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THIRD BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION II

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The present paper is a continuation of the paper [I] where only a special type of the third boundary value problem was investigated. We shall keep the notation introduced in [I]. We also continue the numbering of the paragraphs and formulas. Thus we refer to items in [I] by writing simply (0.1) instead of (0.1) in [I], and so on.

3. THE OPERATORS L, A

As well as in the preceding part let φ be a continuous function on an interval $\langle a, b\rangle$; the sets E, K are defined by (0.1), (0.2). For $\mu \in \mathscr{B}'(\langle a, b\rangle)$ the heat potential $U_\mu = U_\mu^\varphi$ is defined by (0.3). Further, let G^* stand for the adjoint heat kernel, that is, $G^*(x, t) = G(x, -t)$ ($[x, t] \in R^2$). Define the adjoint heat potential U_μ^* of a measure $\mu \in \mathscr{B}'$ by the equality

$$U_\mu^*(x, t) = U_\mu^{*\varphi}(x, t) = \int_a^b G^*(x - \varphi(\tau),\ t - \tau)\,\mathrm{d}\mu(\tau)$$

(for $[x, t] \in R^2$ for which this integral exists). Note that if μ, λ are two measures from $\mathscr{B}'(\langle a, b\rangle)$ such that either both μ and λ are non-negative or the integral

$$\int_a^b U_{|\mu|}(\varphi(t), t)\,\mathrm{d}|\lambda|\,(t)$$

converges, then by the Fubini theorem

$$\int_a^b U_\mu(\varphi(t), t)\,\mathrm{d}\lambda(t) = \int_a^b \left(\int_a^b G(\varphi(t) - \varphi(\tau),\ t - \tau)\,\mathrm{d}\mu(\tau)\right)\mathrm{d}\lambda(t) =$$

$$= \int_a^b \left(\int_a^b G(\varphi(t) - \varphi(\tau),\ t - \tau)\,\mathrm{d}\lambda(t)\right)\mathrm{d}\mu(\tau) = \int_a^b U_\lambda^*(\varphi(\tau), \tau)\,\mathrm{d}\mu(\tau)\,.$$

Let $\lambda \in \mathscr{B}_0'(\langle a, b\rangle)$ be fixed.

For $\mu \in \mathscr{B}'(\langle a, b \rangle)$ let

$$\mathscr{D}(\mu) = \left\{\psi \in \mathscr{D}_b;\ \int_a^b |\psi(\varphi(t), t)|\ U_{|\mu|}(\varphi(t), t)\ \mathrm{d}|\lambda|\ (t) < \infty\right\}. \tag{3.1}$$

On $\mathscr{D}(\mu)$ we define a functional L_μ:

$$\langle \psi, L_\mu \rangle = \int_a^b \psi(\varphi(t), t)\ U_\mu(\varphi(t), t)\ \mathrm{d}\lambda(t)\,, \quad (\psi \in \mathscr{D}(\mu))\,. \tag{3.2}$$

Further, let $A_\mu = H_\mu + L_\mu$. that is

$$\langle \psi, A_\mu \rangle = \langle \psi, H_\mu \rangle + \langle \psi, L_\mu \rangle\,, \quad \psi \in \mathscr{D}(\mu)\,. \tag{3.3}$$

According to the introductory remarks the functional A_μ can be regarded as a weak characterization of the term

$$\frac{\partial U_\mu}{\partial x} + U_\mu(\lambda_0 + \lambda)$$

on K ($\mathrm{d}\lambda_0(t) = \mathrm{d}\varphi(t)$ – provided the function φ is of bounded variation on $\langle a, b \rangle$).

The following assertions are analogous to the relevant assertions from [32]. The proofs of these assertions are also quite similar to the proofs of the assertions in [32] (but [32] deals with sets in R^{n+1} of the form $D \times (\tau_1, \tau_2)$, $D \subset R^n$, instead of the set E).

3.1. Proposition. *The following two conditions are equivalent to each other*:

(i) *For each $\mu \in \mathscr{B}'_0$ there is a unique linear extension of the functional A_μ from $\mathscr{D}(\mu)$ onto the whole $\mathscr{D}_b$.*

(ii) *The potential $U^*_{|\lambda|}$ is bounded on any compact set contained in the set*

$$K_0 = \{[\varphi(t), t];\ t \in \langle a, b)\}\,. \tag{3.4}$$

If one of the conditions (i), (ii) *is fulfilled then $\mathscr{D}(\mu) = \mathscr{D}_b$ for each $\mu \in \mathscr{B}'_0$.*

Proof. Let $\mu \in \mathscr{B}'_0$, $t \in \langle a, b)$. If $\psi(\varphi(t), t) = 0$ for each $\psi \in \mathscr{D}(\mu)$ then

$$\langle \psi, A_\mu \rangle = \langle \psi, A_\mu \rangle + \psi(\varphi(t), t) \quad \text{for} \quad \psi \in \mathscr{D}(\mu)\,.$$

Hence if the condition (i) is fulfilled then it cannot happen that $\psi(\varphi(t), t) = 0$ for each $\psi \in \mathscr{D}(\mu)$. It is thus seen that under the condition (i) for each $\mu \in \mathscr{B}'_0$ and each $t \in \langle a, b)$ there are $\delta > 0$, $\psi \in \mathscr{D}(\mu)$ such that $\psi(\varphi(\tau), \tau) \geqq 1$ for each $\tau \in (t - \delta, t + \delta) \cap \langle a, b)$ and, moreover, if $M \subset \langle a, b)$ is a compact set then there is a $\psi \in \mathscr{D}(\mu)$ such that $\psi(\varphi(\tau), \tau) \geqq 1$ for $\tau \in M$.

Suppose now that the condition (ii) is not fulfilled. Then there are a compact set $M \subset \langle a, b)$ and points $t_i \in M$ $(i = 1, 2, \ldots)$ such that

$$U^*_{|\lambda|}(\varphi(t_i), t_i) \geqq 2^i + 1\,, \quad (i = 1, 2, \ldots)\,. \tag{3.5}$$

There is a point $t_0 < b$, $t_0 > \sup M$ such that

$$[\pi(t_0 - \sup M)]^{-1/2}\, |\lambda|\, (\langle t_0, b \rangle) \leqq 1 \tag{3.6}$$

$(\lambda(\{b\}) = 0$ as $\lambda \in \mathscr{B}'_0$ by the assumption). Put

$$\lambda_1 = |\lambda|\, |_{\langle a, t_0 \rangle}$$

(the restriction of $|\lambda|$ on to the interval $\langle a, t_0 \rangle$). Since

$$G(x, t) \leqq (\pi t)^{-1/2}\,, \quad (t > 0)\,,$$

we get from (3.5), (3.6) that

$$U^*_{\lambda_1}(\varphi(t_i), t_i) \geqq 2^i\,. \tag{3.7}$$

Consider now the measure

$$\mu = \sum_{i=1}^{\infty} 2^{-i} \delta_{t_i}\,.$$

Then $\mu \in \mathscr{B}'_0$, spt $\mu \subset \langle a, t_0 \rangle$ and according to the preceding consideration there is a function $\psi \in \mathscr{D}(\mu)$ such that $\psi(\varphi(\tau), \tau) \geqq 1$ for each $\tau \in \langle a, t_0 \rangle$. For this function ψ we have

$$\int_a^b |\psi(\varphi(t), t)|\, U_\mu(\varphi(t), t)\, \mathrm{d}|\lambda|\, (t) \geqq \int_a^b U_\mu(\varphi(t), t)\, \mathrm{d}\lambda_1(t) =$$

$$= \int_a^b U^*_{\lambda_1}(\varphi(t), t)\, \mathrm{d}\mu(t) \geqq \sum_{i=1}^{\infty} 2^{-i}\, U^*_{\lambda_1}(\varphi(t_i), t_i) = +\infty\,,$$

which means $\psi \notin \mathscr{D}(\mu)$ – a contradiction. We have thus proved that the condition (i) implies (ii).

Suppose now that the condition (ii) is fulfilled. Then for each $\psi \in \mathscr{D}_b$ and each $\mu \in \mathscr{B}'_0$ we have

$$\int_a^b |\psi(\varphi(t), t)|\, U_{|\mu|}(\varphi(t), t)\, \mathrm{d}|\lambda|\, (t) =$$

$$= \int_a^b \left(\int_a^b G(\varphi(t) - \varphi(\tau), t - \tau)\, |\psi(\varphi(t), t)|\, \mathrm{d}|\lambda|\, (t) \right) \mathrm{d}|\mu|\, (\tau)\,.$$

For $\tau > t_0 = \sup \{t \in \langle a, b \rangle;\, [\varphi(t), t] \in \operatorname{spt} \psi\}$ we have

$$\int_a^b G(\varphi(t) - \varphi(\tau), t - \tau)\, |\psi(\varphi(t), t)|\, \mathrm{d}|\lambda|\, (t) = 0$$

and hence (as $t_0 < b$; we suppose furthermore that $t_0 > a$ since else there is nothing to prove)

$$\int_a^b |\psi(\varphi(t), t)\, U_{|\mu|}(\varphi(t), t)\, \mathrm{d}|\lambda|\,(t) \leqq \|\psi\| \int_a^{t_0} U^*_{|\lambda|}(\varphi(\tau), \tau)\, \mathrm{d}|\mu|\,(\tau) \leqq$$
$$\leqq \|\psi\|\, \|\mu\| \sup \{U^*_{|\lambda|}(x, t);\, [x, t] \in K,\ t \leqq t_0\} < \infty\,.$$

It follows that $\mathscr{D}(\mu) = \mathscr{D}_b$ and the condition (i) is fulfilled.

3.2. Lemma. *There is a number $\gamma > 0$ with the following property:*

For each $\tau_0 < b$ there is a function $\psi_{\tau_0} \in \mathscr{D}_b$ such that $0 \leqq \psi_{\tau_0} \leqq 1$ in R^2, $\psi_{\tau_0} = 1$ on $K \cap R_{\tau_0}$ and

(3.8) $$|\langle \psi_{\tau_0}, H_{\delta_\tau} \rangle| \leqq \gamma$$

for each $\tau \in \langle a, b)$.

Proof. Fix $\tau_0 \in \langle a, b)$.

Let $\psi_1 : R^1 \to R^1$ be an infinitely differentiable function with compact support such that $0 \leqq \psi_1 \leqq 1$ on R^1, $\operatorname{spt} \psi_1 \subset (-\infty, b)$, $\psi_1' \leqq 0$ on (τ_0, b), $\psi_1 = 1$ on $\langle a, \tau_0 \rangle$. Choose $\varrho > \sup \{|\varphi(t)|;\, t \in \langle a, b \rangle\}$ and let $\psi_2 : R^1 \to R^1$ be an infinitely differentiable function with compact support such that $0 \leqq \psi_2 \leqq 1$ and $|\psi_2'| \leqq 1$ on R^1, $\psi_2 = 1$ on $\langle -\varrho, \varrho \rangle$. Now define a function ψ_{τ_0}:

$$\psi_{\tau_0}(x, t) = \psi_1(t)\, \psi_2(x)\,, \quad ([x, t] \in R^2)\,.$$

Then $\psi_{\tau_0} \in \mathscr{D}_b$, $\psi_{\tau_0} = 1$ on $K \cap R_{\tau_0}$.

Let $\tau \in \langle a, b)$ be arbitrary. Then

(3.9) $$\langle \psi_{\tau_0}, H_{\delta_\tau} \rangle =$$
$$= \left| \iint_E \left(\frac{\partial G}{\partial x}(x - \varphi(\tau), t - \tau) \frac{\partial}{\partial x} \psi_{\tau_0}(x, t) - G(x - \varphi(\tau), t - \tau) \frac{\partial}{\partial t} \psi_{\tau_0}(x, t) \right) \mathrm{d}x\, \mathrm{d}t \right| \leqq$$
$$\leqq \iint_E \left| \frac{\partial G}{\partial x}(x - \varphi(\tau), t - \tau) \right| \psi_1(t)\, |\psi_2'(x)|\, \mathrm{d}x\, \mathrm{d}t +$$
$$+ \iint_E G(x - \varphi(\tau), t - \tau)\, \psi_2(x)\, |\psi_1'(t)|\, \mathrm{d}x\, \mathrm{d}t = I_1 + I_2\,.$$

As $|\psi_2'| \leqq 1$, $\psi_1 \leqq 1$ we get

(3.10) $$I_1 \leqq \iint_{R_{ab}} \left| \frac{\partial G}{\partial x}(x - \varphi(\tau), t - \tau) \right| \mathrm{d}x\, \mathrm{d}t \leqq \frac{4}{\sqrt{\pi}} \sqrt{(b - a)}$$

according to (0.5).

Further, we have (denoting $E_\tau = E - \bar{R}_\tau$)

$$I_2 = \iint_{E_\tau} \frac{1}{\sqrt{(\pi(t-\tau))}} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) \psi_2(x) \, |\psi_1'(t)| \, \mathrm{d}x \, \mathrm{d}t \,.$$

Putting $\tau_1 = \sup \{t; \psi_1(t) \neq 0\}$ we conclude that $I_2 = 0$ if $\tau \geqq \tau_1$. Suppose now that $\tau < \tau_1$. Then

$$I_2 \leqq \int_\tau^{\tau_1} \frac{|\psi_1'(t)|}{\sqrt{(\pi(t-\tau))}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) \mathrm{d}x \, \mathrm{d}t \,.$$

Using the substitution

$$\frac{x - \varphi(\tau)}{2\sqrt{(t-\tau)}} = z$$

we get

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) \mathrm{d}x = 4\sqrt{(t-\tau)} \int_0^\infty e^{-z^2} \, \mathrm{d}z = 2\sqrt{(\pi(t-\tau))}$$

and hence (for $\psi_1' \leqq 0$ on $\langle a, b\rangle$)

(3.11)
$$I_2 \leqq 2 \int_\tau^{\tau_1} |\psi_1'(t)| \, \mathrm{d}t \leqq 2 \,.$$

Now it suffices to put

$$\gamma = 2 + \frac{4}{\sqrt{\pi}} \sqrt{(b-a)}$$

and (3.8) follows from (3.9), (3.10), (3.11).

3.3. We shall denote

$$\mathbf{m}_\lambda^* = \sup \{U_{|\lambda|}^*(x, t); [x, t] \in K_0\}$$

(where K_0 is defined by (3.4).

3.4. Theorem. *Suppose that λ is a non-negative measure (in $\mathscr{B}_0'$). Then the following two conditions are equivalent to each other:*

(i) *For each $\mu \in \mathscr{B}_0'$ there is a unique measure $\nu_\mu \in \mathscr{B}_0'$ which represents the functional A_μ in the sense that*

$$\langle \psi, A_\mu \rangle = \langle f_\psi, \nu_\mu \rangle \,,$$

$\psi \in \mathscr{D}(\mu)$, $f_\psi(t) = \psi(\varphi(t), t)$ $(t \in \langle a, b\rangle)$.

(ii)
$$\tilde{V}_K + \mathbf{m}_\lambda^* < \infty$$

(*where* $\tilde{V}_K = \sup \{\tilde{v}(\varphi(\tau), \tau); \tau \in \langle a, b\rangle\}$).

Proof. Suppose that (i) is fulfilled. Then for each $\mu \in \mathscr{B}'_0$ the functional A_μ has a unique linear extension from $\mathscr{D}(\mu)$ on to $\mathscr{D}_b$, which means (according to Proposition 3.1) that $\mathscr{D}(\mu) = \mathscr{D}_b$ for each $\mu \in \mathscr{B}'_0$. First, we show that

$$\mathbf{m}^*_\lambda < +\infty . \tag{3.12}$$

Suppose that $\mathbf{m}^*_\lambda = +\infty$. Then there are points $t_i \in \langle a, b)$ such that

$$U^*_\lambda(\varphi(t_i), t_i) > 2^i , \quad (i = 1, 2, \ldots) .$$

Consider a measure

$$\mu = \sum_{i=1}^{\infty} 2^{-i} \delta_{t_i} ;$$

we have $\mu \in \mathscr{B}'_0$. For $\tau < b$ let

$$\lambda_\tau = \lambda|_{\langle a,\tau\rangle} .$$

Given n natural there is a $\tau_0 < b$ such that

$$U^*_{\lambda_{\tau 0}}(\varphi(t_i), t_i) > 2^i \quad \text{for each} \quad i = 1, 2, \ldots, n .$$

Let ψ_{τ_0} be the function from Lemma 3.2. As $\psi_{\tau_0} = 1$ on $K \cap R_{\tau_0}$ we have

$$\int_a^b \psi_{\tau 0}(\varphi(t), t)\, U_\mu(\varphi(t), t)\, \mathrm{d}\lambda(t) \geqq \int_a^b U_\mu(\varphi(t), t)\, \mathrm{d}\lambda_{\tau 0}(t) =$$

$$= \int_a^b U^*_{\lambda_{\tau 0}}(\varphi(t), t)\, \mathrm{d}\mu(t) = \sum_{i=1}^{\infty} 2^{-i}\, U^*_{\lambda_{\tau 0}}(\varphi(t_i), t_i) \geqq n .$$

Since by lemma 3.2

$$|\langle \psi_{\tau_0}, H_{\delta_{t_i}} \rangle| \leqq \gamma ,$$

which implies

$$|\langle \psi_{\tau_0}, H_\mu \rangle| \leqq \gamma \sum_{i=1}^{\infty} 2^{-i} = \gamma ,$$

we have

$$|\langle \psi_{\tau_0}, A_\mu \rangle| \geqq n - \gamma .$$

As $|\psi_{\tau_0}| \leqq 1$ and the number n was arbitrary, it is seen that the functional A_μ is not bounded which means that there is no element from $\mathscr{B}'_0$ representing (in the mentioned sense) the functional A_μ. We see now that if the condition (i) is fulfilled then necessarily (3.12) is valid.

If (3.12) holds then the functional L_μ can be represented by a (unique) measure from $\mathscr{B}'_0$ since in this case

$$|\langle \psi, L_\mu \rangle| \leqq \|\psi\|\, \|\mu\|\, \mathbf{m}^*_\lambda .$$

As $A_\mu = L_\mu + H_\mu$, it follows that then the functional A_μ can be represented by a unique measure from $\mathscr{B}'_0$ if and only if H_μ can. But according to Theorem 2.3 the functional H_μ can be represented by a unique element from $\mathscr{B}'_0$ if and only if the condition $\tilde{V}_K < \infty$ is fulfilled. Hence we obtain the assertion.

4. THE OPERATOR V

For $f \in \mathscr{C}(\langle a, b \rangle)$ let $f . \lambda$ be the product of the function f and the measure λ. For $f \in \mathscr{C}(\langle a, b \rangle)$, $t \in \langle a, b \rangle$ we define

$$(4.1) \qquad Vf(t) = U^*_{f.\lambda}(\varphi(t), t) = \int_a^b f(\tau)\, G^*(\varphi(t) - \varphi(\tau), t - \tau)\, \mathrm{d}\lambda(\tau) =$$

$$= \int_t^b f(\tau) \frac{1}{\sqrt{(\pi(\tau - t))}} \exp\left(-\frac{(\varphi(t) - \varphi(\tau))^2}{4(\tau - t)}\right) \mathrm{d}\lambda(\tau)$$

provided the integrals exist.

If, for instance, the restriction $U^*_\lambda|_K$ is continuous (on K) then for each $f \in \mathscr{C}(\langle a, b \rangle)$ we have $Vf \in \mathscr{C}^\sim_0(\langle a, b \rangle)$ and one can regard V as an operator on $\mathscr{C}(\langle a, b \rangle)$ or as an operator on $\mathscr{C}^\sim_0(\langle a, b \rangle)$ ($V : \mathscr{C} \to \mathscr{C}$, resp. $V : \mathscr{C}^\sim_0 \to \mathscr{C}^\sim_0$).

By the equality (4.1) one can also define Vf for any bounded Baire function f on $\langle a, b \rangle$ and then V can be regarded as an operator on the set of all bounded Baire functions ob $\langle a, b \rangle$ (in this case even supposing only $\mathbf{m}^*_\lambda < \infty$).

4.1. Proposition. *Suppose that $\lambda \in \mathscr{B}'_0$ is a non-negative measure. Then the following two conditions are equivalent to each other:*

(i) *$Vf \in \mathscr{C}^\sim_0(\langle a, b \rangle)$ for each $f \in \mathscr{C}^\sim_0(\langle a, b \rangle)$.*

(ii) *The restriction $U^*_\lambda|_{K_0}$ is continuous and bounded (on K_0).*

Proof. Let (i) be valid and suppose that the restriction $U^*_\lambda|_{K_0}$ is discontinuous at a point $[\varphi(t_0), t_0]$ (where $t_0 \in \langle a, b))$. Choose a point $t' \in (t_0, b)$ such that $\lambda(\{t'\}) = 0$ and put

$$\lambda_1 = \lambda|_{\langle a, t' \rangle}, \quad \lambda_2 = \lambda|_{\langle t', b \rangle}.$$

Let $f \in \mathscr{C}^\sim_0(\langle a, b \rangle)$ be such that $f = 1$ on $\langle a, t' \rangle$. The restriction $U^*_{\lambda_2}|_K$ is certainly continuous at the point $[\varphi(t_0), t_0]$ (for $t_0 \notin \operatorname{spt} \lambda_2$) and $U_{f.\lambda_2}|_K$ is continuous at the point $[\varphi(t_0), t_0]$ as well. As $U^*_\lambda|_K$ is dicontinuous at the point $[\varphi(t_0), t_0]$ and

$$U^*_\lambda = U^*_{\lambda_1} + U^*_{\lambda_2},$$

the restriction $U^*_{\lambda_1}|_K$ is not continuous at the point $[\varphi(t_0), t_0]$. But (as $f = 1$ on $\langle a, t' \rangle$)

$$Vf(t) = U^*_{f.\lambda}(\varphi(t), t) = U^*_{\lambda_1}(\varphi(t), t) + U^*_{f.\lambda_2}(\varphi(t), t)$$

and so Vf is discontinuous at t_0 which contradicts (i). We conclude that under the condition (i) the restriction $U_\lambda^*|_{K_0}$ is continuous on K_0.

Suppose now that the potential U_λ^* is not bounded on K_0. Construct a sequence of points $t_i \in (a, b)$ as follows. Suppose that we have already a point $t_i \in (a, b)$ such that

$$U_\lambda^*(\varphi(t_i), t_i) > 2^{i+1} + 1 .$$

Choose then a point $t_{i+1} \in (a, b)$, $t_{i+1} > t_i$ such that

$$U_\lambda^*(\varphi(t_{i+1}), t_{i+1}) > 2^{i+2} + 1$$

and at the same time

$$\int_{t_{i+1}}^b G^*(\varphi(t_i) - \varphi(\tau), t_i - \tau)\, \mathrm{d}\lambda(\tau) < 1 .$$

There certainly exists such a point t_{i+1}, due to the fact that $\lambda(\{b\}) = 0$ and that by the preceding the restriction $U_\lambda^*|_{K_0}$ is continuous and so it can be unbounded only in a neighbourhood of the point $[\varphi(b), b]$. Moreover, it is ceen that for this sequence of points t_i we have $t_i \to b$ for $i \to +\infty$. Consider now a function $f \in \mathscr{C}_0^{\sim}(\langle a,b\rangle)$ with $f(a) = f(b) = 0$, $f(t_i) = 2^{-i}$ $(i = 1, 2, \ldots)$ and such that f is linear on the intervals $\langle a, t_1\rangle$, $\langle t_i, t_{i+1}\rangle$ $(i = 1, 2, \ldots)$. Then (for each i natural)

$$Vf(t_i) = \int_{t_i}^b f(\tau)\, G^*(\varphi(t_i) - \varphi(\tau),\ t_i - \tau)\, \mathrm{d}\lambda(\tau) \geqq$$

$$\geqq 2^{-(i+1)} \int_{t_i}^{t_{i+1}} G^*(\varphi(t_i) - \varphi(\tau),\ t_i - \tau)\, \mathrm{d}\lambda(\tau) =$$

$$= 2^{-(i+1)} \left\{ \int_{t_i}^b G^*(\varphi(t_i) - \varphi(\tau), t_i - \tau)\, \mathrm{d}\lambda(\tau) - \int_{t_{i+1}}^b G^*(\varphi(t_i) - \varphi(\tau), t_i - \tau)\, \mathrm{d}\lambda(\tau) \right\} >$$

$$> 2^{-(i+1)} (2^{i+1} + 1 - 1) = 1 .$$

Hence $Vf \notin \mathscr{C}_0^{\sim}(\langle a, b\rangle)$ and it is seen that (ii) follows from (i) indeed.

Suppose now that the condition (ii) is fulfilled and that λ is not the zero measure (or else there is nothing to prove). Then $\mathbf{m}_\lambda^* > 0$. Let $f \in \mathscr{C}_0^{\sim}(\langle a, b\rangle)$. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(t)| < \frac{\varepsilon}{2\mathbf{m}_\lambda^*}$$

for each $t \in \langle b - \delta, b\rangle$; furthermore, choose δ such that $\lambda(\{b - \delta\}) = 0$. Putting

$$\lambda_1 = \lambda|_{\langle a,b-\delta\rangle}\,, \quad \lambda_2 = \lambda|_{\langle b-\delta,b\rangle}$$

we have

$$Vf(t) = U_{f.\lambda_1}^*(\varphi(t), t) + U_{f.\lambda_2}^*(\varphi(t), t)$$

for $t \in \langle a, b\rangle$. The restriction $U^*_{f.\lambda_1}|_K$ is continuous on K (since $b \notin \operatorname{spt} \lambda_1$). Further,

$$\left|U^*_{f.\lambda_2}(\varphi(t), t)\right| \leqq \frac{\varepsilon}{2\mathbf{m}^*_\lambda} U^*_\lambda(\varphi(t), t) \leqq \frac{\varepsilon}{2}$$

for any $t \in \langle a, b\rangle$. Hence it is seen that Vf is continuous on $\langle a, b\rangle$ (and, of course, $Vf(b) = 0$). So the condition (i) is fulfilled.

4.2. Remark. Let $\lambda \in \mathscr{B}'_0$ be such that the restriction $U^*_{|\lambda|}|_{K_0}$ is continuous and bounded on K_0. Then also the restrictions $U^*_{\lambda^+}|_{K_0}$, $U^*_{\lambda^-}|_{K_0}$ are continuous and bounded on K_0. Further, $\mathbf{m}^*_\lambda < \infty$. It is seen from the proof of Theorem 3.4 that then for each $\mu \in \mathscr{B}'_0$ the functionl L_μ can be represented by a (unique) measure from $\mathscr{B}'_0$. Especially, the functional L_μ can be then regarded as a functional on $\mathscr{C}^\sim_0(\langle a, b\rangle)$. The Fubini theorem yields that for $\mu \in \mathscr{B}'_0$, $f \in \mathscr{C}^\sim_0$

$$\langle Vf, \mu\rangle = \int_a^b \int_a^b f(\tau)\, G^*(\varphi(t) - \varphi(\tau), t - \tau)\, \mathrm{d}\lambda(\tau)\, \mathrm{d}\mu(t) = \tag{4.2}$$

$$= \int_a^b f(\tau)\, U_\mu(\varphi(\tau), \tau)\, \mathrm{d}\lambda(\tau) = \langle f, L_\mu\rangle\,.$$

Regarding now L as an operator on $\mathscr{B}'_0$ ($L : \mu \mapsto L_\mu$; $L : \mathscr{B}'_0 \to \mathscr{B}'_0$) and V as an operator on $\mathscr{C}^\sim_0$ ($V : f \mapsto Vf$; $V : \mathscr{C}^\sim_0 \to \mathscr{C}^\sim_0$) we see from (4.2) that the operators V, L are adjoint to each other.

4.3. Proposition. *Let $\lambda \in \mathscr{B}'_0$ be non-negative and suppose that the restriction $U^*_\lambda|_{K_0}$ is continuous and bounded on K_0. Then the operator V is a compact operator on $\mathscr{C}^\sim_0$ if and only if the restriction $U^*_\lambda|_K$ is continuous on K.*

Proof. Given n natural, let h_n be a function on R^1 such that $h_n(t) = 0$ for $t \geqq -1/n$, $h_n(t) = 1$ for $t \leqq -2/n$, h_n is linear on the interval $\langle -2/n, -1/n\rangle$. For $[x, t] \in R^2$ put

$$G^*_n(x, t) = G^*(x, t)\, h_n(t)\,, \quad B^*_n(x, t) = G^*(x, t) - G^*_n(x, t)\,.$$

It is seen that for $n \to +\infty$ we have $G^*_n(x, t) \to G^*(x, t)$ monotonically. Putting further for $[x, t] \in R^2$

$$C^n_\lambda(x, t) = \int_a^b G^*_n(x - \varphi(\tau), t - \tau)\, \mathrm{d}\lambda(\tau)\,, \quad D^n_\lambda(x, t) = \int_a^b B^*_n(x - \varphi(\tau), t - \tau)\, \mathrm{d}\lambda(\tau)$$

we have

$$U^*_\lambda(x, t) = C^n_\lambda(x, t) + D^n_\lambda(x, t)\,.$$

On account of the continuity of the kernel G^*_n the potential C^n_λ is continuous (even for each $\lambda \in \mathscr{B}'_0$). At the same time we have that for $[x, t] \in R^2$

$$G^n_\lambda(x, t) \to U^*_\lambda(x, t)$$

$(n \to +\infty)$ and this convergence is monotonous. Then also

$$D^n_\lambda(x, t) \to 0$$

$(n \to +\infty)$ and this convergence is monotone as well.

Suppose now that the restriction $U^*_\lambda|_K$ is continuous on K. As C^n_λ is continuous, the restriction $D^n_\lambda|_K$ is continuous on K. In virtue of the compactness of K the Dini theorem gives that

$$\sup\{D^n_\lambda(\varphi(t), t);\ t \in \langle a, b\rangle\} \to 0 \tag{4.3}$$

for $n \to +\infty$. Consider now operators V_n:

$$V_n f(t) = C^n_{f.\lambda}(\varphi(t), t) = \int_a^b f(\tau)\, G^*_n(\varphi(t) - \varphi(\tau), t - \tau)\, d\lambda(\tau)$$

$(f \in \mathscr{C}_0^\sim, t \in \langle a, b\rangle)$. Due to the fact that the function $G^*_n(\varphi(t) - \varphi(\tau), t - \tau)$ is continuous as a function of the variables t, τ on $\langle a, b\rangle \times \langle a, b\rangle$, the operator V_n is a compact operator on $\mathscr{C}_0^\sim$ (the image of the unit ball of the space $\mathscr{C}_0^\sim$ is a set of equicontinuous and uniformly bounded functions on $\langle a, b\rangle$). For $f \in \mathscr{C}_0^\sim$, $t \in \langle a, b\rangle$ we have

$$Vf(t) - V_n f(t) = U^*_{f.\lambda}(\varphi(t), t) - C^n_{f.\lambda}(\varphi(t), t) = D^n_{f.\lambda}(\varphi(t), t).$$

Hence

$$\|V - V_n\| \leqq \sup\{D^n_\lambda(\varphi(t), t);\ t \in \langle a, b\rangle\}$$

and $\|V - V_n\| \to 0$ $(n \to +\infty)$ (according to (4.3)) which immediately implies that the operator V is compact.

Now suppose that the restriction $U^*_\lambda|_K$ is not continuous on K (so $U^*_\lambda|_K$ is discontinuous at the point $[\varphi(b), b]$). Then there are $\delta > 0$, $t_i \in \langle a, b)$ $(i = 1, 2, \ldots)$, $t_i \to b$ $(i \to +\infty)$ such that

$$U^*_\lambda(\varphi(t_i), t_i) \geqq \delta.$$

Given i natural, choose $t'_i \in \langle a, b)$ such that

$$\int_{t'_i}^b G^*(\varphi(t_i) - \varphi(\tau), t_i - \tau)\, d\lambda(\tau) < \tfrac{1}{2}\delta$$

(there is such a t'_i in virtue of the assumption $\lambda(\{b\}) = 0$). Further, let $f_i \in C_0^\sim$ be such that $0 \leqq f_i \leqq 1$ on $\langle a, b\rangle$, $f_i = 1$ on $\langle a, t'_i\rangle$. Then

$$Vf_i(t_i) = U^*_{f_i.\lambda}(\varphi(t_i), t_i) \geqq$$

$$\geqq U^*_\lambda(\varphi(t_i), t_i) - \int_{t'_i}^b G^*(\varphi(t_i) - \varphi(\tau), t_i - \tau)\, d\lambda(\tau) \geqq \tfrac{1}{2}\delta.$$

Now it is seen that the unit ball of the space $\mathscr{C}_0^\sim$ is not mapped by V on to a set of equicontinuous functions on $\langle a, b\rangle$ and thus V is not a compact operator.

Let us show one auxiliary assertion which will be needed in the following.

4.4. Lemma. *Let $\lambda \in \mathscr{B}'_0(\langle a, b\rangle)$ be a non-negative and continuous measure (that is, $\lambda(\{t\}) = 0$ for each $t \in \langle a, b\rangle$) and suppose that the restriction $U^*_\lambda|_K$ is continuous (on K). Putting for $\delta > 0$, $t \in \langle a, b\rangle$*

$$\lambda_{t,\delta} = \lambda|_{\langle a,b\rangle \cap \langle t, t+\delta\rangle}$$

define on $\langle a, b\rangle$ a function S_δ by

$$S_\delta(t) = U^*_{\lambda_{t,\delta}}(\varphi(t), t)\,, \quad (t \in \langle a, b\rangle)\,.$$

Then for each $\delta > 0$ the function S_δ is continuous on $\langle a, b\rangle$ and $S_\delta \to 0$ for $\delta \to 0+$ uniformly on $\langle a, b\rangle$.

Proof. As λ is a continuous measure on $\langle a, b\rangle$ and U^*_λ is finite on K we certainly have for each $t \in \langle a, b\rangle$ that

$$S_\delta(t) \to 0$$

for $\delta \to 0+$ and this convergence is monotone. Taking into account the Dini theorem it suffices to show that for each $\delta > 0$ the function S_δ is continuous on $\langle a, b\rangle$.

Fix $t \in \langle a, b)$. If $t + \delta \geqq b$ then $S_\delta(t_1) = U^*_\lambda(\varphi(t_1), t_1)$ for $t_1 \in \langle t, b\rangle$ and the continuity of S_δ on $\langle t, b\rangle$ follows from the continuity of $U^*_\lambda|_K$; especially, the function S_δ is continuous from the right at the point t.

Suppose now that $t + \delta < b$ and let $t_1 > t$ be such that $t_1 < t + \delta$, $t_1 + \delta < b$. Consider the term

$$|S_\delta(t) - S_\delta(t_1)| = |U^*_{\lambda_{t,\delta}}(\varphi(t), t) - U^*_{\lambda_{t_1,\delta}}(\varphi(t_1), t_1)| =$$
$$= \left|U^*_{\lambda_{t,\delta}}(\varphi(t), t) - U^*_{\lambda_{t,\delta}}(\varphi(t_1), t_1) - \int_{t+\delta}^{t_1+\delta} G^*(\varphi(t_1) - \varphi(\tau), t_1 - \tau)\,\mathrm{d}\lambda(\tau)\right|.$$

The restriction $U^*_\lambda|_K$ is continuous and so also the restriction $U^*_{\lambda_{t,\delta}}|_K$ is continuous on K (for t fixed). Thus for a given $\varepsilon > 0$ there is a $\delta' > 0$, $\delta' < \delta$ such that

$$|U^*_{\lambda_{t,\delta}}(\varphi(t), t) - U^*_{\lambda_{t,\delta}}(\varphi(t_1), t_1)| < \frac{\varepsilon}{2}$$

for each $t_1 \in (t, t + \delta')$; moreover, δ' can be chosen such that

$$[\pi(\delta - \delta')]^{-1/2}\,\lambda(\langle t + \delta, t + \delta + \delta'\rangle) < \frac{\varepsilon}{2}\,.$$

Then for each $t_1 \in (t, t + \delta')$ $(t_1 + \delta < b)$

$$\left|\int_{t+\delta}^{t_1+\delta} G^*(\varphi(t_1) - \varphi(\tau), t_1 - \tau)\,\mathrm{d}\lambda(\tau)\right| \leqq \int_{t+\delta}^{t+\delta+\delta'} [\pi(\delta - \delta')]^{-1/2}\,\mathrm{d}\lambda(\tau) < \frac{\varepsilon}{2}\,.$$

Hence it is seen that the function S_δ is continuous at the point t from the right. Similarly for the left hand continuity.

Let $\lambda \in \mathscr{B}_0'$ be a measure such that $U^*_{|\lambda|}|_K$ is continuous on K. Then, of course, the restrictions $U^*_{\lambda^+}|_K$, $U^*_{\lambda^-}|_K$ are both continuous on K as well. Further, suppose that

$$\tilde{V}_K = \sup \{\tilde{v}(\varphi(t), t);\ t \in \langle a, b\rangle\} < \infty .$$

Then it follows from (2.10), (4.2) that for $f \in \mathscr{C}_0^\sim$, $\mu \in \mathscr{B}_0'$

$$\langle f, A_\mu \rangle = \langle f, H_\mu \rangle + \langle f, L_\mu \rangle = \langle \overline{W}_- f, \mu \rangle + \langle Vf, \mu \rangle .$$

So the operators A and $(\overline{W}_- + V)$ are adjoint to each other.

In what follows we shall use some more notations from Sections 1 and 2. Especially recall that for $f \in \mathscr{C}_0^\sim$, $\tau \in \langle a, b\rangle$

$$\overline{W}_- f(\tau) = Wf(\varphi(\tau), \tau) - 2f(\tau)\, \mathscr{P}_E^\sim(\varphi(\tau), \tau) =$$

$$= \frac{2}{\sqrt{\pi}} \int_\tau^b f(t) \exp\left(- \frac{(\varphi(t) - \varphi(\tau))^2}{4(t - \tau)} \right) \mathrm{d}_t \frac{\varphi(t) - \varphi(\tau)}{2\sqrt{(t - \tau)}} - 2f(\tau)\, \mathscr{P}_E^\sim(\varphi(\tau), \tau) =$$

$$= \int_a^b f(t)\, \mathrm{d}\nu_\tau(t) - 2f(\tau)\, \mathscr{P}_E^\sim(\varphi(\tau), \tau) ,$$

$$\overline{W}_1 f(\tau) = \overline{W}_- f(\tau) + f(\tau) = \int_a^b f(t)\, \mathrm{d}\nu_\tau(t) - f(\tau)\,(2\mathscr{P}_E^\sim(\varphi(\tau), \tau) - 1) .$$

ν_τ is the measure defined by the equality (2.3). For $r > 0$ we have

$$|\nu_\tau| (\langle \tau, \min\{\tau + r, b\} \rangle) = \frac{2}{\sqrt{\pi}}\, \tilde{v}^r(\varphi(\tau), \tau) .$$

5.1. Lemma. *Suppose that $\lambda \in \mathscr{B}_0'$ is a continuous measure, let $U^*_{|\lambda|}|_K$ be continuous and let*

$$\text{(5.1)} \qquad \lim_{r \to 0+} \sup_{\tau \in \langle a, b)} \left(\frac{2}{\sqrt{\pi}}\, \tilde{v}^r(\varphi(\tau), \tau) + |2\mathscr{P}_E^\sim(\varphi(\tau), \tau) - 1| \right) < 1 .$$

Then the equation

$$(\overline{W}_- + V) f = 0$$

has in $\mathscr{C}_0^\sim$ only the trivial solution.

Proof. For $\tau \in \langle a, b\rangle$, $r > 0$ denote

$$\lambda_{\tau,r} = |\lambda|\,|_{\langle a,b\rangle \cap \langle \tau, \tau + r\rangle}, \quad S_r(\tau) = U^*_{\lambda_{\tau,r}}(\varphi(\tau), \tau) .$$

According to Lemma 4.4, for each $\varepsilon > 0$ there is an $r > 0$ such that

$$\sup \{S_r(\tau);\ \tau \in \langle a, b\rangle\} < \varepsilon .$$

Since (5.1) is supposed to be fulfilled there is an $r_0 > 0$ such that

$$(5.2) \qquad \sup_{\tau \in \langle a, b)} \left(\frac{2}{\sqrt{\pi}} \, \tilde{v}^{r_0}(\varphi(\tau), \tau) + \left| 2\mathscr{P}_E^{\sim}(\varphi(\tau), \tau) - 1 \right| + S_{r_0}(\tau) \right) < 1 .$$

Let $f \in \mathscr{C}_0^{\sim}(\langle a, b \rangle)$ be a function for which

$$(\overline{W}_- + V) f = 0 .$$

There is a point $\tau' \in \langle b - r_0, b)$ such that

$$|f(\tau')| = \sup \{ |f(\tau)| ; \ \tau \in \langle b - r_0, b \rangle \} .$$

As $|v_{\tau'}| (\langle a, \tau' \rangle) = 0$, we have

$$(5.3) \qquad |(\overline{W}_1 + V) f(\tau')| =$$

$$= \left| \int_{\tau'}^{b} f(t) \, \mathrm{d}v_{\tau'}(t) - f(\tau') \left(2\mathscr{P}_E^{\sim}(\varphi(\tau'), \tau') - 1 \right) + U_{f.\lambda}^{*}(\varphi(\tau'), \tau') \right| \leqq$$

$$\leqq |f(\tau')| \left(\frac{2}{\sqrt{\pi}} \, \tilde{v}^{r_0}(\varphi(\tau'), \tau') + \left| 2\mathscr{P}_E^{\sim}(\varphi(\tau'), \tau') - 1 \right| + S_{r_0}(\tau') \right) .$$

But, since

$$0 = (\overline{W}_- + V) f(\tau') = (\overline{W}_1 + V) f(\tau') - f(\tau') ,$$

we get from (5.3), (5.2) that $f(\tau') = 0$, that is $f(\tau) = 0$ for each $\tau \in \langle b - r_0, b \rangle$. Continuing by induction we obtain that $f(\tau) = 0$ for each $\tau \in \langle a, b \rangle$ (see also [4] – the proof of Lemma 2.1).

5.2. Theorem. *Let $\lambda \in \mathscr{B}_0'$ be a continuous measure such that the restriction $U_{|\lambda|}^{*}|_K$ is continuous on K and suppose that the condition (5.1) is fulfilled. Then for each $v \in \mathscr{B}_0'$ the equation*

$$(5.4) \qquad A_\mu = v$$

has in $\mathscr{B}_0'$ a unique solution.

Proof. First, let us consider the operator $(\overline{W}_1 + V)$. We have found in paragraph 2.5 that the Fredholm radius of the operator $\overline{W}_1$ is equal to the reciprocal value of the number

$$\omega W_1 = \lim_{r \to 0+} \sup_{\tau \in \langle a, b \rangle} \left(\frac{2}{\sqrt{\pi}} \, \tilde{v}^{r}(\varphi(\tau), \tau) + \left| \mathscr{P}_E^{\sim}(\varphi(\tau), \tau) - 1 \right| \right) .$$

According to Proposition 4.3 the operator V is compact under our assumption (more precisely, the operators corresponding to the measures λ^+ and λ^- are compact, but V is equal to the difference of those operators) and so

$$\omega(\overline{W}_1 + V) = \omega W_1 .$$

By Lemma 5.1 the equation

$$(\overline{W}_- + V)f = [(\overline{W}_1 + V) + I]f = 0$$

has in $\mathscr{C}_0^\sim$ only the zero solution. It follows from the Riesz-Schauder theory that for each $g \in \mathscr{C}_0^\sim$ the equation

$$(\overline{W}_- + V)f = g$$

has a unique solution in $\mathscr{C}_0^\sim$ and since the operators A and $(\overline{W}_- + V)$ are adjoint to each other, the assertion follows.

5.3. Remark. Suppose that the assumptions from Theorem 5.2 are fulfilled and let μ be the solution of the equation (5.4). As A_μ is a weak characterization of the term

$$\frac{\partial U_\mu}{\partial x} + U_\mu(\lambda_0 + \lambda)$$

on K (where $\mathrm{d}\lambda_0(t) = \mathrm{d}\lambda(t)$) then the potential U_μ considered on the set E is a solution of the third boundary value problem for the heat equation on E with the boundary condition

$$\frac{\partial U_\mu}{\partial x} + U_\mu(\lambda_0 + \lambda) = \nu$$

prescribed on K.

Let us also note that in a similar way one can solve the third boundary value problem of the given form for the heat equation on the set

$$E_- = \{[x, t] \in R^2;\ t \in (a, b),\ x < \varphi(t)\}$$

and also on sets of the form

$$E_1 = \{[x, t] \in R^2;\ t \in (a, b),\ \varphi_1(t) < x < \varphi_2(t)\}$$

(where φ_1, φ_2 are some suitable functions on $\langle a, b \rangle$) – see also [4] where the first boundary value problem for the heat equation is solved on the sets of the above mentioned forms.

5.4. Remark. Suppose that the condition (5.1) is fulfilled and suppose, in addition, that the restriction $U^*_{|\lambda_0|}|_K$ is continuous on K. As λ_0 is a continuous measure we can take $\lambda = -\lambda_0$ in Theorem 5.2. In this case A_μ is a weak characterization of the derivative

$$\frac{\partial U_\mu}{\partial x}$$

on K (A_μ may be called the flow of heat in this case — see [15]). If μ is the solution of the equation (5.4) then the potential U_μ considered on the set E is a solution of the second boundary value problem for the heat equation on the set E with the boundary condition

$$\frac{\partial U_\mu}{\partial x} = \nu$$

on K.

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