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# ON DISJOINT SUBSETS OF A COMPLETE LATTICE ORDERED GROUP

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*Summary.* In this paper we investigate a cardinal function $\nu$ which is defined on the class of all lattice ordered groups. It is proved that each complete lattice ordered group is isomorphic to a completely subdirect product of $\nu$-homogeneous lattice ordered groups.

*Keywords:* lattice ordered group, completely subdirect product, disjoint subset.

*AMS Subject Classification:* 06F15

For a lattice ordered group $G$ we denote by $vG$ the least cardinal $\alpha$ such that, whenever $A$ is a bounded disjoint subset of $G$, then card $A \leqq \alpha$. The lattice ordered group $G$ is said to be $v$-homogeneous if $vH = vG$ for each nonzero convex $l$-subgroup $H$ of $G$. (Cf. [8].)

Next, for $0 < x \in G$ let $v(x)$ be the least cardinal $\beta$ such that card $B \leqq \beta$ for each disjoint subset $B$ of the interval $[0, x]$ of $G$.

The notion of a completely subdirect product of lattice ordered groups was introduced in [7].

In the present paper the following results will be established.

(A) *For each infinite cardinal $\alpha$ there exists a complete lattice ordered group $G$ with $G \neq \{0\}$ such that $v(x) \geqq \alpha$ for each strictly positive element $x$ of $G$.*

(B) *Each complete lattice ordered group is isomorphic to a completely subdirect product of $v$-homogeneous lattice ordered groups.*

From (A) it follows that two results of [8] and [9] concerning completely subdirect product decompositions and ideal subdirect product decompositions of lattice ordered groups (cf. (C) and (D) below) are not correct.

## 1. PRELIMINARIES

We apply the standard notation for lattice ordered groups. (Cf., e.g., [1], [4].)

We recall some notation from [9].

Let $G$ be a lattice ordered group. Let $vG$ be defined as above. If $vG = \aleph_0$, then $G$ is said to be of countable type. $G$ is called continuous if for each $0 < x \in G$ there are nonzero elements $x_1$ and $x_2$ in $G$ such that $x = x_1 + x_2$ and $|x_1| \wedge |x_2| = 0$. Next, let $R$ (the real group) and $Z$ (the integer group) be the additive group of all reals or all integers, respectively, with the natural linear order.

The lattice ordered group $G$ is said to be an ideal subdirect product of lattice ordered groups $G_i$ ($i \in I$) if $G$ is a subdirect product of the system $\{G_i: i \in I\}$ (i.e., if $G$ is an $l$-subgroup of $\prod_{i\in I} G_i$ such that for each $i \in I$ and each $x^i \in G_i$ there is $x \in G$ with $x(i) = x^i$) and if, moreover, $G$ is an $l$-ideal of $\prod_{i\in I} G_i$.

It is clear that if $G$ is an ideal subdirect product of a system $\{G_i: i \in I\}$, then $G$ is a completely subdirect product of this system.

Let us remark that in [8] and [9] the terms subdirect sum (ideal subdirect sum, completely subdirect sum) are used instead of subdirect product (ideal subdirect product, completely subdirect product).

The following two assertions are contained in [8] and [9] (we modify the terminology according to the above remark).

(C) (Cf. [8], or a quotation in [9], p. 292.) *Each complete lattice ordered group is isomorphic to an ideal subdirect product of real groups, integer groups and continuous complete lattice ordered groups of countable type.*

(D) (Cf. [9], Theorem 3.2.) *Each $\sigma$-complete lattice ordered group is isomorphic to a completely subdirect sum of real groups, integer groups and continuous complete lattice ordered groups of countable type.*

The mapping $v: G \to vG$ is a cardinal function defined on the class of all lattice ordered groups. An analogously defined cardinal function concerning Boolean algebras was investigated by several authors (cf., e.g., [5], [6]).

## 2. AN EXAMPLE

Let $\beta$ be a non-limit cardinal, $\beta > \aleph_0$, $\beta = \alpha^+$. Let $I$ be a set with $\operatorname{card} I = \beta$. We put $G = \prod_{i\in I} G_i$, where $G_i = R$ for each $i \in I$.

For $g \in G$ we denote

$$\operatorname{Sup} g = \{i \in I: g(i) \neq 0\} .$$

Next, let $H$ be the set of all $g \in G$ with $\operatorname{Sup} g \leqq \alpha$. Thus $H$ is an $l$-ideal of $G$. Put $K = G/H$. The set of all positive integers will be denoted by $N$.

**2.1. Lemma.** *The lattice ordered group $K$ is archimedean.*

Proof. For $x \in G$ we denote $x + H = x^*$. By way of contradiction, assume that $K$ fails to be archimedean. Then there are $x, y \in G$ such that

(1) $$0^* < nx^* < y^*$$

for each $n \in N$. Without loss generality we can suppose that $0 < x < y$ is valid.

Put $I_0 = \{i \in I: x(i) > 0\}$. Since $x^* > 0^*$, we must have $\operatorname{card} I_0 = \beta$. For $i \in I_0$ let $m(i)$ be the least positive integer with $m(i)\,x(i) > y(i)$.

Let $n \in N$. We denote

$$I(n) = \{i \in I_0: m(i) = n\} .$$

If card $I(n) = \beta$, then $nx^* \nleqq y$, which contradicts (1). Thus we have

(2) $$\operatorname{card} I(n) \leqq \alpha \quad \text{for each} \quad n \in N\,.$$

If $n_1$ and $n_2$ are distinct positive integers, then $I(n_1) \cap I(n_2) = \emptyset$. Each $i \in I_0$ belongs to some $I(n)$, whence $I_0 = \cup_{n \in N} I(n)$. Thus in view of (2), card $I_0 \leqq \alpha$, which is a contradiction.

Let us remark that there exists an infinite number of $l$-ideals $H_j$ in $G$ such that $G/H_j$ is not archimedean.

**2.2. Lemma.** *Let $x^*$ be a strictly positive element of $K$. Then $v(x^*) \geqq \beta$.*

Proof. Without loss of generality we can assume that $x > 0$. Let $I_0$ be as in the proof of 2.1. Hence card $I_0 = \beta$. Thus there are subsets $I_j$ $(j \in J)$ of $I_0$ such that

$$\operatorname{card} J = \beta\,,$$

$$\operatorname{card} I_j = \beta \quad \text{for each} \quad j \in J\,,$$

and $I_{j(1)} \cap I_{j(2)} = 0$ whenever $j(1), j(2)$ are distinct elements of $J$.

For each $j \in J$ we define $x^j$ in $G$ as follows: $x^j(t) = x(t)$ if $t \in I_j$, and $x^j(t) = 0$ otherwise. Then $0^* < (x^j)^* < x^*$ for each $j \in J$, and

$$x^*_{j(1)} \wedge x^*_{j(2)} = 0^*$$

whenever $j(1)$ and $j(2)$ are distinct elements of $J$. Therefore $v(x^*) \geqq \beta$.

In view of 2.1 we can construct the Dedekind completion of the lattice ordered group $K$; let us denote it by $G_\beta$.

**2.3. Proposition.** *$G_\beta$ is a complete lattice ordered group and $G_\beta \neq \{0\}$. Let $y$ be a strictly positive element of $G_\beta$. Then $v(y) \geqq \beta$.*

Proof. The first and the second assertions are obvious. Consider the canonical embedding of $K$ into $G_\beta$. There exists $0 < x^* \in K$ such that $x^* \leqq y$. Thus in view of 2.2 we have $v(y) \geqq \beta$.

As a corollary we obtain that (A) is valid.

**2.4. Lemma.** *Let $T$ be a nonempty set, and for each $t \in T$ let $H_t$ be a lattice ordered group such that some of the following conditions is fulfilled:* (i) $H_t = R$; (ii) $H_t = Z$; (iii) *$H_t$ is of countable type and $H_t \neq \{0\}$. Assume that $G$ is a completely subdirect product of the system $\{H_t : t \in T\}$. Then for each $0 < g \in G$ there exists $0 < x \in G$ with $x \leqq y$ such that $v(x) \leqq \aleph_0$.*

Proof. Let $0 < g \in G$. There exists $t \in T$ such that $g(t) > 0$. Next, there exists $y \in G$ such that $y(t) = g(t)$ and $y(t') = 0$ for each $t' \in T \setminus \{t\}$. Then we have $v(y) \leqq \aleph_0$.

From 2.3 and 2.4 we obtain that neither (C) nor (D) hold.

## 3. DISJOINTNESS IN COMPLETE LATTICE ORDERED GROUPS

In this section we assume that $G$ is a complete lattice ordered group. For each $X \subseteq G$ we put

$$X^d = \{g \in G: |g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

It is well-known that in view of the completeness of $G$ we have a direct product decomposition

$$(1) \qquad G = X^d \times X^{dd}.$$

For $y \in G$ we denote by $[y]$ the convex $l$-subgroup of $G$ generated by the element $y$. A strictly positive element $x$ of $G$ is said to be basic if the interval $[0, x]$ of $G$ is a chain. Then (cf. [2]) $[x]$ is a linearly ordered group and the relation

$$(2) \qquad G = [x] \times [x]^d$$

is valid. If $g \in G$, then $g([x])$ will denote the component of $g$ in the direct factor $[x]$ (a similar notation will be applied also for other direct factors).

Let $B$ be the set of all basic elements of $G$. Next, let $A_i$ ($i \in I(1)$) be the set of all nonzero convex $l$-subgroups of $G$ which are linearly ordered. For each $i \in I(1)$ there is $x \in B$ with $A_i = [x]$; hence in view of (2), $A_i$ is a direct factor of $G$.

Put $H_1 = B^{dd}$ and $H_2 = B^d$. Assume that $H_1 \neq \{0\}$. Then $I(1) \neq \emptyset$ and $[x] \subseteq H_1$ for each $x \in B$. Let $h_1 \in H_1$; we denote by $\varphi(h_1)$ the element of $\prod_{i \in I(1)} A_i$ such that $\varphi(h_1)(i) = h_1(A_i)$ for each $i \in I(1)$. The mapping $\varphi$ is a homomorphism of $H_1$ into $\prod_{i \in I(1)} A_i$. If $\varphi(h_1) = 0$, then $|h_1| \wedge x = 0$ for each $x \in B$, thus $h_1 \in B^d$, and therefore $h_1 = 0$. Hence $\varphi$ is an isomorphism of $H_1$ into $\prod_{i \in I(1)} A_i$. Next, for each $i \in I(1)$ and each $z \in A_i$ we have $z \in H_1$, $\varphi(z)(A_i) = z$ and $\varphi(z)(A_j) = 0$ whenever $j \in I(1) \setminus \{i\}$. Thus we obtain

**3.1. Lemma.** *Let $H_1 \neq \{0\}$. Then $H_1$ is isomorphic to a completely subdirect product of linearly ordered groups $A_i$ ($i \in I(1)$).*

Now let us deal with $H_2$ (if $H_1 = \{0\}$, then $H_2 = G$). In view of (1),

$$(3) \qquad G = H_1 \times H_2.$$

The case $H_2 = \{0\}$ would be trivial for the above consideration; thus let us assume that $H_2 \neq \{0\}$. Next, since $H_2$ has no basic element, for each strictly positive element $x$ of $G$ the relation

$$(4) \qquad v(x) \geqq \aleph_0$$

is valid.

Let $[h, h']$ be an interval of $H_2$ and let $C \subseteq [h, h']$. If $c_1 \wedge c_2 = h$ whenever $c_1$ and $c_2$ are distinct elements of $C$, then $C$ will be said to be disjoint in $[h, h']$. Let us denote by $v[h, h']$ the least cardinal $\gamma$ such that card $C \leqq \gamma$ whenever $C$ is a disjoint subset of $[h, h']$. If $0 < x \in G$, then clearly $v(x) = v[0, x]$.

**3.2. Lemma.** *Let $x$ and $y$ be strongly positive elements of $H_2$ such that $v[0, x] = v[0, y]$. Then $v[0, x + y] = v[0, x]$.*

Proof. Let $\{z_i\}_{i \in I}$ be a disjoint subset of $[0, x + y]$. Let $i \in I$. There are $x_i \in [0, x]$ and $y_i \in [0, y]$ with $z_i = x_i + y_i$. Put

$$I_1 = \{i \in I: x_i \neq 0\} , \quad I_2 = I \setminus I_1 .$$

If $i$ and $j$ are distinct elements of $I_1$, then $x_i \wedge x_j = 0$, hence $\operatorname{card} I_1 \leqq v[0, x]$. Next, if $i$ and $j$ are distinct elements of $I_2$, then $y_i > 0$, $y_j > 0$ nd $y_i \wedge y_j = 0$, thus $\operatorname{card} I_2 \leqq v[0, y]$. Therefore in view of (4),

$$\operatorname{card} I = \operatorname{card} I_1 + \operatorname{card} I_2 \leqq v[0, x] .$$

Hence $v[0, x + y] \leqq v[0, x]$. On the other hand, we obviously have $v[0, x] \leqq v[0, x + y]$.

**3.3. Lemma.** *Let $t_1, t_2, t_3, \ldots \in G$, $0 < t_1 \leqq t_2 \leqq \ldots$, $vt_n = t$, $v[0, t_n] = v[0, t_1]$ for each $n \in N$. Then $v[0, t] = v[0, t_1]$.*

Proof. Let $\{z_i\}_{i \in I}$ be a disjoint subset of $[0, t]$. Let $i \in I$. There exists the first ositive integer $n$ such that $z_i \wedge t_n > 0$; we denote this $n$ by $n(i)$. Next, for each $n \in N$ we put

$$I(n) = \{i \in I: n(i) = n\} .$$

Then we have

$$I = \cup_{n \in N} I(n) . \tag{5}$$

Let $n \in N$ be fixed. If $i(1)$ and $i(2)$ are distinct elements of $I(n)$, then $0 < z_{i(1)} \wedge t_n$ $0 < z_{i(2)} \wedge t_n$ and $(z_{i(1)} \wedge t_n) \wedge (z_{i(2)} \wedge t_n) = 0$. Thus $\operatorname{card} I(n) \leqq v[0, t_n] = v[0, t_1]$. Therefore in view of (4) and (5) we obtain $\operatorname{card} I \leqq v[0, t_1]$ and hence $v[0, t] \leqq v[0, t_1]$. On the other hand, the relation $t_1 \leqq t$ yields $v[0, t_1] \leqq v[0, t]$.

**3.4. Lemma.** *Let $H'$ be a convex l-subgroup of $H$, $H' \neq \{0\}$. Then the following conditions are equivalent:*

(i) *$H'$ is v-homogeneous.*

(ii) *If $x$ and $y$ are strictly positive elements of $H'$, then $v[0, x] = v[0, y]$.*

Proof. This is an immediate consequence of the definition of $v[p, q]$ for $p, q \in H_2$ $p < q$.

From 3.2, 3.3, 3.4 and from [3] (Theorem 1.21) we obtain

**3.5. Lemma.** *The lattice ordered group $H_2$ is a completely subdirect product of v-homogeneous lattice ordered groups.*

Since each linearly ordered group is $v$-homogeneous, Lemmas 3.1, 3.5 and the relation (3) yield that Theorem (B) above is valid.

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## Súhrn


### O DISJUNKTNÝCH PODMNOŽINÁCH ÚPLNEJ ZVÄZOVO USPORIADANEJ GRUPY

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V článku sa vyšetruje kardinálna funkcia $\nu$ definovaná na triede všetkých zväzovo usporiadaných grúp. Dokazuje sa, že každá úplná zväzovo usporiadaná grupa je úplne polopriamym súčinom $\nu$-homogénnych zväzovo usporiadaných grúp.


## Резюме


### О НЕПРЕСЕКАЮЩИХСЯ ПОДМНОЖЕСТВАХ ПОЛНОЙ РЕШЕТОЧНО УПОРЯДОЧЕННОЙ ГРУППЫ

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В статье рассматривается кардинальная функция $\nu$, определенная на классе всех решеточно упорядоченных групп. Доказано, что каждая полная решеточно упорядоченная группа изоморфна вполне полупрямому произведению $\nu$-однородных решеточно упорядоченных групп.



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