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Eva Špániková

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# ASYMPTOTIC PROPERTIES OF SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

EVA ŠPÁNIKOVÁ, Žilina

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*Summary.* The purpose of this paper is to study the asymptotic behavior of solutions of the nonlinear differential system (S) which is either superlinear or sublinear and $\int^\infty p_i(t)\,dt < \infty$, $i = 1, 2, \ldots, n-1$.

*Keywords:* Nonlinear differential system with deviating arguments, superlinear nonlinear differential system, sublinear nonlinear differential system, oscillatory solution, nonoscillatory solution.

*Classification AMS:* 34K25.

## INTRODUCTION

In this paper we consider the differential system with deviating arguments

$$\text{(S)}\qquad \begin{aligned} y_i'(t) &= p_i(t)\,y_{i+1}(t)\,, \quad i = 1, 2, \ldots, n-2\,; \quad n \geqq 2\\ y_{n-1}'(t) &= p_{n-1}(t) f_{n-1}(y_n(g_n(t)))\,,\\ y_n'(t) &= f_n(t, y_1(g_1(t)))\,, \end{aligned}$$

where the following conditions are assumed to be fulfilled:

(a) $p_i(t)$ is continuous and nonnegative on $[a, \infty)$; $p_i(t) \not\equiv 0$ on any infinite subinterval of $[a, \infty)$, $\int^\infty p_i(t)\,dt < \infty$, $i = 1, 2, \ldots, n-1$.

(b) $g_i(t)$ is continuous on $[a, \infty)$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, n$; $g_n(t) \leqq t$ for $t \geqq a$.

(c) $f_{n-1}(u)$ is continuous on $R$; $|f_{n-1}(u)| \leqq K|u|^\beta$, for $0 < \beta \leqq 1$, $0 < K$ – const.

(d) $f_n(t, v)$ is continuous on $[a, \infty) \times R$ and $|f_n(t, v)| \leqq \omega(t, |v|)$ for $(t, v) \in [a, \infty) \times \times R$ where $\omega(t, z)$ is continuous on $[a, \infty) \times [0, \infty)$ and nondecreasing in $z$ for every $t \in [a, \infty)$.

System (S) is called superlinear or sublinear according to whether $\omega(t, z)/z$ is nondecreasing or nonincreasing in $z$ for $z > 0$. The purpose of this paper is to study the asymptotic behavior of solutions of system (S) which is either superlinear or sublinear. Hereafter the term "solution" will be understood to mean a solution $y(t) = \{y_1(t), y_2(t), \ldots, y_n(t)\}$ of (S) which exists on some half-line $[\tau, \infty)$, $\tau > a$, and satisfies $\sup\{\sum_{i=1}^n |y_i(t)| : t \geqq \tau'\} > 0$ for any $\tau' \geqq \tau$. Such a solution is said to be oscillatory [weakly oscillatory] if each of its components [at least one component] has arbitrarily large zeros. A solution is said to be nonoscillatory [weakly nonoscillatory] if each of its components [at least one component] is eventually of constant sign.

Let $i_k \in \{1, 2, \ldots, n-1\}$, $k = 1, 2, \ldots, n-1$, and $t, s \in [a, \infty]$. We define:

$$I_0(t, s) = J_0(t, s) \equiv 1 ;$$

$$I_k(t, s; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = \int_s^t p_{i_1}(x) I_{k-1}(x, s; p_{i_2}, p_{i_3}, \ldots, p_{i_k}) \, dx ;$$

$$J_k(t, s; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = \int_s^t p_{i_k}(x) J_{k-1}(t, x; p_{i_1}, p_{i_2}, \ldots, p_{i_{k-1}}) \, dx ;$$

$$\delta_0^k(t) = \bar{\delta}_0^k(t) \equiv 1 ;$$

$$\delta_j^k(t) = J_j(\infty, t; p_{k+j-1}, \ldots, p_{k+1}, p_k) ;$$

$$\bar{\delta}_j^k(t) = I_j(\infty, t; p_k, p_{k+1}, \ldots, p_{k+j-1}) \quad \text{for} \quad j = 1, 2, \ldots, n-k ;$$

$$g^*(t) = \max\{g_1(t), t\}, \quad g_*(t) = \min\{g_1(t), t\},$$

$$h^*(t) = \sup_{a \leq s \leq t} g^*(s), \qquad h_*(t) = \inf_{s \geq t} g_*(s) .$$

It is easy to prove that the following identities hold:

$$I_k(t, s; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = J_k(t, s; p_{i_1}, p_{i_2}, \ldots, p_{i_k})$$
$$\text{for} \quad k = 1, 2, \ldots, n-1 .$$

## OSCILLATION THEOREMS

We first prove a theorem which enables us to classify all solutions of (S) according to their behavior as $t \to \infty$.

**Theorem 1.** *Assume that either* (S) *is superlinear and*

$$(1) \qquad \int^\infty \prod_{j=1}^{n-1} \delta_1^j(g_*(t)) \, \omega(t, c) \, dt < \infty \quad \textit{for all} \quad c > 0$$

*or* (S) *is sublinear and*

$$(2) \qquad \int^\infty \frac{\delta_1^i(g_*(t))}{\delta_1^i(g_1(t))} \, \omega\Big(t, c \prod_{j=1}^{n-1} \delta_1^j(g_*(t))\Big) \, dt < \infty \quad \textit{for all} \quad c > 0 ,$$
$$i = 1, 2, \ldots, n-1 .$$

*If* $y(t) = \{y_1(t), y_2(t), \ldots, y_n(t)\}$ *is a solution of* (S), *then exactly one of the following cases occurs:*

(I) $\limsup_{t \to \infty} |y_i(t)| = \infty$, $i = 1, 2, \ldots, n$;

(II) *there exists an integer* $k \in \{1, 2, \ldots, n-1\}$ *and a nonzero constant* $\alpha_k$ *such that*

$$\lim_{t\to\infty} \frac{y_i(t)}{\delta^t_{k-i}(t)} = (-1)^{k-i}\,\alpha_k \quad \textit{for} \quad i = 1, 2, \ldots, k\,;$$

$$\lim_{t\to\infty} y_{i+k}(t)\,\delta^k_i(t) = 0 \quad \textit{for} \quad i = 1, 2, \ldots, n-k\,;$$

(III) *there exists a constant* $\alpha_n$ *such that*

$$\lim_{t\to\infty} \frac{y_i(t)}{\delta^i_{n-i}(t)} = (-1)^{n-i} f_{n-1}(\alpha_n) \quad \textit{for} \quad i = 1, 2, \ldots, n-1\,;$$

$$\lim_{t\to\infty} y_n(t) = \alpha_n\,.$$

Proof. We assume that (S) is superlinear and (1) holds. Let $y(t)$ be a solution of (S) defined on $[\tau, \infty)$. Let $T \geqq \tau$ be such that $h_*(T) \geqq \tau$ and $\delta^i_1(T) \leqq 1$ for $i = 1, 2, \ldots, n-1$. Integrating the first $(n-1)$ equations of (S) from $T$ to $t$ and combining them we have

$$(3_l) \qquad |y_1(t)| \leqq \sum_{j=1}^{l} |y_j(T)|\, I_{j-1}(t, T; p_1, p_2, \ldots, p_{j-1}) + \\ + I_l(t, T; p_1, p_2, \ldots, p_l |y_{l+1}|)\,, \quad l = 1, 2, \ldots, n-2\,; \quad t \geqq T;$$

$$(3_{n-1}) \qquad |y_1(t)| \leqq \sum_{j=1}^{n-1} |y_j(T)|\, I_{j-1}(t, T; p_1, p_2, \ldots, p_{j-1}) + \\ + I_{n-1}(t, T; p_1, p_2, \ldots, p_{n-1} |f_{n-1}(y_n(g_n))|)\,, \quad t \geqq T.$$

Integrating the last equation of (S) from $T$ to $t$ and using (d) we obtain

$$(4_n) \qquad |y_n(t)| \leqq |y_n(T)| + \textstyle\int_T^t \omega(s, |y_1(g_1(s))|)\, ds\,, \quad t \geqq T.$$

Using (b), (c), $(4_n)$ and Taylor's theorem we get

$$(5) \qquad |f_{n-1}(y_n(g_n(t)))| \leqq M + N \textstyle\int_T^t \omega(s, |y_1(g_1(s))|)\, ds\,, \quad t \geqq T,$$

where $M = K|y_n(t)|^\beta\,, \quad N = K\beta|y_n(T)|^{\beta-1}\,.$

Integrating the first $(n-1)$ equations of (S) from $T$ to $t$ and combining them (using (5)) we have

$$(4_l) \qquad |y_l(t)| \leqq \sum_{j=l}^{n-1} |y_j(T)|\, I_{j-l}(t, T; p_l, p_{l+1}, \ldots, p_{j-1}) + \\ + M\, I_{n-l}(t, T; p_l, \ldots, p_{n-1}) + \\ + N\, I_{n-l+1}(t, T; p_l, p_{l+1}, \ldots, p_{n-1}, \omega(\cdot, |y_1(g_1(\cdot))|))\,, \\ l = 1, 2, \ldots, n-1\,; \quad t \geqq T.$$

Suppose that $\limsup_{t\to\infty} |y_1(t)| = \infty$. Let there exist an integer $i \in \{2, 3, \ldots, n\}$ such that $\limsup_{t\to\infty} |y_i(t)| < \infty$. Then from $(3_{i-1})$ we get a contradiction with the assumption. Case (I) is proved.

Suppose that $\limsup_{t\to\infty} |y_1(t)| < \infty$. From (1) we get that the function $\prod_{j=1}^{n-1} \delta_1^j(t)\, \omega(t, c)$ is integrable on $[T, \infty)$ for all $c > 0$. It is easy to prove that $p_1(t)\, y_2(t) \in L^1[T, \infty)$ (from $(4_2)$ we have

$$\int_T^\infty p_1(t)\, |y_2(t)|\, dt \leqq \sum_{j=2}^{n-1} |y_j(T)|\, \delta_{j-1}^1(T) + M\, \delta_{n-1}^1(T) +$$
$$+ N \int_T^\infty \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, |y_1(g_1(s))|)\, ds)\,.$$

From the first equation of (S) we obtain

(6) $$y_1(t) = \alpha_1 - \int_t^\infty p_1(s)\, y_2(s)\, ds\,, \quad t \geqq T\,,$$
$$\text{where} \quad \alpha_1 = y_1(T) + \int_T^\infty p_1(s)\, y_2(s)\, ds\,.$$

From (6) we have $\lim_{t\to\infty} y_1(t) = \alpha_1$.

From $(4_{i+1})$ for $i = 1, 2, \ldots, n-1$, we obtain

$$\delta_i^1(t)\, |y_{i+1}(t)| \leqq \delta_i^1(t) \Big[ \sum_{j=i+1}^{n-1} |y_j(T)|\, \delta_{j-i-1}^{i+1}(T) + M\, \delta_{n-i-1}^{i+1}(T) +$$
$$+ N \int_T^{t_1} \omega(s, |y_1(g_1(s))|) \prod_{j=i+1}^{n-1} \delta_1^j(s)\, ds\Big] +$$
$$+ N \int_{t_1}^t \omega(s, |y_1(g_1(s))|) \prod_{j=1}^{n-1} \delta_1^j(s)\, ds\,, \quad t \geqq T\,.$$

Hence we see that $\delta_i^1(t)\, |y_{i+1}(t)|$ can be made arbitrarily small by taking $t_1$ sufficiently large and then letting $t$ increase without bound. Thus, $\lim_{t\to\infty} \delta_i^1(t)\, y_{i+1}(t) = 0$, $i = 1, 2, \ldots, n-1$; and we arrive at Case (II) (with $k = 1$) if $\alpha_1 \neq 0$.

Suppose that $\alpha_1 = 0$. From (1) we get that the functions $\delta_1^1(g_1(t)) \prod_{j=2}^{n-1} \delta_1^j(t)\, \omega(t, 1)$, $\prod_{j=1}^{n-1} \delta_1^j(t)\, \omega(t, 1)$ and $\prod_{j=2}^{n-1} \delta_1^j(t)\, \omega(t, c\, \delta_1^1(g_1(t)))$ are integrable on $[T, \infty)$ for all $c > 0$. Choose a $T_1 \geqq T$ such that $T_1^* = h_*(T_1) \geqq T$, $|y_1(g_1(t))| \leqq 1$ for $t \geqq T_1$,

$$N \int_{T_1}^\infty \delta_1^1(g_1(s)) \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, 1)\, ds \leqq \tfrac{1}{3}$$

and

$$N \int_{T_1}^\infty \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, 1)\, ds \leqq \tfrac{1}{3}\,.$$

From (6) with regard to $(4_2)$ we get

$$(7)\qquad |y_1(t)| \leqq \delta_1^1(t)\,[\sum_{j=2}^{n-1}|y_j(T)|\,\bar{\delta}_{j-2}^2(T) + M\,\bar{\delta}_{n-2}^2(T) +$$

$$+ N \int_T^t \prod_{j=2}^{n-1} \delta_1^j(s)\,\omega(s, |y_1(g_1(s))|)\,ds] +$$

$$+ N \int_t^\infty \prod_{j=1}^{n-1} \delta_1^j(s)\,\omega(s, |y_1(g_1(s))|)\,ds\,, \quad t \geqq T\,.$$

Putting $u_1(t) = \sup_{s \geqq t} |y_1(s)|$ and using the decreasing nature of the right-hand side of (7), we obtain

$$(8)\qquad \frac{u_1(t)}{\delta_1^1(t)} \leqq K_1 + N\int_{T_1}^t \prod_{j=2}^{n-1} \delta_1^j(s)\,\omega(s, |y_1(g_1(s))|)\,ds +$$

$$+ \frac{N}{\delta_1^1(t)}\int_t^\infty \prod_{j=1}^{n-1} \delta_1^j(s)\,\omega(s, |y_1(g_1(s))|)\,ds\,, \quad t \geqq T_1\,,$$

where $K_1$ is a positive constant.

For each $t \geqq T_1$ we denote

$$R_t^1 = \{s \in [T_1, \infty);\ g_1(s) \leqq t\}\,, \quad A_t^1 = \{s \in [T_1, \infty);\ g_1(s) > t\}\,.$$

We then have

$$\frac{u_1(g_1(s))}{\delta_1^1(g_1(s))} \leqq \sup_{T_1^* \leqq \sigma \leqq t} \frac{u_1(\sigma)}{\delta_1^1(\sigma)} \quad \text{for} \quad s \in R_t^1\,,$$

$$u_1(g_1(s)) \leqq u_1(t) \quad \text{for} \quad s \in A_t^1.$$

Using the inequality

$$(9)\qquad \omega(s, ab) \leqq a\,\omega(s, b), \quad 0 < a \leqq 1\,, \quad b > 0\,,$$

which is a consequence of the superlinearity of (S), we can derive the following inequality from (8):

$$\frac{u_1(t)}{\delta_1^1(t)} \leqq K_1 + N \sup_{T^*_1 \leqq s \leqq t} \frac{u_1(s)}{\delta_1^1(s)}\Bigg[\int_{R_t^1 \cap [T_1, t)} \delta_1^1(g_1(s)) \prod_{j=2}^{n-1} \delta_1^j(s)\,\omega(s, 1)\,ds +$$

$$+ \frac{1}{\delta_1^1(t)}\int_{R_t^1 \cap [t, \infty)} \delta_1^1(g_1(s)) \prod_{j=1}^{n-1} \delta_1^j(s)\,\omega(s, 1)\,ds\Bigg] +$$

$$+ N\,\frac{u_1(t)}{\delta_1^1(t)}\Bigg[\delta_1^1(t)\int_{A_t^1 \cap [T_1, t)} \prod_{j=2}^{n-1} \delta_1^j(s)\,\omega(s, 1)\,ds +$$

$$+ \int_{A_t^1 \cap [t, \infty)} \prod_{j=1}^{n-1} \delta_1^j(s)\,\omega(s, 1)\,ds\Bigg] \leqq$$

$$\leqq K_1 + \sup_{T_1^* \leqq s \leqq t} \frac{u_1(s)}{\delta_1^1(s)}\,N\int_{T_1}^\infty \delta_1^1(g_1(s)) \prod_{j=2}^{n-1} \delta_1^j(s)\,\omega(s, 1)\,ds +$$

$$+\frac{u_1(t)}{\delta_1^1(t)} N \int_{T_1}^{\infty} \prod_{j=1}^{n-1} \delta_1^j(s)\,\omega(s, 1)\,ds \leqq$$

$$\leqq K_1 + \tfrac{1}{3} \sup_{T_1^* \leqq s \leqq t} \frac{u_1(s)}{\delta_1^1(s)} + \tfrac{1}{3}\frac{u_1(t)}{\delta_1^1(t)}, \quad t \geqq T_1,$$

and consequently

$$\frac{u_1(t)}{\delta_1^1(t)} \leqq \bar{K}_1 + \tfrac{1}{2} \sup_{T_1 \leqq s \leqq t} \frac{u_1(s)}{\delta_1^1(s)}, \quad t \geqq T_1,$$

where $\bar{K}_1 = \frac{3}{2}K_1 + \frac{1}{2} \sup_{T_1^* \leqq s \leqq T_1} u_1(s)/\delta_1^1(s)$. It follows that

$$\frac{|y_1(t)|}{\delta_1^1(t)} \leqq \frac{u_1(t)}{\delta_1^1(t)} \leqq \sup_{T_1 \leqq s \leqq t} \frac{u_1(s)}{\delta_1^1(s)} \leqq 2\bar{K}_1, \quad t \geqq T_1.$$

From the last inequality we have

(10) $$|y_1(g_1(t))| \leqq K_1^* \, \delta_1^1(g_1(t)) \quad \text{for} \quad t \geqq T,$$

where $K_1^*$ is a positive constant.

The function $p_2(t)\, y_3(t)$ is integrable on $[T, \infty)$ (because $(4_3)$ with regard to (10) yields

$$\int_T^\infty p_2(t)\,|y_3(t)|\,dt \leqq \sum_{j=3}^{n-1} |y_j(T)|\, \delta_{j-2}^2(T) + M\, \delta_{n-2}^2(T) +$$

$$+ N \int_T^\infty \prod_{j=2}^{n-1} \delta_1^j(s)\,\omega(s, K_1^*\,\delta_1^1(g_1(s)))\,ds),$$

and we have

(11) $$y_2(t) = \alpha_2 - \int_t^\infty p_2(s)\, y_3(s)\,ds, \quad t \geqq T,$$

where $\alpha_2 = y_2(T) + \int_T^\infty p_2(s)\, y_3(s)\,ds$.

Therefore, $\lim_{t\to\infty} y_2(t) = \alpha_2$.

From (6) with regard to (11) we obtain

$$y_1(t) = -\alpha_2 \int_t^\infty p_1(x_1)\,dx_1 + \int_t^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2)\, y_3(x_2)\,dx_2\,dx_1.$$

The last inequality implies

$$\lim_{t\to\infty} \frac{y_1(t)}{\delta_1^1(t)} = -\alpha_2.$$

From $(4_l)$ $(l = i + 2,\ i = 1, 2, \ldots, n-2)$ in view of (10) we get

$$\delta_i^2(t)\,|y_{i+2}(t)| \leqq \delta_i^2(t) \Big[\sum_{j=i+2}^{n-1} |y_j(T)|\, \delta_{j-i-2}^{i+2}(T) + M\, \delta_{n-i-2}^{i+2}(T)\Big] +$$

$$+ N\, \delta_i^2(t) \int_T^{t_1} \omega(s, K_1^*\,\delta_1^1(g_1(s))) \prod_{j=i+2}^{n-1} \delta_1^j(s)\,ds +$$

$$+ N \int_{t_1}^{t} \omega(s, K_1^* \delta_1^1(g_1(s))) \prod_{j=2}^{n-1} \delta_1^j(s) \, ds , \quad t \geqq T ,$$

$i = 1, 2, \ldots, n - 3$, and

$$\delta_{n-2}^2(t) \left|y_n(t)\right| \leqq \delta_{n-2}^2(t) \left|y_n(T)\right| + \delta_{n-2}^2(t) \int_T^{t_1} \omega(s, K_1^* \delta_1^1(g_1(s))) \, ds +$$

$$+ \int_{t_1}^{t} \prod_{j=2}^{n-1} \delta_1^j(s) \, \omega(s, K_1^* \delta_1^1(g_1(s))) \, ds , \quad t \geqq T .$$

Hence we see that $\delta_i^2(t) \left|y_{i+2}(t)\right|$, $i = 1, 2, \ldots, n - 2$ can be made arbitrarily small by taking $t_1$ sufficiently large and then letting $t$ increase without bound. Thus, $\lim_{t \to \infty} \delta_i^2(t) \, y_{i+2}(t) = 0$, $i = 1, 2, \ldots, n - 2$; and we arrive at Case (II) (with $k = 2$) if $\alpha_2 \neq 0$.

Further, we suppose that there exists an integer $m \in \{2, 3, \ldots, n - 3\}$ such that $\alpha_i = y_i(T) + \int_T^\infty p_i(s) \, y_{i+1}(s) \, ds = 0$ for $i = 1, 2, \ldots, m$. We show that either Case (II) or Case (III) can occur. We have

$$(12_i) \qquad y_i(t) = - \int_t^\infty p_i(s) \, y_{i+1}(s) \, ds \, ; \quad i = 1, 2, \ldots, m \, ; \quad t \geqq T .$$

From (1) we get that the functions

$$\omega(t, c \prod_{j=1}^{m-1} \delta_1^j(g_1(t))) \, \delta_1^m(g_1(t)) \prod_{j=m+1}^{n-1} \delta_1^j(t) ,$$

$$\omega(t, c \prod_{j=1}^{m-1} \delta_1^j(g_1(t))) \prod_{j=m}^{n-1} \delta_1^j(t) \quad \text{and}$$

$$\omega(t, c \prod_{j=1}^{m} \delta_1^j(g_1(t))) \prod_{j=m+1}^{n-1} \delta_1^j(t) \quad \text{are integrable on} \quad [T, \infty)$$

for all $c > 0$. Choose a $T_m$ such that $T_m^* = h_*(T_m) \geqq T$,

$$\frac{\left|y_1(t)\right|}{\prod_{j=1}^{m-1} \delta_1^j(t)} \leqq 1 \quad \text{for} \quad t \geqq T_m ,$$

$$N \int_{T_m}^\infty \omega(s, \prod_{j=1}^{m-1} \delta_1^j(g_1(s))) \, \delta_1^m(g_1(s)) \prod_{j=m+1}^{n-1} \delta_1^j(s) \, ds \leqq \tfrac{1}{3} \quad \text{and}$$

$$N \int_{T_m}^\infty \omega(s, \prod_{j=1}^{m-1} \delta_1^j(g_1(s))) \prod_{j=m}^{n-1} \delta_1^j(s) \, ds \leqq \tfrac{1}{3} .$$

From $(12_i)$ $(i = 1, 2, \ldots, m)$ with regard to $(4_{m+1})$ we get

$$\left|y_1(t)\right| \leqq \delta_{m-1}^1(t) \, \delta_1^m(t) \left[ \sum_{j=m+1}^{n-1} \left|y_j(T)\right| \delta_{j-m-1}^{m+1}(T) + M \, \delta_{n-m-1}^{m+1}(T) + \right.$$

$$\left. + N \int_T^t \omega(s, \left|y_1(g_1(s))\right|) \prod_{j=m+1}^{n-1} \delta_1^j(s) \, ds \right] +$$

$$+ N \, \delta_{m-1}^1(t) \int_t^\infty \omega(s, \left|y_1(g_1(s))\right|) \prod_{j=m}^{n-1} \delta_1^j(s) \, ds , \quad t \geqq T .$$

The last inequality implies

$$(13)\qquad \frac{|y_1(t)|}{\prod_{j=1}^{m-1}\delta_1^j(t)} \leqq \frac{|y_1(t)|}{\delta_{m-1}^1(t)}\,\delta_1^m(t)\,\Big[\sum_{j=m+1}^{n-1}|y_j(T)|\,\delta_{j-m-1}^{m+1}(T) + M\,\delta_{n-m-1}^{m+1}(T) +$$

$$+ N\int_T^t \omega(s, |y_1(g_1(s))|)\prod_{j=m+1}^{n-1}\delta_1^j(s)\,\mathrm{d}s\Big] +$$

$$+ N\int_t^\infty \omega(s, |y_1(g_1(s))|)\prod_{j=m}^{n-1}\delta_1^j(s)\,\mathrm{d}s\,, \quad t \geqq T.$$

We now define

$$u_m(t) = \sup_{s\geqq t}\frac{|y_1(s)|}{\prod_{j=1}^{m-1}\delta_1^j(s)}\,.$$

For each $t \geqq T_m$ we denote

$$R_t^m = \{s \in [T_m, \infty);\ g_1(s) \leqq t\}\,,\quad A_t^m = \{s \in [T_m, \infty);\ g_1(s) > t\}\,.$$

Proceeding similarly as in the last case we obtain

$$(14)\qquad |y_1(g_1(t))| \leqq K_m^* \prod_{j=1}^{m}\delta_1^j(g_1(t))\,,\quad t \geqq T,$$

where $K_m^*$ is a positive constant.

It is easy to prove that $p_{m+1}(t)\,y_{m+2}(t) \in L^1[T, \infty)$ and

$$(15)\qquad y_{m+1}(t) = \alpha_{m+1} - \int_t^\infty p_{m+1}(s)\,y_{m+2}(s)\,\mathrm{d}s\,,\quad t \geqq T,$$

where $\alpha_{m+1} = y_{m+1}(T) + \int_T^\infty p_{m+1}(s)\,y_{m+2}(s)\,\mathrm{d}s\,.$

From (15) we have $\lim_{t\to\infty} y_{m+1}(t) = \alpha_{m+1}$. From $(12_i)$ $(i = 1, 2, \ldots, m)$ with regard to (15) we get

$$y_i(t) = (-1)^{m-i+1}\,\alpha_{m+1} J_{m-i+1}(\infty; t; p_m, p_{m-1}, \ldots, p_i) +$$
$$+ (-1)^{m-i}\,J_{m-i+2}(\infty; t; p_{m+1}y_{m+2}, p_m, \ldots, p_i)\,,\quad t \geqq T.$$

Using L'Hospital's rule we have

$$\lim_{t\to\infty}\frac{y_i(t)}{\delta_{m-i+1}^i(t)} = (-1)^{m-i+1}\,\alpha_{m+1}\,,\quad i = 1, 2, \ldots, m\,.$$

From $(4_{m+1+i})$ $(i = 1, 2, \ldots, n-m-1)$ with regard to (14) we can prove that $\lim_{t\to\infty} y_{m+1+i}(t)\,\delta_i^{m+1}(t) = 0$ for $i = 1, 2, \ldots, n-m-1$. If $\alpha_{m+1} \neq 0$ then Case (II) occurs for $k = m + 1$.

Suppose that $\alpha_{n-2} = 0$. It is easy to prove that in this case $(12_i)$ (for $i = 1, 2, \ldots, n-2$) and (14) (for $m = n - 2$) hold, $p_{n-1}(t) f_{n-1}(y_n(g_n(t))) \in L^1[T, \infty)$ and

$$(16)\qquad y_{n-1}(t) = \alpha_{n-1} - \int_t^\infty p_{n-1}(s) f_{n-1}(y_n(g_n(s)))\,\mathrm{d}s\,,\quad t \geqq T,$$

where

$$\alpha_{n-1} = y_{n-1}(T) + \int_T^\infty p_{n-1}(s) f_{n-1}(y_n(g_n(s)))\,ds .$$

From (16) we have $\lim_{t\to\infty} y_{n-1}(t) = \alpha_{n-1}$. Using ($12_i$) ($i = 1, 2, \ldots, n-2$) and (16) we obtain

$$\lim_{t\to\infty} \frac{y_i(t)}{\delta^i_{n-i-1}(t)} = (-1)^{n-i-1}\,\alpha_{n-1}\,, \quad i = 1, 2, \ldots, n-2 .$$

From ($4_n$) we have $\lim_{t\to\infty} \delta_1^{n-1}(t)\, y_n(t) = 0$. If $\alpha_{n-1} \neq 0$ then Case (II) occurs for $k = n - 1$.

Suppose that $\alpha_{n-1} = 0$. In this case we easily see that ($12_i$) (for $i = 1, 2, \ldots, n-2$) and (14) (for $m = n-1$) hold, $f_n(t, y_1(g_1(t))) \in L^1[T, \infty)$ and

(17) $$y_n(t) = \alpha_n - \int_t^\infty f_n(s, y_1(g_1(s)))\,ds\,, \quad t \geqq T,$$

where $\alpha_n = y_n(T) + \int_T^\infty f_n(s, y_1(g_1(s)))\,ds$.

From (17) we have $\lim_{t\to\infty} y_n(t) = \alpha_n$.

From ($12_i$) ($i = 1, 2, \ldots, n-2$) with regard to (16) we have

$$y_i(t) = (-1)^{n-i} J_{n-i}(\infty, t;\, p_{n-1} f_{n-1}(y_n(g_n)),\, p_{n-2}, \ldots, p_{i+1}, p_i)$$

$$i = 1, 2, \ldots, n-1\,,$$

and

$$\lim_{t\to\infty} \frac{y_i(t)}{\delta^i_{n-i}(t)} = (-1)^{n-i} f_{n-1}(\alpha_n)\,, \quad i = 1, 2, \ldots, n-1 .$$

Case (III) occurs.

Next, suppose that (S) is sublinear and (2) holds. Case (I) and Case (II) (for $k = 1$) can be proved analogously as in the case when system (S) is superlinear. Suppose that $\alpha_1 = 0$. From (2) we get that functions

$$\prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, c\, \delta_1^1(g_1(s))) \quad \text{and} \quad \frac{\delta_1^1(s)}{\delta_1^1(g_1(s))} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))$$

are integrable on $[T, \infty)$ for all $c > 0$. We shall show that $y_1(t) = O(\delta_1^1(t))$ as $t \to \infty$. Suppose the contrary. Then it is possible to select $T_{1,1}$, $T_{2,1}$ and $T_{3,1}$ in such a way that

$$T < T_{1,1} < T_{2,1} < T_{3,1}\,, \quad T^*_{1,1} = h_*(T_{1,1}) \geqq T,$$

$$|y_1(T^*_{1,1})| \geqq \delta_1^1(T^*_{1,1})\,,$$

$$\sup_{T^*_{1,1} \leqq s \leqq t} \frac{|y_1(s)|}{\delta_1^1(s)} = \sup_{T_{2,1} \leqq s \leqq t} \frac{|y_1(s)|}{\delta_1^1(s)} \quad \text{for} \quad t \geqq T_{2,1}\,,$$

$$N \int_{T_{2,1}}^\infty \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\,ds \leqq \tfrac{1}{4}\,;$$

$$N \int_{T_{2,1}}^{\infty} \frac{\delta_1^1(s)}{\delta_1^1(g_1(s))} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, ds \leqq \tfrac{1}{4} \quad \text{and}$$

$$\sum_{j=2}^{n-1} |y_j(T)|\, \bar{\delta}_{j-2}^2(T) + M\, \bar{\delta}_{n-2}^2(T) +$$

$$+ N \int_T^{T_{2,1}} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, |y_1(g_1(s))|)\, ds \leqq \frac{1}{4} \frac{|y_1(T_{3,1})|}{\delta_1^1(T_{3,1})}.$$

We rewrite (7) as follows:

$$\frac{|y_1(t)|}{\delta_1^1(t)} \leqq \sum_{j=2}^{n-1} |y_i(T)|\, \bar{\delta}_{j-2}^2(T) + M\, \bar{\delta}_{n-2}^2(T) + \tag{18}$$

$$+ N \int_T^{T_{2,1}} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, |y_1(g_1(s))|)\, ds + N \int_{T_{2,1}}^{t} \prod_{j=2}^{n-1} \delta_1^j(s)\, .$$

$$.\, \omega(s, |y_1(g_1(s))|)\, ds + \frac{N}{\delta_1^1(t)} \int_t^{\infty} \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, |y_1(g_1(s))|)\, ds,$$

$$t \geqq T_{2,1}.$$

Define

$$v_1(t) = \sup_{T^*_{1,1} \leqq s \leqq t} \frac{|y_1(s)|}{\delta_1^1(s)} \quad \text{for} \quad t \geqq T^*_{1,1}.$$

Noting that the right-hand side of (18) is an increasing function of $t$ and using the inequality

$$\omega(s, ab) \leqq a\, \omega(s, b), \quad a \geqq 1, \quad b > 0, \tag{19}$$

which is a consequence of the sublinearity of (S), we obtain

$$\tfrac{3}{4} \delta_1^1(t)\, v_1(t) \leqq N\, \delta_1^1(t) \int_{T_{2,1}}^{t} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, v_1(g_1(s))\, ds + \tag{20}$$

$$+ N \int_t^{\infty} \prod_{j=1}^{n-1} \delta_1^j(s)\, v_1(g_1(s))\, \omega(s, \delta_1^1(g_1(s)))\, ds, \quad t \geqq T_{3,1}.$$

For each $t \geqq T_{1,1}$ we denote

$$\bar{R}_t^1 = \{s \in [T_{1,1}; \infty);\ g_1(s) \leqq t\}, \quad \bar{A}_t^1 = \{s \in [T_{1,1}; \infty);\ g_1(s) > t\}.$$

We then have

$$v_1(g_1(s)) \leqq v_1(t) \quad \text{for} \quad s \in \bar{R}_t^1,$$

$$\delta_1^1(g_1(s))\, v_1(g_1(s)) \leqq \sup_{\sigma \geqq t} [\delta_1^1(\sigma)\, v_1(\sigma)] \quad \text{for} \quad s \in \bar{A}_t^1.$$

From (20) we have

$$\tfrac{3}{4} \delta_1^1(t)\, v_1(t) \leqq N\, \delta_1^1(t)\, v_1(t) \left[ \int_{\bar{R}_t^1 \cap [T_{2,1}; t)} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, ds + \right.$$

$$+ \frac{1}{\delta_1^1(t)} \int_{R_t^1 \cap [t,\infty)} \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, ds \Bigg] +$$

$$+ N \sup_{s \geqq t} [\delta_1^1(s)\, v_1(s)] \Bigg[ \delta_1^1(t) \int_{A_t^1 \cap [T_{2,1}; t)} \frac{1}{\delta_1^1(g_1(s))} \prod_{j=2}^{n-1} \delta_1^j(s) \,.$$

$$. \,\omega(s, \delta_1^1(g_1(s)))\, ds + \int_{A_t^1 \cap [t,\infty)} \frac{1}{\delta_1^1(g_1(s))} \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, ds \Bigg] \leqq$$

$$\leqq \delta_1^1(t)\, v_1(t)\, N \int_{T_{2,1}}^{\infty} \prod_{j=2}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, ds +$$

$$+ \sup_{s \geqq t} [\delta_1^1(s)\, v_1(s)]\, N \int_{T_{2,1}}^{\infty} \frac{1}{\delta_1^1(g_1(s))} \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, \delta_1^1(g_1(s)))\, ds \leqq$$

$$\leqq \tfrac{1}{4}\, \delta_1^1(t)\, v_1(t) + \tfrac{1}{4} \sup_{s \geqq t} [\delta_1^1(s)\, v_1(s)]\,, \quad t \geqq T_{3,1}\,.$$

Thus we arrive at $0 < \sup_{s \geqq t} [\delta_1^1(s)\, v_1(s)] \leqq \frac{1}{2} \sup_{s \geqq t} [\delta_1^1(s)\, v_1(s)]$, $t \geqq T_{3,1}$; a contradiction. Further, we proceed in the same way as in the proof of the case when system (S) is superlinear. This completes the proof.

We now turn to an investigation of the behavior of oscillatory solutions of system (S).

Condition (G*): There exists a sequence $\{\bar{t}_n\}_{n=1}^{\infty}$ such that $\bar{t}_n \to \infty$ as $n \to \infty$ and

$$h^*(\bar{t}_n) = \bar{t}_n \text{ for } n = 1, 2, \ldots.$$

Condition ($H_*$): There exists a sequence $\{\tilde{t}_n\}_{n=1}^{\infty}$ such that $\tilde{t}_n \to \infty$ as $n \to \infty$ and

$$h_*(H(\tilde{t}_n)) = \tilde{t}_n \text{ for } n = 1, 2, \ldots, \text{ where } H(t) = \inf_{s \geqq t} g_n(s).$$

**Theorem 2.** (i) *Assume that* (S) *is superlinear,* $\beta = 1$ *and conditions* (1) *and* ($H_*$) *are satisfied. Then every oscillatory solution* $y(t)$ *of* (S) *fulfils Case* (I) *of Theorem* 1.

(ii) *Assume that* (S) *is sublinear and conditions* (2) *and* (G*) *are satisfied. Then every oscillatory solution* $y(t)$ *of* (S) *fulfils Case* (III) *of Theorem* 1 *with* $\alpha_n = 0$.

Proof. Let $y(t)$ be an oscillatory solution of (S) defined on $[\tau, \infty)$. Choose a $T \geqq \tau$ such that $h_*(T) \geqq \tau$ and $\delta_1^i(T) \leqq 1$, $i = 1, 2, \ldots, n-1$. Since the solution $y(t)$ is oscillatory, Case (II) and Case (III) (with $\alpha_n \neq 0$) of Theorem 1 can never occur, so that it must fulfil Case (I) or Case (III) of Theorem 1 with $\alpha_n = 0$.

(i) Consider the case where (S) is superlinear, $\beta = 1$ and conditions (1) and ($H_*$) are satisfied. Suppose that Case (III) with $\alpha_n = 0$ holds. Because ($12_i$) ($i = 1, 2, \ldots, n-2$), (16) and (17) hold we obtain

$$|y_1(t)| \leqq K \int_t^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2) \ldots$$
$$\ldots \int_{x_{n-2}}^\infty p_{n-1}(x_{n-1}) \int_{g_n x_{n-1}}^\infty \omega(s, |y_1(g_1(s))|)\, ds\, dx_{n-1} \ldots dx_1\,, \quad t \geqq T,$$

which implies

$$(21)\qquad \frac{|y_1(t)|}{\prod_{j=1}^{n-1}\delta_1^j(t)} \leqq K\int_{H(t)}^{\infty}\omega(s,|y_1(g_1(s))|)\,ds\,,\quad t\geqq T\,.$$

Let us put

$$v(t)=\sup_{s\geqq t}\frac{|y_1(s)|}{\prod_{j=1}^{n-1}\delta_1^j(s)}$$

and choose $t_1$ and $t_2$ such that

$$T<t_1<t_2\,,\quad \frac{|y_1(t)|}{\prod_{j=1}^{n-1}\delta_1^j(t)}\leqq 1\quad\text{for}\quad t\geqq t_1\quad\text{and}\quad h_*(H(t_2))\geqq t_1\,.$$

With the aid of (9) we derive from (21) the inequalities

$$v(t)\leqq K\int_{H(t)}^{\infty}\omega(s,v(g_1(s))\prod_{j=1}^{n-1}\delta_1^j(g_1(s)))\,ds\leqq$$
$$\leqq K\,v(h_*(H(t)))\int_{H(t)}^{\infty}\prod_{j=1}^{n-1}\delta_1^j(g_1(s))\,\omega(s,1)\,ds\,,\quad t\geqq t_2\,,$$

which implies

$$(22)\qquad \frac{v(t)}{v(h_*(H(t)))}\leqq K\int_{H(t)}^{\infty}\prod_{j=1}^{n-1}\delta_1^j(g_1(s))\,\omega(s,1)\,ds\,,\quad t\geqq t_2\,.$$

But this is a contradiction, because the right-hand side of (22) tends to zero as $t\to\infty$, while the left-hand side equals 1 along a sequence diverging to infinity by condition $(H_*)$.

(ii) Consider the case when (S) is sublinear and the conditions (2) and $(G^*)$ are satisfied. Suppose that Case (I) holds. We can select $t_3$, $t_4$ and $t_5$ in the following manner:

$$T<t_3<t_4<t_5\,,\quad t_3^*=h_*(t_3)\geqq T,\quad |y_1(t_3^*)|\geqq 1\,,$$
$$\sup_{t_3^*\leqq s\leqq t}|y_1(s)|=\sup_{t_4\leqq s\leqq t}|y_1(s)|\quad\text{for}\quad t\geqq t_4\,,$$
$$N\int_{t_4}^{\infty}\prod_{j=1}^{n-1}\delta_1^j(s)\,\omega(s,1)\,ds\leqq\tfrac{1}{4}\quad\text{and}$$
$$\sum_{j=1}^{n-1}|y_j(T)|\,\bar\delta_{j-1}^1(T)+M\,\bar\delta_{n-1}^1(T)+$$
$$+N\int_T^{t_4}\prod_{j=1}^{n-1}\delta_1^j(s)\,\omega(s,|y_1(g_1(s))|)\,ds\leqq\tfrac{1}{2}|y_1(t_5)|\,.$$

We define $u(t)=\sup_{t_3^*\leqq s\leqq t}|y_1(s)|$.

Using $u(t)$ and (19) in the inequality

$$|y_1(t)| \leqq \sum_{j=1}^{n-1} |y_j(T)| \,\bar{\delta}_{j-1}^1(T) + M\, \bar{\delta}_{n-1}^1(T) +$$

$$+ N \int_T^{t_4} \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, |y_1(g_1(s))|)\, \mathrm{d}s +$$

$$+ N \int_{t_4}^t \omega(s, |y_1(g_1(s))|) \prod_{j=1}^{n-1} \delta_1^j(s)\, \mathrm{d}s\,, \quad t \geqq t_4\,,$$

which follows from $(4_1)$, we find

$$u(t) \leqq \tfrac{1}{2}\, u(t) + N \int_{t_4}^t \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, u(g_1(s)))\, \mathrm{d}s\,, \quad t \geqq t_5$$

and hence

$$\text{(23)} \qquad \tfrac{1}{2}\, u(t) \leqq N \int_{t_4}^t u(g_1(s)) \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, 1)\, \mathrm{d}s \leqq$$

$$\leqq N\, u(h^*(t)) \int_{t_4}^t \prod_{j=1}^{n-1} \delta_1^j(s)\, \omega(s, 1)\, \mathrm{d}s\,, \quad t \geqq t_5\,.$$

From (23) we obtain

$$\frac{u(t)}{u(h^*(t))} \leqq \tfrac{1}{2} \quad \text{for} \quad t \geqq t_5\,.$$

Because of $(G^*)$ this is a contradiction. This completes the proof of Theorem 2.

**Theorem 3.** *Assume that*

$$\text{(24)} \qquad |f_n(t, x)| \leqq q(t)\, |x| \quad \textit{for} \quad (t, x) \in [a, \infty) \times R,$$

*where $q(t)$ is continuous and nonnegative on $[a, \infty)$. Assume moreover that conditions $(G^*)$ and $(H_*)$ hold and $\beta = 1$. If*

$$\text{(25)} \qquad \int^\infty \frac{\delta_1^i(g_*(t))}{\delta_1^i(g_1(t))} \prod_{j=1}^{n-1} \delta_1^j(g_*(t))\, q(t)\, \mathrm{d}t < \infty \quad \textit{for} \quad i = 1, 2, \ldots, n-1\,;$$

*then all solutions of* (S) *are weakly nonoscillatory.*

Proof. Suppose to the contrary that there exists an oscillatory solution $y(t)$ of (S). Since by (24) system (S) is both superlinear and sublinear, and since (25) is equivalent to (1) or (2), we can apply Theorem 2 to conclude that $y(t)$ fulfils both Case (I) and Case (III) (with $\alpha_n = 0$). But this is impossible, and so (S) has no oscillatory solutions.

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## Súhrn

## ASYMPTOTICKÉ VLASTNOSTI RIEŠENÍ DIFERENCIÁLNYCH SYSTÉMOV S POSUNUTÝMI ARGUMENTAMI

EVA ŠPÁNIKOVÁ

Cieľom článku je vyšetrovať asymptotické vlastnosti riešení nelineárneho systému diferenciálnych rovníc (S), ktorý je superlineárny alebo sublineárny a $\int^\infty p_i(t)\,dt < \infty$, $i = 1, 2, \ldots, n - 1$.

## Резюме

## АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

EVA ŠPÁNIKOVÁ

В этой статье рассматриваются асимптотические свойства решений суперлинейной или сублинейной нелинейной системы дифференциальных уравнений (S) и $\int^\infty p_i(t)\,dt < \infty$, $i = 1, 2, \ldots, n - 1$.

*Author's address:* VŠDS fakulta PEDaS, Marxa-Engelsa 15, 010 01 Žilina.