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ON SKELETAL AND IRREDUCIBLE ELEMENTS IN TOLERANCE LATTICES OF FINITE DISTRIBUTIVE LATTICES

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In this paper, skeletal elements and irreducible elements in tolerance lattices of finite distributive lattices are investigated. Some statements are valid for infinite distributive lattices as well. The main result is that the problem of characterizing finite distributive lattices that are isomorphic to tolerance lattices of finite distributive lattices is equivalent to the problem of characterizing finite partially ordered sets that are isomorphic to interval sets of finite partially ordered sets.

Terminology from [1] will be used.

Notation. $TL(\mathfrak{L})$ denotes the lattice of all compatible tolerances on a lattice $\mathfrak{L}$; it is called the *tolerance lattice* of $\mathfrak{L}$.

$CL(\mathfrak{L})$ denotes the lattice of all congruences on a lattice $\mathfrak{L}$; it is called the *congruence lattice* of $\mathfrak{L}$.

$\mathscr{C}(T)$ is the transitive hull of a tolerance T.

$\mathscr{J}(\mathfrak{L})$ is the set of all join-irreducible elements in $\mathfrak{L}$.

$\mathscr{I}nt(M)$ denotes the set of all intervals in a partially ordered set M, partially ordered by set inclusion.

$\langle a, b\rangle$ is the interval with the least element a and the greatest element b.

$[a, b]$ is an ordered pair of elements.

$\Theta(a, b)$ is the principal congruence generated by $[a, b]$.

Δ is the diagonal ($=$ binary relation defined by $a\Delta b :\Leftrightarrow a = b$)

x^* is the pseudocomplement of x; the elements of the set $\{x^* \mid x \in |\mathfrak{L}|\}$ are called the *skeletal* elements of the lattice $\mathfrak{L}$.

Compatible tolerance relations are shortly called *compatible tolerances* or *tolerances.*

Congruence lattices of lattices are compactly generated distributive lattices, therefore they satisfy the Join Infinite Distributive Identity

$$T \wedge \bigvee_{CL}\{S_i\}_{i\in I} = \bigvee_{CL}\{T \wedge S_i\}_{i\in I}\,.$$

Since for lattices the operator $\mathscr{C}$ is a join-complete lattice homomorphism ([4]), we have, for a given tolerance T,

$$\mathscr{C}(T \wedge \bigvee_{TL}\{S|\ T \wedge S = \Delta\}) = \mathscr{C}(T) \wedge \mathscr{C}(\bigvee_{TL}\{S|\ T \wedge S = \Delta\}) =$$
$$= \mathscr{C}(T) \wedge \bigvee_{CL}\{\mathscr{C}(S)|\ T \wedge S = \Delta\} = \bigvee_{CL}\{\mathscr{C}(T) \wedge \mathscr{C}(S)|\ T \wedge S = \Delta\} = \Delta\,.$$

It follows that tolerance lattices of lattices are pseudocomplemented. This result is due to H.-J. Bandelt.

Proposition 1. *Let $\mathfrak{L}$ be a lattice. If $T \in TL(\mathfrak{L})$ is skeletal in $TL(\mathfrak{L})$ then it is a congruence on $\mathfrak{L}$.*

Proof. Let $T \in TL(\mathfrak{L})$ be skeletal, for instance $T = S^*$ for some $S \in TL(\mathfrak{L})$. By [4], $\mathscr{C}(T) \wedge \mathscr{C}(S) = \mathscr{C}(T \wedge S) = \mathscr{C}(\Delta) = \Delta$, and so $\mathscr{C}(T) \wedge S = \Delta$. Thus $\mathscr{C}(T) \leqq T$ and consequently $\mathscr{C}(T) = T$. Q.E.D.

Lemma 1. *Let T be a congruence on a lattice $\mathfrak{L}$. Then the pseudocomplement of T in $TL(\mathfrak{L})$ is identical with the pseudocomplement of T in $CL(\mathfrak{L})$.*

Proof. Let T^* be the pseudocomplement of T in $TL(\mathfrak{L})$. Then for any $C \in CL(\mathfrak{L})$, $C \wedge T = \Delta$ implies $C \leqq T^*$, so that T^* is the pseudocomplement of T in $CL(\mathfrak{L})$. Conversely, let T^* be the pseudocomplement of T in $CL(\mathfrak{L})$. Then for any $S \in TL(\mathfrak{L})$, $S \wedge T = \Delta$ implies $\mathscr{C}(S) \wedge T = \Delta$, hence $S \leqq \mathscr{C}(S) \leqq T^*$, and so T^* is the pseudocomplement of T in $TL(\mathfrak{L})$. Q.E.D.

Proposition 2. *Let $\mathfrak{L}$ be a lattice. Then $T \in TL(\mathfrak{L})$ is skeletal in $TL(\mathfrak{L})$ if and only if T is skeletal in $CL(\mathfrak{L})$.*

Proof. $\Rightarrow$ Let $S^* = T$ for a tolerance $S \in TL(\mathfrak{L})$. Then $\mathscr{C}(S) \wedge T = \Delta$ and $C \wedge \wedge\, \mathscr{C}(S) = \Delta$ implies $C \wedge S = \Delta$ for any $C \in TL(\mathfrak{L})$, so that $C \leqq T$. Thus $(\mathscr{C}(S))^* = = T$. Combining Proposition 1 and Lemma 1, we conclude that T is skeletal in $CL(\mathfrak{L})$. $\Leftarrow$ Let $C^* = T$ for a congruence C. Then T is obviously skeletal in $TL(\mathfrak{L})$. Q.E.D.

Proposition 3. *Let $\mathfrak{L}$ be a locally finite distributive lattice. Then $T \in TL(\mathfrak{L})$ is skeletal in $TL(\mathfrak{L})$ if and only if T is a congruence on $\mathfrak{L}$.*

Proof. $\Rightarrow$ Apply Proposition 1.

$\Leftarrow$ Let T be a congruence on $\mathfrak{L}$. $CL(\mathfrak{L})$ is Boolean, thus T is skeletal in $CL(\mathfrak{L})$. According to Proposition 2, T is skeletal in $TL(\mathfrak{L})$. Q.E.D.

Another formulation of this statement is given in [4].

Remark. If $\mathfrak{L}$ is not locally finite, the statement of Proposition 3 is not true; there are congruences on $\mathfrak{L}$ that are not skeletal.

Before starting the investigation of join-irreducible elements in tolerance lattices of finite distributive lattices, we define a new type of elements of a lattice.

Definition. Let a, b be elements of a lattice $\mathfrak{L}$. The element b is said to be *relatively join-irreducible with respect to* a if whenever $b \leqq u \vee v$, then either $b \leqq a \vee u$ or $b \leqq a \vee v$.

Let a, b be elements of a lattice $\mathfrak{L}$. The element a is said to be *relatively meet-irreducible with respect to* b if whenever $u \wedge v \leqq a$, then either $b \wedge u \leqq a$ or $b \wedge v \leqq a$.

Lemma 2. *Let a, b be elements of a distributive lattice $\mathfrak{L}$, $a < b$. Then b is relatively join-irreducible with respect to a if and only if $b = x \vee y$ implies $x = b$ or $y = b$ for any pair of elements $x, y \in \langle a, b\rangle$.*

Proof is obvious.

A dual statement may be formulated for relatively meet-irreducible elements.

Lemma 3. *Let a, b be elements of a lattice $\mathfrak{L}$. Then b is relatively join-irreducible with respect to a if and only if for any $n \in N$ and for any n-tuple $u_1, \ldots, u_n \in L$, $u_1 \vee \ldots \vee u_n \geqq b$ implies $a \vee u_i \geqq b$ for some $i \in \{1, \ldots, n\}$.*

Proof. $\Leftarrow$ Obvious.

$\Rightarrow$ The statement is true for $n = 1$.

Assume it is true for $n = 1, \ldots, m$. Let $n = m + 1$, and let $u_1 \vee \ldots \vee u_m \vee u_{m+1} \geqq b$. Then either $a \vee (u_1 \vee \ldots \vee u_m) \geqq b$ or $a \vee u_{m+1} \geqq b$. In the latter case, put $i = m + 1$. In the former, $(a \vee u_1) \vee \ldots \vee (a \vee u_m) \geqq b$, thus, according to the assumption, there exists $i \in \{1, \ldots, m\}$ such that $a \vee (a \vee u_i) \geqq b$, and so $a \vee u_i \geqq b$. Q.E.D.

A dual statement may be formulated for relatively meet-irreducible elements.

Lemma 4. *Let $\mathfrak{L}$ be a finite distributive lattice. Then for any pair a, b of elements of $\mathfrak{L}$ satisfying $a < b$, a being relatively meet-irreducible with respect to b and b being relatively join-irreducible with respect to a, elements c, d can be found such that $c < d$, c is relatively meet-irreducible with respect to d and d is join-irreducible, and $\Theta(a, b) = \Theta(c, d)$ holds.*

Proof. Let $\mathfrak{L}$, a, b satisfy the assumptions. Let d be the greatest lower bound of the set $X = \{x \in |\mathfrak{L}| \mid x \vee a = b\}$ and let $c = a \wedge d$. Since $b \in X$, $d \leqq b$. Because of distributivity, $d \in X$. Let $d = u \vee v$. Then $b = d \vee a = u \vee v \vee a = (u \vee a) \vee (v \vee a)$. Since b is relatively join-irreducible with respect to a, either $u \vee a = b$ or $v \vee a = b$. Hence $u \geqq d$ or $v \geqq d$, and so $u = d$ or $v = d$. Thus d is join-irreducible. Now, assume $c = u \wedge v$ for some $u, v \in \langle c, d\rangle$. Then $(a \vee u) \wedge (a \vee v) = a \vee (u \wedge v) = a \vee c = a$ and by the assumption either $u \leqq a$ or $v \leqq a$. Thus $u = d \wedge u \leqq d \wedge a = c$ or $v = d \wedge v \leqq d \wedge a = c$. Consequently, c is relatively meet-irreducible with respect to d. $[a, b] = [c \vee a, d \vee a]$ implies $[a, b] \in \Theta(c, d)$, and so $\Theta(a, b) \subseteq \Theta(c, d)$. $[c, d] = [a \wedge d, b \wedge d]$ implies $[c, d] \in \Theta(a, b)$, and so $\Theta(c, d) \subseteq \Theta(a, b)$. Thus $\Theta(a, b) = \Theta(c, d)$. Q.E.D.

Lemma 5. *Let a, b, c, d be elements of a finite distributive lattice $\mathfrak{L}$, $a < b$, $c < d$, a relatively meet-irreducible with respect to b, b join-irreducible, c relatively meet-irreducible with respect to d and d join-irreducible. Then $\Theta(a, b) = \Theta(c, d)$ implies $a = c$ and $b = d$.*

Proof. $\Theta(a, b) = \Theta(c, d)$ implies the following identities: $a \wedge c = b \wedge c$, $a \vee d = b \vee d$, $a \wedge c = a \wedge d$, $b \vee c = b \vee d$. (G. Grätzer and E. T. Schmidt, cf. [1], p. 74.) Hence $a \vee d \geqq b$, and so $d \geqq b$; $b \vee c \geqq d$ and so $b \geqq d$. Thus $b = d$, Further, $c = d \wedge c = b \wedge c = a \wedge c = a \wedge d = a \wedge b = a$. Q.E.D.

Lemma 6. *Let c, d be elements of a finite distributive lattice $\mathfrak{L}$, $c < d$, c relatively meet-irreducible with respect to d and d join-irreducible. Then $\Theta(c, d)$ is a join-irreducible element in $TL(\mathfrak{L})$.*

Proof. Let $S, T \in TL(\mathfrak{L})$ be such that $S \vee T = \Theta(c, d)$. By [3], there exist a lattice polynomial p and elements $c_1, \ldots, c_n, d_1, \ldots, d_n \in |\mathfrak{L}|$ such that $c_i\, S\, d_i$ or $c_i\, T d_i$ for $i = 1, \ldots, n$, and $c = p(c_1, \ldots, c_n)$, $d = p(d_1, \ldots, d_n)$. But in the case of distributive lattices, any lattice polynomial is equivalent to a lattice polynomial of the form "join of meets". We can suppose p is of this form. Since d is join-irreducible, there exists a meet polynomial q and a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $d = q(d_{i_1}, \ldots, d_{i_k})$ and $c \geqq q(c_{i_1}, \ldots, c_{i_k})$. Because of relative meet-irreducibility of c with respect to d, there exists i_j such that $d \wedge c_{i_j} \leqq c$. Hence $c = c \vee (d \wedge c_{i_j})$, $d = c \vee (d \wedge d_{i_j})$, and so cSd or cTd. Thus $\Theta(c, d) = S$ or $\Theta(c, d) = T$. Q.E.D.

Lemma 7. *Let $\mathfrak{L}$ be a finite distributive lattice, T a join-irreducible element in $TL(\mathfrak{L})$. Then there exist elements $a, b \in \mathfrak{L}$ such that $a < b$, a is relatively meet-irreducible with respect to b and b is relatively join-irreducible with respect to a, and $\Theta(a, b) = T$.*

Proof. By [2], each compatible tolerance on a distributive lattice is a join of principal congruences. If $T \in TL(\mathfrak{L})$ is join-irreducible, it is a principal congruence. Since $\Theta(x, y) = \Theta(x \wedge y, x \vee y)$, $T = \Theta(a, b)$ where $a < b$. Now, whenever $x, y \leqq b$ are such elements that $a = x \wedge y$, then $\Theta(a, b) = \Theta(x, b) \vee \Theta(y, b)$. Because of the irreducibility of $\Theta(a, b)$, either $\Theta(a, b) = \Theta(x, b)$ or $\Theta(a, b) = \Theta(y, b)$. But this is only possible if either $a = x$ or $a = y$. Hence a is relatively meet-irreducible with respect to b. Similarly, b is relatively join-irreducible with respect to a. Q.E.D.

Proposition 4. *Let T be a compatible tolerance on a finite distributive lattice $\mathfrak{L}$. The following conditions are equivalent:*

(i) *T is a join-irreducible element in $TL(\mathfrak{L})$;*

(ii) *$T = \Theta(a, b)$, where $a < b$, a is relatively meet-irreducible with respect to b and b is relatively join-irreducible with respect to a;*

(iii) *$T = \Theta(c, d)$, where $c < d$, c is relatively meet-irreducible with respect to d and d is join-irreducible; the elements c and d are uniquely determined.*

Proof. This proposition is a combination of the preceding lemmas.

Remark. The least element of a lattice, if it exists, is not considered to be join-irreducible, or to be relatively join-irreducible with respect to an element of the lattice. Similarly for the dual concepts.

Lemma 8. *Let a, b, c, d be elements of a finite distributive lattice $\mathfrak{L}$ such that $a < b$, $c < d$, b and d are join-irreducible, a is relatively meet-irreducible with respect to b and c is relatively meet-irreducible with respect to d. Then $\Theta(a, b) \leqq \Theta(c, d)$ if and only if $b \wedge c \leqq a < b \leqq d$.*

Proof. $\Leftarrow$ Obvious.

$\Rightarrow$ Let $\Theta(a, b) \leqq \Theta(c, d)$. Then $a \wedge c = b \wedge c$ and $a \vee d = b \vee d$. Hence $b \wedge c \leqq a$ and because of the join-irreducibility of b, $b \leqq d$, since $b \leqq a$ is impossible. Q.E.D.

Lemma 9. *Let $\mathfrak{L}$ be a finite distributive lattice. Let d be a join-irreducible element in $\mathfrak{L}$ and let c be a relatively meet-irreducible element with respect to d, $c < d$. Then there exists $c^d \in |\mathfrak{L}|$ such that $c^d \leqq d$, $c^d \nleqq c$ and whenever $x \leqq d$, $x \nleqq c$, then $c^d \leqq x$. The element c^d is join-irreducible.*

Proof. Put $X = \{x \in |\mathfrak{L}| \mid x \leqq d,\ x \nleqq c\}$ and let $c^d = \bigwedge X$. Then $c^d \leqq d$, since $d \in X$. Because of the relative meet-irreducibility of c with respect to d, $c^d \nleqq c$. If $c^d = u \vee v$, then either both $u \leqq c$ and $v \leqq c$, which is impossible, or $u \in X$ or $v \in X$. Hence $u = c^d$ or $v = c^d$. Thus c^d is join-irreducible. Q.E.D.

Lemma 10. *Let x, y be join-irreducible elements of a finite distributive lattice $\mathfrak{L}$, $x \leqq y$. Then there exists exactly one element $c \in |\mathfrak{L}|$ such that $c^y = x$ and c is relatively meet-irreducible with respect to y.*

Proof. Let c be the least upper bound of the set $Z = \{z \in |\mathfrak{L}| \mid z \leqq y,\ x \nleqq z\}$. Then obviously $c \leqq y$, $x \nleqq c$. If $c = u \wedge v$ for $u, v \leqq y$, then either $x \nleqq u$ or $x \nleqq v$, and so $u \in Z$ or $v \in Z$. Hence $u = c$ or $v = c$. Thus c is relatively meet-irreducible with respect to y. Now, $c^y = \bigwedge\{z \in |\mathfrak{L}| \mid z \leqq y,\ z \nleqq c\}$. Hence $c^y \leqq x$. Since $c^y < x$ implies $c^y \in Z$, which is impossible because $c = \bigvee Z$ and $c^y \nleqq c$, it follows that

$c^y = x$. Now let $p \in |\mathfrak{L}|$ be relatively meet-irreducible with respect to y, such that $p^y = c^y$. Then $c \leqq p$ and $p \leqq c$, because $c \nleqq c^y = p^y$ and $p \nleqq p^y = c^y$, respectively, and so $c = p$. Q.E.D.

Proposition 5. *Let $\mathfrak{L}$ be a finite distributive lattice. Then the set of all join-irreducible elements in $TL(\mathfrak{L})$ is order isomorphic to the set of all intervals in $\mathscr{J}(\mathfrak{L})$ ordered by set inclusion.*

Proof. Let $f = (\Theta(c, d) \mapsto \langle c^d, d \rangle)$, where c, d is the unique pair of elements generating the given join-irreducible tolerance, such that d is join-irreducible and c is relatively meet-irreducible with respect to d, $c < d$. The mapping f is isotone, since $\Theta(c_1, d_1) \leqq \Theta(c_2, d_2)$ implies $c_2 \wedge d_1 \leqq c_1 < d_1 \leqq d_2$, and so $c_2^{d_2} \leqq c_1^{d_1} \leqq \leqq d_1 \leqq d_2$, hence $\langle c_1^{d_1}, d_1 \rangle \subseteq \langle c_2^{d_2}, d_2 \rangle$. Now, let $x, y \in \mathscr{J}(\mathfrak{L})$, $x \leqq y$. By Lemma 10, there exists exactly one element c such that c is relatively meet-irreducible with respect to y and $c^y = x$. Thus f is bijective. Assume $\langle c_1^{d_1}, d_1 \rangle \subseteq \langle c_2^{d_2}, d_2 \rangle$, that is $c_2^{d_2} \leqq c_1^{d_1} \leqq d_1 \leqq d_2$. Then $d_1 \wedge c_2 \leqq c_1 < d_1 \leqq d_2$, since $d_1 \wedge c_2 \nleqq c_1$ implies $c_1^{d_1} \leqq d_1 \wedge c_2$ and so $c_2^{d_2} \leqq c_1^{d_1} \leqq d_1 \wedge c_2 \leqq c_2$, which is impossible. Hence $\Theta(c_1, d_1) \leqq \Theta(c_2, d_2)$. Thus f is a bijective strong isotone map of $\mathscr{J}(TL(\mathfrak{L}))$ onto $\mathscr{Int}(\mathscr{J}(\mathfrak{L}))$. Q.E.D.

Corollary. *A finite distributive lattice $\mathfrak{D}$ is isomorphic to the tolerance lattice of a finite distributive lattice if and only if the set of all join-irreducible elements of $\mathfrak{D}$ is order isomorphic to the set of all intervals of a finite partially ordered set, partially ordered by inclusion.*

Proof. $\Rightarrow$ Let $\mathfrak{D} \cong TL(\mathfrak{L})$ for a finite distributive lattice $\mathfrak{L}$. Then $\mathscr{J}(\mathfrak{D}) \cong \cong \mathscr{J}(TL(\mathfrak{L})) \cong \mathscr{Int}(\mathscr{J}(\mathfrak{L}))$.

$\Leftarrow$ Let $\mathscr{J}(\mathfrak{D}) \cong \mathscr{Int}(M)$ for a finite partially ordered set M. Then there exists a finite distributive lattice $\mathfrak{L}$ such that $\mathscr{J}(\mathfrak{L}) \cong M$. Then $\mathscr{J}(\mathfrak{D}) \cong \mathscr{Int}(M) \cong \cong \mathscr{Int}(\mathscr{J}(\mathfrak{L})) \cong \mathscr{J}(TL(\mathfrak{L}))$ by the assumption, the preceding argument and Proposition 5. Q.E.D.

A tolerance T on an algebra is called a *relatively maximal tolerance* if there exists an ordered pair $[a, b]$ of elements of the algebra such that T is a maximal element among all tolerances not containing $[a, b]$ ([5]). Every tolerance is a (possibly infinite) meet of a family of relatively maximal tolerances.

Proposition 6. *Let T be a tolerance on an algebra $\mathfrak{A}$. Then T is completely meet-irreducible in $TL(\mathfrak{A})$ if and only if it is a relatively maximal tolerance.*

Proof. $\Rightarrow$ Let T be completely meet-irreducible in $TL(\mathfrak{A})$. Then T is a relatively maximal tolerance, since T is the meet of a family of relatively maximal tolerances.

$\Leftarrow$ Let T be a relatively maximal tolerance, $T = \bigwedge_{i \in I} T_i$ for some $T_i \in TL(\mathfrak{A})$ $(i \in I)$. There exist elements $a, b \in |\mathfrak{A}|$ such that T is a maximal element among all toleranceson $\mathfrak{A}$ that do not contain the pair $[a, b]$. There exists $i_0 \in I$ such that $[a, b] \notin T_{i_0}$, and so $T = T_{i_0}$. Q.E.D.

As shown in [5], relatively maximal tolerances on distributive lattices are exactly the tolerances of the type τ, or *τ-tolerances*, defined in [6]. Thus we have

Corollary. *Let T be a tolerance on a finite distributive lattice $\mathfrak{L}$. Then T is meet-irreducible if and only if T is a τ-tolerance.*

Remark. The set of all join-irreducible elements of a finite distributive lattice $\mathfrak{L}$ and the set of all meet-irreducible elements of $\mathfrak{L}$ are order isomorphic. So we can prove Proposition 5 also by constructing an order isomorphism between the set of all τ-tolerances on $\mathfrak{L}$ and $\mathcal{I}nt(\mathcal{J}(\mathfrak{L}))$. A tolerance T on a finite distributive lattice $\mathfrak{L}$ is a τ-tolerance if and only if an ideal P and a dual ideal Q exist, both P and Q prime, such that $P \cup Q = |\mathfrak{L}|$ and $T = (P \times P) \cup (Q \times Q)$. Then $|\mathfrak{L}| \setminus P$ is also a dual prime ideal, $|\mathfrak{L}| \setminus P \subseteq Q$. A dual ideal on a finite distributive lattice is prime if and only if its least element is join-irreducible. The map $g = (T \mapsto \langle \bigwedge Q, \bigwedge(|\mathfrak{L}| \setminus P) \rangle$ is the desired isomorphism.

References

[1] *G. Grätzer:* General lattice theory. Akademie-Verlag Berlin, 1978.

[2] *I. Chajda, B. Zelinka:* Minimal compatible tolerances on lattices. Czech. Math. J. *27* (1977), 452—459.

[3] *I. Chajda, B. Zelinka:* Lattices of tolerances. Čas. pěst. mat. *102* (1977), 10—24.

[4] *J. Niederle:* On atoms in tolerance lattices of distributive lattices. Čas. pěst. mat. *106* (1981), 311—315.

[5] *J. Niederle:* A note on tolerance lattices. To appear.

[6] *J. Niederle:* Relative bicomplements and tolerance extension property in distributive lattices. Čas. pěst. mat. *103* (1978), 250—254.

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