\int_{H \cap \mathrm{U}(x^i; r/2)} \partial_j(\alpha_i \varphi)\, \mathrm{d}\varkappa_m;\ \varphi \in \mathscr{D},\ |\varphi| \leqq 1\} \leqq$$
$$\leqq P(H \cap \mathrm{U}(x^i; r/2)) < \infty\ .$$

Now suppose that there are points $y^1, \ldots, y^{m+1} \in \partial H \cap \mathrm{U}(x^i; r/2)$ which are not situated in a single hyperplane. Denote by L_k the hyperplane containing $\{y^s;\ s \neq k\}$. Then $\bigcup_{k=1}^{m+1} (R^m - L_k) = R^m$ and there exist $a_k \in \mathscr{D}$ such that $0 \leqq a_k \leqq 1$, $\mathrm{spt}\, a_k \cap L_k = \emptyset$ and $\sum_{k=1}^{m+1} a_k = 1$ in a neighbourhood of $\mathrm{cl}\, H$. Now $\varphi \in \mathscr{D}$ satisfies

$$\int_{H \cap \mathrm{U}(x^i; r/2)} \partial_j(\alpha_i \varphi)\, \mathrm{d}\varkappa_m = \sum_{k=1}^{m+1} \int_H \partial_j(\alpha_i \varphi a_k)\, \mathrm{d}\varkappa_m$$

and thus it suffices to prove

$$\sup \{\int_H \partial_j(\alpha_i a_k \varphi)\, \mathrm{d}\varkappa_m;\ \varphi \in \mathscr{D},\ |\varphi| \leqq 1\} < \infty \quad \text{for} \quad k = 1, \ldots, m + 1\ .$$

Let us fix k. The vectors $x - y^1, \ldots, x - y^{k-1}, x - y^{k+1}, \ldots, x - y^{m+1}$ are linearly independent for each $x \in R^m - L_k$. There are infinitely differentiable functions b_s

on $R^m - L_k$ such that

$$(\delta_{1j}, \delta_{2j}, \ldots, \delta_{mj}) = \sum_{\substack{s=1\\ s\neq k}}^{m+1} b_s(x) \frac{y^s - x}{|y^s - x|^m},$$

where

$$\delta_{rs} = \begin{matrix} 1 & \text{for} & r = s \\ 0 & \text{for} & r \neq s \end{matrix}$$

is the Kronecker delta. For $\varphi \in \mathcal{D}$, $|\varphi| \leqq 1$ we have

$$\int_H \partial_j(\alpha_i a_k \varphi)\, d\varkappa_m =$$

$$= \int_H \sum_{\substack{s=1\\ s\neq k}}^{m+1} b_s(x) \frac{y^s - x}{|y^s - x|^m} \,.\, \operatorname{grad}(\alpha_i(x)\, a_k(x)\, \varphi(x))\, d\varkappa_m(x) =$$

$$= \sum_{\substack{s=1\\ s\neq k}}^{m+1} \int_H \frac{y^s - x}{|y^s - x|^m} \,.\, \operatorname{grad}(\alpha_i(x)\, a_k(x)\, \varphi(x)\, b_s(x))\, d\varkappa_m(x) -$$

$$- \sum_{\substack{s=1\\ s\neq k}}^{m+1} \int_H \alpha_i(x)\, a_k(x)\, \varphi(x) \frac{y^s - x}{|y^s - x|^m} \,.\, \operatorname{grad} b_s(x)\, d\varkappa_m(x) \leqq$$

$$\leqq \sum_{\substack{s=1\\ s\neq k}}^{m+1} A \sup_{x \in \operatorname{spt} a_k} |b_s(x)|\, v_r^H(y^s) +$$

$$+ \sum_{\substack{s=1\\ s\neq k}}^{m+1} \int_H |\alpha_i(x)\, a_k(x)|\, |y^s - x|^{1-m}\, |\operatorname{grad} b_s(x)|\, d\varkappa_m(x),$$

because $\operatorname{grad} h_y(x) = (y - x)/(A|y - x|^m)$. Thus

$$\sup\left\{\int_H \partial_j(\alpha_i a_k \varphi)\, d\varkappa_m;\ \varphi \in \mathcal{D},\ |\varphi| \leqq 1\right\} \leqq \sum_{\substack{s=1\\ s\neq k}}^{m+1} A \sup_{x \in \operatorname{spt} a_k} |b_s(x)|\, V_r^H +$$

$$+ \sum_{\substack{s=1\\ s\neq k}}^{m+1} \int_H |\alpha_i(x)\, a_k(x)|\, |x - y^s|^{1-m}\, |\operatorname{grad} b_s(x)|\, d\varkappa_m(x) < \infty.$$

The set H has a finite perimeter.

Lemma 5. *Let $H \subset R^m$ be a Borel set with a compact boundary ∂H. Then for every $z \in R^m - \partial H$ we have*

$$v_r^H(z) \leqq A^{-1} P(H)\, [\operatorname{dist}(z, \partial H)]^{1-m}.$$

Proof. See [K1], Proposition 2.11, Remark 2.3.

Lemma 6. *Let $H \subset R^m$ be a Borel set with a compact boundary. Then $V_0^H < \infty$ if and only if $V^H < \infty$.*

Proof. It suffices to prove that $V_0^H < \infty$ implies $V^H < \infty$. If $V_0^H < \infty$ then there is $r > 0$ for which $V_r^H < \infty$. Let $x \in \partial H$. Then there are infinitely differentiable functions α_1, α_2 such that $\alpha_1 + \alpha_2 = 1$ in R^m, $0 \leqq \alpha_i \leqq 1$ for $i = 1, 2$, $\operatorname{spt} \alpha_1 \subset$ $\subset \mathrm{U}(x; r)$, $\operatorname{spt} \alpha_2 \subset R^m - \operatorname{cl} \mathrm{U}(x; r/2)$. If $\varphi \in \mathscr{D}$, $|\varphi| \leqq 1$, $\operatorname{spt} \varphi \subset R^m - \{x\}$, then

$$\int_H \operatorname{grad} \varphi(y) \, . \operatorname{grad} h_x(y) \, \mathrm{d}\varkappa_m(y) =$$
$$= \int_H \operatorname{grad} (\alpha_1(y) \varphi(y)) \, . \operatorname{grad} h_x(y) \, \mathrm{d}\varkappa_m(y) +$$
$$+ \int_H \operatorname{grad} (\alpha_2(y) \varphi(y)) \, . \operatorname{grad} h_x(y) \, \mathrm{d}\varkappa_m(y) \leqq v_r^H(x) + v^{H - \mathrm{U}(x;r/2)}(x) \, .$$

By Lemma 5 we have

$$v^H(x) \leqq v_r^H(x) + v^{H - \mathrm{U}(x;r/2)}(x) \leqq$$
$$\leqq v_r^H(x) + A^{-1} P(H - \mathrm{U}(x; r/2)) \, (r/2)^{1-m} \, .$$

Thus by Lemma 4 we conclude

$$V^H \leqq V_r^H + A^{-1} 2^{m-1} r^{1-m} [P(H) + P(\mathrm{U}(0; r/2))] < \infty \, .$$

Notation. For a Borel set $H \subset R^m$, $z \in R^m$ we denote by

$$d_H(z) = \lim_{r \to 0+} \frac{\varkappa_m(\mathrm{U}(z; r) \cap H)}{\varkappa_m(\mathrm{U}(z; r))}$$

the m-dimensional density of H at the point z (if it exists).

Definition 3. Let $H \subset R^m$ be a Borel set, $y \in R^m$. A unit vector Θ is termed the *interior normal* of H at y in *Federer's sense*, if the symmetric difference of H and the half-space $\{x \in R^m; \, (x - y) \cdot \Theta > 0\}$ has m-dimensional density zero at y. If there is such a vector Θ, then it is unique and we shall denote it by $n^H(y)$; if there is no interior normal of H at y in this sense, we denote by $n^H(y)$ the zero vector in R^m. The set $\{y \in R^m; \, |n^H(y)| > 0\}$ is called the *reduced boundary* of H and will be denoted by $\partial_r H$.

Notation. Let $D \subset R^m$ be an open set, $\psi: D \to R^k$ a mapping. If the differential of ψ at $z \in D$ exists, we denote it by $\mathrm{D}\psi(z)$.

In the rest of the paper we will consider an open set $D \subset R^m$, a homeomorphism $\psi: D \to R^m$ and a bounded Borel set H such that $\operatorname{cl} H \subset D$ and the mapping ψ is a diffeomorphism of class C^1 in a neighbourhood of ∂H. Further, we will suppose that g is a lower-semicontinuous function on ∂H such that $0 < \inf g \leqq \sup g < \infty$.

Lemma 7. *$\partial_r \psi(H) = \psi(\partial_r H)$, and $n^{\psi(H)}(\psi(x))$ is normal vector to the surface $\psi(\{z \in D; \, (z - x) \cdot n^H(x) = 0\}$ at $\psi(x)$ for each $x \in \partial_r H$.*

Proof. Let $x \in \partial_r H$. Since ψ belongs to the class C^1 in a neighbourhood of x, the surface $\psi(\{z \in D; \, (z - x) \cdot n^H(x) = 0\})$ has a unit normal n at $\psi(x)$. We choose the normal n oriented to the set $\psi(\{z \in D; \, (z - x) \cdot n^H(x) > 0\})$. We prove that $n^{\psi(H)}(\psi(x)) = n$.

Denote $Y = \{z \in D; (z - x) \cdot n^H(x) < 0\}$, $Z = \{z \in R^m; (z - \psi(x)) \cdot n < 0\}$. Then

(4) $$d_{(Z \div \psi(Y))}(\psi(x)) = 0 .$$

Now we prove

(5) $$d_{(\psi(H) \div \psi(Y))}(\psi(x)) = 0 .$$

There are positive constants $L \geqq 1$, ϱ such that ψ is a Lipschitz mapping on $U(x; \varrho)$ with the constant L and ψ^{-1} is a Lipschitz mapping on $U(\psi(x); \varrho)$ with the constant L. If $0 < r < \varrho$ then $\psi^{-1}(U(\psi(x); r)) \subset U(x; rL)$ and thus

$$\varkappa_m(U(\psi(x); r) \cap (\psi(H) \div \psi(Y))/\varkappa_m(U(\psi(x); r)) \leqq$$
$$\leqq \varkappa_m(\psi(U(x; rL) \cap (H \div Y)))/\varkappa_m(U(x; r)) .$$

Therefore

$$d_{\psi(H) \div \psi(Y)}(\psi(x)) \leqq L^m d_{H \div Y}(x) = 0$$

and thus

$$d_{Z \div \psi(H)}(\psi(x)) \leqq d_{Z \div \psi(Y)}(\psi(x)) + d_{\psi(Y) \div \psi(H)}(\psi(x)) = 0$$

and $n^{\psi(H)}(\psi(x)) = n$, $\psi(x) \in \partial_r \psi(H)$. Hence $\psi(\partial_r H) \subset \partial_r \psi(H)$. Similarly, $\psi^{-1}(\partial_r \psi(H)) \subset \partial_r H$ and thus $\partial_r \psi(H) = \psi(\partial_r H)$.

Lemma 8. *If $y \in \partial_r H$, $u \in R^m$ then*

$$n^{\psi(H)}(\psi(y)) \cdot D\psi(y)\, u = (u \cdot n^H(y)) (n^{\psi(H)}(\psi(y)) \cdot D\psi(y)\, n^H(y)) .$$

Proof. By virtue of the linearity of the operator $D\psi(y)$ it suffices to suppose $u \neq (u \cdot n^H(y))\, n^H(y)$. Since ψ belongs to the class C^1 in a neighbourhood of y, there are positive constants L, r such that $|\psi(x) - \psi(y)| \leqq L|x - y|$ for each $x \in U(y; r)$. Since $n^H(y) [u - (u \cdot n^H(y))\, n^H(y)] = 0$, $n^{\psi(H)}(\psi(y))$ is a normal vector of the surface $\psi(\{z \in D; (z - y) \cdot n^H(y) = 0\})$ at $\psi(y)$ by Lemma 7, and

$$0 = \lim_{t \to 0+} n^{\psi(H)}(\psi(y)) \cdot \frac{\psi(y + t[u - (u \cdot n^H(y))\, n^H(y)]) - \psi(y)}{|\psi(y + t[u - (u \cdot n^H(y))\, n^H(y)]) - \psi(y)|} =$$
$$= \lim_{t \to 0+} n^{\psi(H)}(\psi(y)) \cdot \frac{\psi(y + t[u - (u \cdot n^H(y))\, n^H(y)]) - \psi(y)}{Lt|u - (u \cdot n^H(y))\, n^H(y)|} =$$
$$= \frac{n^{\psi(H)}(\psi(y)) \cdot D\psi(y) [u - (u \cdot n^H(y))\, n^H(y)]}{L|u - (u \cdot n^H(y))\, n^H(y)|} .$$

The linearity of $D\psi(y)$ implies

$$n^{\psi(H)}(\psi(y)) \cdot D\psi(y)\, u = n^{\psi(H)}(\psi(y)) \cdot D\psi(y) [(u \cdot n^H(y))\, n^H(y)] .$$

Lemma 9. *Let $x \in R^m$, let $B \subset R^m$ be a Borel set, C a positive constant such that $\varkappa_{m-1}(B \cap U(x; r)) \leqq Cr^{m-1}$ for each positive r. Then for every $\alpha > 0$, $r \in (0, 1)$*

we have

$$\int_{B\cap U(x;r)} |x-y|^{\alpha+1-m}\,d\varkappa_{m-1}(y) \leqq \frac{C\,2^{m-1}}{1-2^{-\alpha}}\,r^{\alpha}.$$

An elementary Proof. Since $r < 1$ there is a natural number i such that $2^{-i} < r \leqq$ $\leqq 2^{-i+1}$. Calculation yields

$$\int_{B\cap U(x;r)} |x-y|^{\alpha+1-m}\,d\varkappa_{m-1}(y) \leqq$$
$$\leqq \sum_{j=1}^{\infty} (2^{-j})^{\alpha+1-m}\,\varkappa_{m-1}(B\cap(U(x;2^{-j+1}) - U(x;2^{-j}))) \leqq$$
$$\leqq C\sum_{j=1}^{\infty}(2^{-j})^{\alpha+1-m}(2^{-j+1})^{m-1} \leqq \frac{C\,2^{m-1}}{1-2^{-\alpha}}\,r^{\alpha}.$$

Lemma 10. *Let $B \subset R^m$ be a Borel set with a compact boundary, $z \in R^m$, $r > 0$. Then*

$$\varkappa_{m-1}(U(z;r)\cap\partial_r B) \leqq Am(m+2)^m(\tfrac{1}{2}+V^B)\,r^{m-1}.$$

Proof. See [K1], Corollary 2.17 and Remark 2.3.

In the rest of the paper we will suppose that ψ is a diffeomorphism of class $C^{1+\alpha}$ in a neighbourhood of ∂H, where $\alpha \in (0, 1\rangle$.

Lemma 11. *Let $V^H < \infty$. Then there are positive numbers r_0, C_1, C_2, L such that for every $x \in D$, $r \in (0, r_0)$*

$$v^{\psi(H)}_{r,g\psi^{-1}}(\psi(x)) \leqq C_1\,v^H_{Lr,g}(x) + C_2 r^{\alpha}.$$

Proof. There are $r_1 \in (0,1)$ and a positive constant L such that for every $x, y \in$ $\in \{z;\operatorname{dist}(z,\partial H) \leqq r_1\}$ we have $|\psi(x)-\psi(y)| \leqq L|x-y|$, $\|D\psi(x)\| \leqq L$, $\|D\psi(x) -$ $- D\psi(y)\| \leqq L|x-y|^{\alpha}$ and for every $x, y \in \{z;\operatorname{dist}(z,\psi(\partial H)) \leqq r_1\}$ we have $|\psi^{-1}(x)-\psi^{-1}(y)| \leqq L|x-y|$, where $\|D\psi(x)\|$ denotes the norm of the operator $D\psi(x)$. Since $V^H < \infty$, we have $P(H) < \infty$ by virtue of Lemma 4. Theorem 1 implies $P(\psi(H)) < \infty$. According to [K1], Lemma 2.15, Proposition 2.5, Definition 2.2, Remark 2.3 we have for every $x \in \partial H$, $z \in \partial\psi(H)$ and $r > 0$

$$v^H_{r,g}(x) = \int_{\partial_r H\cap U(x;r)} |n^H(y)\cdot \operatorname{grad} h_x(y)|\,g(y)\,d\varkappa_{m-1}(y)/g(x),$$
$$v^{\psi(H)}_{r,g,\psi^{-1}}(z) = \int_{\partial_r\psi(H)\cap U(z;r)} |n^{\psi(H)}(y)\cdot\operatorname{grad} h_z(y)|\,g(\psi^{-1}(y))\,.$$
$$.\,d\varkappa_{m-1}(y)/g(\psi^{-1}(z)).$$

Now let $y \in \partial_r H$, $x \in D - \{y\}$, $|x-y| < r_1/L$. We wish to estimate $|\operatorname{grad} h_{\psi(x)}(\psi(y))\,.$ $.\,n^{\psi(H)}(\psi(y))|$ in terms of $|\operatorname{grad} h_x(y)\cdot n^H(y)|$:

$$|\operatorname{grad} h_{\psi(x)}(\psi(y))\cdot n^{\psi(H)}(\psi(y))| =$$
$$= A^{-1}\left|\frac{\psi(x)-\psi(y)}{|\psi(x)-\psi(y)|^m}\cdot n^{\psi(H)}(\psi(y))\right| \leqq$$

$$\leqq L^m A^{-1}|x-y|^{-m}\left|n^{\psi(H)}(\psi(y))\int_0^1 D\psi(y+t(x-y))(x-y)\,dt\right| \leqq$$
$$\leqq L^m A^{-1}|x-y|^{-m}\left|n^{\psi(H)}(\psi(y))\,D\psi(y)(x-y)\right| +$$
$$+ L^m A^{-1}|x-y|^{-m}\int_0^1 \left|D\psi(y+t(x-y))(x-y) - D\psi(y)(x-y)\right| dt.$$

Lemma 8 yields

$$\left|\operatorname{grad} h_{\psi(x)}(\psi(y)) \cdot n^{\psi(H)}(\psi(y))\right| \leqq$$
$$\leqq L^m A^{-1}|x-y|^{-m}\left|n^{\psi(H)}(\psi(y)) \cdot D\psi(y)\,n^H(y)\right| \left|n^H(y)(x-y)\right| +$$
$$+ L^m A^{-1}|x-y|^{-m}\int_0^1 L|x-y|\,|t(x-y)|^\alpha\,dt \leqq$$
$$\leqq L^{m+1}\left|\operatorname{grad} h_x(y)\cdot n^H(y)\right| + L^{m+1}A^{-1}|x-y|^{\alpha+1-m}.$$

Thus if $x \in D$, $y \in \partial_r H - \{x\}$, $|x-y| < r_1/L$ then

(6) $$\left|\operatorname{grad} h_{\psi(x)}(\psi(y))\, n^{\psi(H)}(\psi(y))\right| \leqq$$
$$\leqq L^{m+1}\left|\operatorname{grad} h_x(y)\cdot n^H(y)\right| + L^{m+1}A^{-1}|x-y|^{\alpha+1-m}.$$

Put $r_0 = r_1/L^2$. Let $x \in D$, $r \in (0, r_0)$. From Lemma 7 and (6) we obtain

$$v^{\psi(H)}_{r,g\cdot\psi^{-1}}(\psi(x)) =$$
$$= \int_{\partial_r\psi(H)\cap U(\psi(x);r)} \left|n^{\psi(H)}(z)\cdot\operatorname{grad} h_{\psi(x)}(z)\right| g(\psi^{-1}(z))\,d\varkappa_{m-1}(z)/g(x) \leqq$$
$$\leqq \int_{\psi(\partial_r H\cap U(x;rL))} \left[L^{m+1}\left|n^H(\psi^{-1}(z))\cdot\operatorname{grad} h_x(\psi^{-1}(z))\right| +\right.$$
$$\left.+ L^{m+1}A^{-1}|x-\psi^{-1}(z)|^{1+\alpha-m}\right] g(\psi^{-1}(z))\,d\varkappa_{m-1}(z)/g(x) \leqq$$
$$\leqq L^{m-1}\int_{\partial_r H\cap U(x;rL)} \left[L^{m+1}\left|n^H(y)\cdot\operatorname{grad} h_x(y)\right| +\right.$$
$$\left. L^{m+1}A^{-1}|x-y|^{1+\alpha-m}\right] g(y)\,d\varkappa_{m-1}(y)/g(x) \leqq L^{2m} v^H_{Lr,g}(x) +$$
$$+ L^{2m+\alpha} m(m+2)^m (1/2 + V^H)\,2^{m-1}(1-2^{-\alpha})^{-1}\, r^\alpha/\inf g$$

by virtue of Lemma 10 and Lemma 9.

Theorem 2. $V^{\psi(H)} < \infty \Leftrightarrow V^H < \infty$.

Proof. Let $V^H < \infty$. We may assume that $\alpha \leqq 1$. Lemma 11 yields

$$V_0^{\psi(H)} \leqq C_1 V_0^H < \infty.$$

Thus $V^{\psi(H)} < \infty$ by Lemma 6. Since ψ^{-1} has the same character as the function ψ, $V^{\psi(H)} < \infty$ implies $V^H < \infty$.

Theorem 3. *If* $V_0^H = 0$ *then* $V_0^{\psi(H)} = 0$.

Proof. Since $V_0^H = 0$ we have $V^H < \infty$ by Lemma 6. Lemma 11 implies $0 \leqq \leqq V_0^{\psi(H)} \leqq C_1 V_0^H = 0$.

Lemma 12. *Let $B \subset R^m$ be a Borel set with a compact boundary. Then for every $z \in R^m$*

$$v^B(z) \leqq V^B + 1/2 .$$

Proof. See [K1], Theorem 2.16, Remark 2.3.

Lemma 13. *Let $B \subset R^m$ be a Borel set. If $z^1, \ldots, z^{m+1} \in R^m$ are not situated in a single hyperplane and*

$$\sum_{j=1}^{m+1} v^B(z^j) < \infty ,$$

then $P(B) < \infty$.

Proof. See [K1], Theorem 2.12, Remark 2.3.

Notation. Let $B \subset R^m$ be a Borel set with a compact boundary. We denote

$$\tau_B = \{z \in \partial B;\ \exists \varrho > 0\colon \lim_{r \to 0+} \{\sup v_r^B(y);\ y \in \mathrm{U}(z; \varrho) \cap \partial B\} = 0\} .$$

Lemma 14. $\tau_{\psi(H)} = \psi(\tau_H)$.

Proof. We may suppose $\alpha \leqq 1$. Let $z \in \tau_H$. Then there is $\varrho > 0$ such that

$$K = \sup \{v_{5\varrho}^H(y);\ y \in \mathrm{U}(z; 5\varrho) \cap \partial H\} < \infty , \tag{7}$$

$$\lim_{r \to 0+} \sup \{v_r^H(y);\ y \in \mathrm{U}(z; 5\varrho) \cap \partial H\} = 0 . \tag{8}$$

We prove $V^{H \cap \mathrm{U}(z;\varrho)} < \infty$. If $\partial H \cap \mathrm{U}(z; \varrho)$ lies in a single hyperplane then $V^{H \cap \mathrm{U}(z;\varrho)} < \infty$. Suppose that $\partial H \cap \mathrm{U}(z; \varrho)$ does not lie in a single hyperplane. If $x \in \partial H \cap \mathrm{cl}\, \mathrm{U}(z; 3\varrho)$ then

$$v^{H \cap \mathrm{U}(z;\varrho)}(x) \leqq v_{5\varrho}^H(x) + v^{\mathrm{U}(z;\varrho)}(x) \leqq K + 1 , \tag{9}$$

$$v^{\mathrm{U}(z;\varrho) - H}(x) \leqq K + 1 . \tag{10}$$

Since $\partial H \cap \mathrm{U}(z; \varrho)$ does not lie in a single hyperplane, the set $H \cap \mathrm{U}(z; \varrho)$ has a finite perimeter. If $x \in \partial \mathrm{U}(z; \varrho) \cap \mathrm{int}\, H$ then

$$v^{H \cap \mathrm{U}(z;\varrho)}(x) \leqq 1 + v^{\mathrm{U}(z;\varrho) - H}(x) .$$

Let $\varphi \in \mathscr{D}$, $|\varphi| \leqq 1$. Then the function

$$y \mapsto \int_{\mathrm{U}(z;\varrho) - H} \mathrm{grad}\, \varphi \,.\, \mathrm{grad}\, h_y \,\mathrm{d}\varkappa_m$$

is continuous in R^m and harmonic in $\mathrm{U}(z; 3\varrho) \cap \mathrm{int}\, H$. If $y \in \partial \mathrm{U}(z; 3\varrho)$ then Lemma 5 yields

$$\left|\int_{\mathrm{U}(z;\varrho) - H} \mathrm{grad}\, \varphi \,.\, \mathrm{grad}\, h_y \,\mathrm{d}\varkappa_m\right| \leqq v^{U(z;\varrho) - H}(y) \leqq$$
$$\leqq A^{-1}[P(\mathrm{U}(z; \varrho)) + P(H \cap \mathrm{U}(z; \varrho))]\, (2\varrho)^{1-m} .$$

If $y \in \partial H \cap U(z; 3\varrho)$ then [K1], Proposition 2.5, Remark 2.3 and (10) imply

$$\left|\int_{U(z;\varrho)-H} \operatorname{grad} \varphi \, . \operatorname{grad} h_y \, d\varkappa_m\right| \leqq 1 + v^{U(z;\varrho)-H}(y) \leqq K + 2 .$$

Since $x \in \operatorname{int} H \cap U(z, 3\varrho)$, we have

$$\left|\int_{U(z;\varrho)-H} \operatorname{grad} \varphi \, . \operatorname{grad} h_x \, d\varkappa_m\right| \leqq$$
$$\leqq \max (2 + K, A^{-1}[P(U(z; \varrho)) + P(H \cap U(z; \varrho))] (2\varrho)^{1-m})$$

due to the maximum principle. Hence

$$v^{H \cap U(z;\varrho)}(x) \leqq 1 + v_{3\varrho}^{U(z;\varrho)-H}(x) \leqq$$
$$\leqq 1 + \max (2 + K, A^{-1}[P(U(z; \varrho)) + P(H \cap U(z; \varrho)) (2\varrho)^{1-m}) .$$

This inequality and the relation (9) yield

$$V^{H \cap U(z;\varrho)} \leqq 1 + \max (2 + K, A^{-1}[P(U(z; \varrho)) +$$
$$+ P(H \cap U(z; \varrho))] (2\varrho)^{1-m}) < \infty .$$

According to Lemma 11 there are numbers r_0, C_1, C_2, L such that for every $x \in \partial H$, $r \in (0, r_0)$ we have

$$v_r^{\psi(H \cap U(z;\varrho))}(\psi(x)) \leqq C_1 v_{Lr}^{H \cap U(z;\varrho)}(x) + C_2 r^{\alpha} .$$

Since ψ is a homeomorphism, there is $R \in (0, \min(r_0, \varrho/(2L)))$ such that $U(\psi(z); 2R) \subset \subset \psi(U(z; \varrho/2))$. If $y \in U(\psi(x); R) \cap \psi(\partial H)$ then there is $x \in \partial H \cap U(z; \varrho/2)$ such that $y = \psi(x)$. If $r \in (0, R)$ then

$$v_r^{\psi(H)}(y) = v_r^{\psi(H \cap U(z;\varrho))}(\psi(x)) \leqq C_1 v_{Lr}^H(x) + C_2 r^{\alpha} .$$

Therefore

$$\limsup_{r \to 0+} \{v_r^{\psi(H)}(y); \; y \in \partial\psi(H) \cap U(\psi(z); R)\} \leqq$$
$$\leqq \lim_{r \to 0+} [C_1 \sup \{v_{Lr}^H(x); \; x \in \partial H \cap U(z; \varrho)\} + C_2 r^{\alpha}] = 0 .$$

Thus $\psi(z) \in \tau_{\psi(H)}$. Therefore $\psi(\tau_H) \subset \tau_{\psi(H)}$. Since ψ^{-1} has the same character as ψ, we have $\psi^{-1}(\tau_{\psi(H)}) \subset \tau_H$. Hence $\psi(\tau_H) = \tau_{\psi(H)}$.

Lemma 15. *Let $B \subset R^m$ be a Borel set with a compact boundary. Then*

$$V_0^B = 0 \Leftrightarrow \tau_B = \partial B .$$

Proof. If $V_0^B = 0$ then

$$\limsup_{r \to 0+} \{v_r^B(y); \; y \in \partial B\} = 0$$

and thus $\tau_B = \partial B$. Now let $\tau_B = \partial B$. Let $\varepsilon > 0$. Then for every $z \in \partial B$ there are positive numbers $\varrho(z), r(z)$ such that $v_{r(z)}^B(y) < \varepsilon$ for each $y \in \partial B \cap U(z; \varrho(z))$.

Since ∂B is compact there exists a finite set $\{z^1, \ldots, z^n\}$ of points of ∂B such that for every $y \in \partial B$ there is $i \in \{1, \ldots, n\}$ for which $|z^i - y| < \varrho(z^i)$. If $r \in (0, \min\{r(z^i); i = 1, \ldots, n\})$ then $v_r^B(y) < \varepsilon$ for each $y \in \partial B$. Hence $V_0^B = 0$.

Definition 4. Let $B \subset R^m$, $z \in B$. A mapping $\varphi: B \to R^m$ is called *conformal at the point* z if there is $\delta > 0$ such that $\mathrm{U}(z; \delta) \subset B$, $\varphi \in \mathscr{C}^1(\mathrm{U}(z; \delta))$ and the angle of the curves $\{\varphi(z + t\Theta_j); 0 \leqq t < \delta\}$ $(j = 1, 2)$ at the point $\varphi(z)$ is the same as the angle of the curves $\{z + t\Theta_j; 0 \leqq t < \delta\}$ $(j = 1, 2)$ at the point z for all pairs of unit vectors Θ_j, Θ_2.

Lemma 16. *Let $B \subset R^m$, let $\varphi: B \to R^m$ be an injective mapping which is conformal at the point $z \in B$. Then φ^{-1} is conformal at the point $\varphi(z)$.*

Proof. φ is conformal at the point z if and only if there is $\delta > 0$ such that $\mathrm{U}(z, \delta) \subset B$, $\varphi \in \mathscr{C}^1(\mathrm{U}(z; \delta))$ and for every two vectors $u, v \neq 0$ we have $\mathrm{D}\varphi(z)\, u \neq 0$, $\mathrm{D}\varphi(z)\, v \neq 0$ and

$$\frac{\mathrm{D}\varphi(z)\, u}{|\mathrm{D}\varphi(z)\, u|} \cdot \frac{\mathrm{D}\varphi(z)\, v}{|\mathrm{D}\varphi(z)\, v|} = \frac{u}{|u|} \cdot \frac{v}{|v|}. \tag{11}$$

Thus $\mathrm{J}\varphi(z) \neq 0$, where $\mathrm{J}\varphi(z)$ is the Jacobian of the mapping φ at the point z. Since the mapping $f(x, y) = y - \varphi(x)$ is a mapping of class C^1 in $\mathrm{U}(z; \delta) \times R^m$ and $f(x, y) = 0$ if and only if $y \in \varphi(B)$, $x = \varphi^{-1}(y)$, the implicit function theorem implies that φ^{-1} is a mapping of class C^1 in a neighbourhood of $\varphi(z)$. The mapping $\mathrm{D}\varphi^{-1}(\varphi(z))$ is the inverse mapping to the mapping $\mathrm{D}\varphi(z)$. Thus we obtain from (11) a similar relation for $\mathrm{D}\varphi^{-1}(\varphi(z))$. Hence the mapping φ^{-1} is conformal at the point $\varphi(z)$.

Lemma 17. Let K be a positive constant. For a positive r put $B_r = \{x \in \partial H; v_{r,g}^H(x) > K\}$, $\varrho(r) = \sup\{\operatorname{dist}(x, \partial H - \tau_H); x \in B_r\}$ when both sets B_r, $\partial H - \tau_H$ are nonempty. In the opposite case put $\varrho(r) = 0$. Then $\varrho(r) \searrow 0$ for $r \to 0_+$.

Proof. The function $\varrho(r)$ is nondecreasing on the interval $(0, \infty)$. If the limit of $\varrho(r)$ at the origin were different from zero, there would exist a positive ε such that $\varrho(r) > \varepsilon$ for each r. Thus there would exist a sequence $x_n \in B_{1/n}$ such that $\operatorname{dist}(x_n, \partial H - \tau_H) > \varepsilon$. Since ∂H is compact there exists a subsequence $\{x_n'\}$ of the sequence $\{x_n\}$ and a point $x \in \partial H$ such that $x_n' \to x$ for $n \to \infty$. Now let $\delta > 0$. For every $r > 0$ there is a natural number n such that $x_n' \in \mathrm{U}(x; \delta) \cap \partial H$ and $1/n < r$. Then

$$\sup\{v_r^H(y); y \in \mathrm{U}(x; \delta) \cap \partial H\} \geqq v_r^H(x_n') \geqq v_{r,g}^H(x_n')(\min g)/(\sup g) \geqq$$
$$\geqq K(\inf g)/(\sup g).$$

Thus for every $\delta > 0$

$$\limsup_{r\to 0+} \{v_r^H(y);\ y \in U(x;\delta) \cap \partial H\} \geqq K(\inf g)/(\sup g) > 0$$

and hence $x \in \partial H - \tau_H$. Therefore $\operatorname{dist}(x_n', \partial H - \tau_H) \to 0$ for $n \to \infty$, a contradiction.

Note 1. Let $C \subset R^m$ be an open set, let $\varphi\colon C \to R^m$ be a diffeomorphism of class C^1, let B be a compact subset of C. Then there is a constant K such that $\|D\varphi^{-1}(x)\| \leqq K$ for each $x \in \varphi(B)$, because $\varphi(B)$ is compact. Since

$$D\varphi^{-1}(\varphi(x))\, D\varphi(x) = I\,,$$

where I is the identity operator, we have $|u| \leqq K|D\varphi(x)\,u|$ for each vector $u \in R^m$. Thus $\|D\varphi(x)\| \geqq 1/K$ for each $x \in B$.

Lemma 18. *Let K, L be positive constants. For r positive put $B_r = \{x \in \partial H; v_{r,g}^H(x) > K\}$. Let $\varrho(r)$ be the function from Lemma 17, $\varphi(r) = [r + \varrho(Lr)]^\alpha$ for $r > 0$. Then there are positive constants r_0, C such that for every $r \in (0, r_0)$, $x \in B_{Lr}$ there exists $z \in (\partial H - \tau_H) \cap U(x; \varrho(Lr))$ such that for every $y \in \partial_r H$, $0 < |x - y| < r$ we have*

$$\begin{aligned}
&|\operatorname{grad} h_{\psi(x)}(\psi(y)) \cdot n^{\psi(H)}(\psi(y))| \leqq \\
&\leqq |\operatorname{grad} h_x(y) \cdot n^H(y)| \left[\|D\psi(z)\| \left|D\psi(z)\,\frac{x-y}{|x-y|}\right|^{-m} (1 + C\,\varphi(r))^m + \right.\\
&\left. + C\,\varphi(r)\right] + C|x-y|^{1+\alpha-m}\,.
\end{aligned}$$

Proof. By the assumption there are positive constants r_1, M such that ψ is a diffeomorphism of class $C^{1+\alpha}$ on $\{x; \operatorname{dist}(x, \partial H) < 2r_1\} \subset D$, ψ^{-1} is a diffeomorphism of class $C^{1+\alpha}$ on $\{x; \operatorname{dist}(x, \psi(\partial H)) < 2r_1\} \subset \psi(D)$, for every $x, y \in \{z; \operatorname{dist}(z, \psi(\partial H)) \leqq r_1\}$ we have $|\psi^{-1}(x) - \psi^{-1}(y)| \leqq M|x - y|$, for every $x, y \in \{z; \operatorname{dist}(z, \partial H) \leqq r_1\}$ we have $|D\psi(x) - D\psi(y)| \leqq M|x - y|$. First, let $\partial H = \tau_H$. Then $V_0^H = 0$ due to Lemma 15 and thus $V_{0,g}^H = 0$. There is $R > 0$ for which $\sup\{v_{LR,g}^H(x); x \in \partial H\} < K$ and thus $B_{Lr} = \emptyset$ for $r \in (0, R)$. Now let us assume $\partial H - \tau_H \neq \emptyset$. Let $r \in (0, r_1)$, $x \in B_{Lr}$. Since $\partial H - \tau_H$ is compact there exists $z \in \partial H - \tau_H$ for which $|x - z| \leqq \varrho(Lr)$. If $y \in \partial_r H$, $0 < |x - y| < r$ then

$$\begin{aligned}
&|n^{\psi(H)}(\psi(y)) \cdot \operatorname{grad} h_{\psi(x)}(\psi(y))| = A^{-1} \left|n^{\psi(H)}(\psi(y)) \cdot \frac{\psi(x) - \psi(y)}{|\psi(x) - \psi(y)|^m}\right| \leqq \\
&\leqq A^{-1}|\psi(x) - \psi(y)|^{-m}\, |n^{\psi(H)}(\psi(y)) \cdot D\psi(y)\,(x - y)| + \\
&+ A^{-1}|\psi(x) - \psi(y)|^{-m}\, |n^{\psi(H)}(\psi(y))\,. \\
&\cdot \textstyle\int_0^1 (D\psi(y) - D\psi(y + t(x - y)))\,(x - y)\,dt| \leqq \\
&\leqq A^{-1}|\psi(x) - \psi(y)|^{-m}\, |n^H(y)\,(x - y)|\, |n^{\psi(H)}(\psi(y)) \cdot D\psi(y)\, n^H(y)| + \\
&+ A^{-1}M^{m+1}|x - y|^{-m}\, |x - y|^{1+\alpha}
\end{aligned}$$

by Lemma 8. Therefore

$$(12)\qquad \left|\operatorname{grad} h_{\psi(x)}(\psi(y)) \cdot n^{\psi(H)}(\psi y))\right| \leqq$$
$$\leqq \left|\operatorname{grad} h_x(y) \cdot n^H(y)\right| |x - y|^m |\psi(x) - \psi(y)|^{-m} (\|D\psi(y) - D\psi(z)\| +$$
$$+ \|D\psi(z)\|) + A^{-1} M^{m+1} |x - y|^{1+\alpha-m} .$$

Now we estimate the expressions in (12):

$$(13)\qquad |x - y|^m |\psi(x) - \psi(y)|^{-m} \|D\psi(y) - D\psi(z)\| \leqq M^{m+1} |z - y|^\alpha \leqq$$
$$\leqq M^{m+1}(r + \varrho(rL))^\alpha .$$

Further,

$$|x - y|^m |\psi(x) - \psi(y)|^{-m} \|D\psi(z)\| \leqq$$
$$\leqq \|D\psi(z)\| \, |x - y|^m \left[\frac{1}{|D\psi(z)(x - y)|} + \right.$$
$$\left. + \left|\frac{1}{|\psi(x) - \psi(y)|} - \frac{1}{|D\psi(z)(x - y)|}\right|\right]^m \leqq$$
$$\leqq \|D\psi(z)\| \, |x - y|^m \left[\frac{1}{|D\psi(z)(x - y)|} + \right.$$
$$+ |\psi(x) - \psi(y)|^{-1} |D\psi(z)(x - y)|^{-1} \left|\int_0^1 D\psi(z)(x - y) \, dt - \right.$$
$$\left.\left. - \int_0^1 D\psi(y + t(x - y))(x - y) \, dt\right|\right]^m .$$

According to Note 1 we have

$$(14)\qquad |x - y|^m |\psi(x) - \psi(y)|^{-m} \|D\psi(z)\| \leqq$$
$$\leqq \|D\psi(z)\| \left|D\psi(z) \frac{x - y}{|x - y|}\right|^{-m} [1 + M^2(r + \varrho(rL))^\alpha]^m .$$

Puting $C = \max(M^2, M^{m+1})$ we obtain from (12), (13), (14)

$$\left|\operatorname{grad} h_{\psi(x)}(\psi(y)) \cdot n^{\psi(H)}(\psi(y))\right| \leqq$$
$$\leqq \left|\operatorname{grad} h_x(y) \cdot n^H(y)\right| \left\{\|D\psi(z)\| \left|D\psi(z) \frac{x - y}{|x - y|}\right|^{-m} (1 + C\varphi(r))^m + \right.$$
$$\left. + C\,\varphi(r)\right\} + C|x - y|^{1+\alpha-m} .$$

Theorem 4. *If ψ is conformal on $\partial H - \tau_H$ then $V_{0,f}^{\psi(H)} = V_{0,g}^H$, where $f = g \circ \psi^{-1}$.*

Proof. We may assume that $\alpha \leqq 1$. Since by Lemma 14 and Lemma 16 ψ^{-1} has the same character as ψ, it suffices to prove $V_{0,f}^{\psi(H)} \leqq V_{0,g}^H$. If $V_{0,g}^H = 0$ then $V_0^H = 0$ and $V_0^{\psi(H)} = 0$ by Theorem 2 and thus $V_{0,f}^{\psi(H)} = 0$. Hence it suffices to suppose

$0 < V^H_{0,g} < \infty$. Then $V^H_0 < \infty$ and $V^H < \infty$ according to Lemma 6. Lemma 11 implies that there are positive numbers r_0, C_1, C_2, L such that for every $x \in D$, $r \in (0, r_0)$

$$(15) \qquad v^{\psi(H)}_{r,f}(\psi(x)) \leqq C_1 v^H_{Lr,g}(x) + C_2 r^\alpha .$$

For $r > 0$ we denote $B_r = \{x \in \partial H;\ v^H_{r,g}(x) > V^H_{0,g} C_1^{-1}\}$. If $x \in \partial H - B_{Lr}$, where $r \in (0, r_0)$, then (15) yields

$$(16) \qquad v^{\psi(H)}_{r,f}(\psi(x)) \leqq V^H_{0,g} + C_2 r^\alpha .$$

Now we estimate $v^{\psi(H)}_{r,f}(\psi(x))$ on the set B_{Lr}. By the assumption there are positive constants $r_1 \in (0, r_0), K$ such that ψ is a diffeomorphism of class $C^{1+\alpha}$ on $\{x; \operatorname{dist}(x, \partial H) < 2r_1\} \subset D, \psi^{-1}$ is a diffeomorphism of class $C^{1+\alpha}$ on $\{x; \operatorname{dist}(x; \psi(\partial H)) < 2r_1\} \subset \subset \psi(D)$, for every $x, y \in \{z; \operatorname{dist}(z, \partial H) \leqq r_1\}$ we have $|\psi(x) - \psi(y)| \leqq K|x - y|$, $\|D\psi(x)\| \leqq K$, $\|D\psi(x) - D\psi(y)\| \leqq K|x - y|^\alpha$ and for all $x, y \in \{z; \operatorname{dist}(z, \psi(\partial H) \leqq \leqq r_1\}$ we have $|\psi^{-1}(x) - \psi^{-1}(y)| \leqq K|x - y|$, $\|D\psi^{-1}(x)\| \leqq K$. By Lemma 18 there are positive constants $r_2 \in (0, r_1/K)$, C_3 such that for every $r \in (0, r_2)$, $x \in B_{LKr}$ there exists $z \in (\partial H - \tau_H) \cap U(x; \varrho(LKr))$ such that for every $y \in \partial_r H, 0 < |x - y| < < rK$ we have

$$(17) \qquad |\operatorname{grad} h_{\psi(x)}(\psi(y)) \cdot n^{\psi(H)}(\psi(y))| \leqq$$
$$\leqq |\operatorname{grad} h_x(y) \cdot n^H(y)| \left[\|D\psi(z)\| \left| D\psi(z) \frac{x - y}{|x - y|} \right|^{-m} (1 + C_3 \varphi(Kr))^m + \right.$$
$$\left. + C_3 \varphi(Kr) \right] + C_3 |x - y|^{1+\alpha-m}$$

where $\varphi(r)$ is the function defined in Lemma 18. Now let $r_3 = \min(r_2, 1/K)$, $r \in (0, r_3)$, $x \in B_{rL} \subset B_{rLK}$. Then there exists $z \in (\partial H - \tau_H) \cap U(x; \varrho(LKr))$ for which (17) holds for each $y \in \partial_r H$, $0 < |x - y| < rK$. If $v, w \in U(x; Kr)$ then

$$|\psi(v) - \psi(w)| \leqq |D\psi(z)(v - w)| +$$
$$+ \left| \int_0^1 [D\psi(v + t(w - v)) - D\psi(z)] (v - w) \, dt \right| \leqq$$
$$\leqq |v - w| [\|D\psi(z)\| + K(rK + \varrho(LKr))^\alpha] .$$

Note 1 yields

$$(18) \qquad |\psi(v) - \psi(w)| \leqq |v - w| \, \|D\psi(z)\| \, [1 + K^2 \varphi(Kr)] .$$

According to ([K1], Lemma 2.15, Proposition 2.5, Definition 2.2, Remark 2.3), Lemma 7 and (17)

$$v^{\psi(H)}_{r,f}(\psi(x)) = \int_{\partial_r \psi(H) \cap U(\psi(x); r)} |n^{\psi(H)}(y) \cdot \operatorname{grad} h_{\psi(x)}(y)| \, g(\psi^{-1}(y))$$
$$d\varkappa_{m-1}(y)/g(x) \leqq \int_{\psi(\partial_r H \cap U(x; rK))} \Big\{ |\operatorname{grad} h_x(\psi^{-1}(y)) \cdot n^H(\psi^{-1}(y))|$$

$$\left[\left|D\psi(z)\frac{x-\psi^{-1}(y)}{|x-\psi^{-1}(y)|}\right|^{-m}\|D\psi(z)\|\,(1+C_3\varphi(Kr))^m + C_3\varphi(Kr)\right] +$$
$$+ C_3|x-\psi^{-1}(y)|^{1+\alpha-m}\Big\}\, g(\psi^{-1}(y))\,d\varkappa_{m-1}(y)/g(x)\,.$$

Using the Lipschitz condition (see (18)) we estimate this integral by the expression

$$\|D\psi(z)\|^{m-1}\,[1+K^2\varphi(Kr)]^m \int_{\partial H\cap U(x;Kr)}\Big\{|\text{grad}\, h_x(y)\cdot n^H(y)|\,.$$
$$\cdot\left[\|D\psi(z)\|\left|D\psi(z)\frac{x-y}{|x-y|}\right|^{-m}(1+C_3\varphi(Kr))^m + C_3\varphi(Kr)\right] +$$
$$+ C_3|x-y|^{1+\alpha-m}\Big\}\, g(y)\,d\varkappa_{m-1}(y)/g(x)\,.$$

Since $z\in\partial H - \tau_H$ and therefore ψ is conformal at z, we have $|D\psi(z)\,u| = \|D\psi(z)\|\,|u|$ for each $u\in R^m$. Hence

$$v_{r,f}^{\psi(H)}(\psi(x)) \leqq [1+K^2\varphi(Kr)]^{m-1}\int_{\partial_r H\cap U(x;Kr)} \text{grad}\, h_x(y)\cdot n^H(y)|$$
$$[(1+C_3\varphi(Kr))^m + C_3\,\|D\psi(z)\|^{m-1}\,\varphi(Kr)]\; g(y)\,d\varkappa_{m-1}(y)/g(x) +$$
$$+ C_3[1+K^2\varphi(Kr)]^{m-1}\,K^{m-1}(\sup g)\,.$$
$$\cdot\int_{\partial_r H\cap U(x;Kr)}|x-y|^{1+\alpha-m}\,d\varkappa_{m-1}(y)/g(x)\,.$$

Lemma 10 and Lemma 9 imply

$$v_{r,f}^{\psi(H)}(\psi(x)) \leqq (1+K^2\varphi(Kr))^{m-1}\,[(1+C_3\varphi(Kr))^m + K^{m-1}C_3\varphi(Kr)]\; v_{Kr,g}^H(x) + C_4(1+K^2\varphi(Kr))^{m-1}\, r^\alpha\,, \tag{19}$$

where $C_4 = C_3K^{m-1+\alpha}Am(m+2)\,(1/2+V^H)\,2^{m-1}(1-2^{-\alpha})^{-1}\,(\sup g)/(\inf g)$. Since $\partial\psi(H) = \psi(\partial H)$, we conclude that $V_{0,f}^{\psi(H)} \leqq V_{0,g}^H$.

References

[AKK1] *T. S. Angell, R. E. Kleinman, J. Král:* Double layer potentials on boundaries with corners and edges. Comment. Math. Univ. Carolinae *27* (1989).

[AKK2] *T. S. Angell, R. E. Kleinman, J. Král:* Layer potentials on boundaries with corners and edges. Časopis pěst. mat. *113* (1988), 387—402.

[DG1] *E. De Giorgi:* Su una teoria generale della misura $(r-1)$-dimensionale in uno spazi ad r dimensioni. Annali di Mat. Pura ed Appl. Ser. *4, 36* (1954), 191—213.

[DG2] *E. De Giorgi:* Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazi ad r dimensioni. Ricerche Mat. *4* (1955), 95—113.

[Do] *M. Dont, E. Dontová:* Invariance of the Fredholm radius of an operator in potential theory. Časopis pěst. mat. *112* (1987), 269—283.

[Fel] *H. Federer:* A note on the Gauss-Green theorem. Proc. Amer. Math. Soc. *9* (1958), 447—451.

[Fe2] *H. Federer:* Geometric measure theory. Springer-Verlag 1969.
[Fe3] *H. Federer:* The Gauss-Green theorem. Trans. Amer. Math. Soc. *58* (1945), 44—76.
[K1] *J. Král:* Integral Operators in Potential Theory. Lecture Notes in Mathematics 823, Springer-Verlag, Berlin 1980.
[K2] *J. Král:* Flows of heat and the Fourier problem. Czechoslovak Math. J. *20* (*95*) (1970), 556—597.
[K3] *J. Král:* Note on sets whose characteristic functions have signed measure for their partial derivatives (Czech). Časopis pěst. mat. *86* (1961), 178—194.
[K4] *J. Král:* The Fredholm method in potential theory. Trans. Amer. Math. Soc. *125* (1966), 511—547.
[K5] *J. Král:* The Fredholm radius of an operator in potential theory. Czechoslovak Math.J. *15* (*90*) (1965), 565—588.
[KW] *J. Král, W. Wendland:* Some examples concerning applicability of the Fredholm-Radon method in potential theory. Aplikace Matematiky *31* (1986), 293—308.
[O] *M. Ohtsuka:* Reading of De Giorgi's papers, Gakushuin University 1980.

Souhrn

INVARIANCE FREDHOLMOVA POLOMĚRU NEUMANNOVA OPERÁTORU

DAGMAR MEDKOVÁ

Jednou z klasických metod řešení Dirichletovy a Neumannovy úlohy v R^m je metoda integrálních rovnic. Jestliže chceme použít při řešení úlohy Fredholm-Radonovu teorii, musíme znát Fredholmův poloměr Neumannova operátoru. V článku je ukázáno, že při deformaci zkoumané oblasti difeomorfním zobrazením, které je konformní (tj. zachovává úhly) na přesně specifikované části hranice, se nemění Fredholmův poloměr Neumannova operátoru.

Резюме

ИНВАРИАНТНОСТЬ РАДИУСА ФРЕДГОЛЬМА ОПЕРАТОРА НЕЙМАНА

DAGMAR MEDKOVÁ

Одним из классических методов решения задачи Дирихле и Неймана в R^m является метод интегральных уравнений. Для того, чтобы при решении задачи можно было воспользоваться теорией Фредгольма и Радона, необходимо знать радиус Фредгольма оператора Неймана. В статье показано, что при деформации исследуемой области диффеоморфизмом, который является конформным (т.е. сохраняет углы) на точно определенной части границы, радиус Фредгольма оператора Неймана не меняется.

Author's address: Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.