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GENERAL REPRESENTABILITY PROBLEM FOR THE LAPLACE TRANSFORM OF EXPONENTIALLY BOUNDED VECTOR-VALUED FUNCTIONS

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The purpose of this paper is to study general test sequences permitting to decide whether a given function is the Laplace transform of an exponentially bounded measurable vector-valued function. These test sequences will be called later Laplace representative sequences (see Section 4). Special cases were studied earlier in [1] and [2]. Related results connected with our special case of the so called summatory representative sequences (see Section 25), but concerning integrable originals, are to be found in [3].

We attempt to axiomatize the properties of Laplace representative sequences in the form as general as possible, or more precisely, as we can. This means that certain simplifications or modifications are not excluded. But our conviction, based on careful analysis of the situation, is that the possibility of essential changes in the characteristic properties of Laplace representative sequences seems very improbable.

Our main result, Theorem 13, is given for reflexive Banach spaces. The reason for this restriction to reflexive Banach spaces is to direct the attention to the properties of Laplace representative sequences rather than to the technical problems connected with nonreflexivity which involves sizable complications and probably necessitates a strenghtening of the notion of Laplace representative sequences.

1. We shall use the following notation: (1) $\mathbb{R}$ — the real number field, (2) (ω, ∞) — the set of all real numbers greater than ω for $\omega \in \mathbb{R}$, (3) $\mathbb{C}$ — the complex number field, (4) $(\mathrm{Re} > \omega)$ — the set of all complex numbers whose real part is greater than ω if $\omega \in \mathbb{R}$, (5) $M_1 \to M_2$ — the set of all mappings of the whole set M_1 into the set M_2.

2. In the whole paper, E denotes a Banach space with the norm $\|\cdot\|$.

3. The functional analysis (including the theory of vector valued functions) is used in the extent of the first three Chapters of [6] except for certain special topics (e.g. II.4, III.3). The reader interested only in the numerical case needs nothing more than the basic facts from the modern differential and integral calculus.

4. Let $\Phi_k \in (0, \infty) \times \mathbb{R} \to \mathbb{C}$, $k \in \{1, 2, \ldots\}$. The sequence Φ_k, $k \in \{1, 2, \ldots\}$, will be called a *Laplace representative sequence* if

(R_1) $\displaystyle\int_{-\infty}^{\infty} |\Phi_k(t, \beta)| \, d\beta < \infty$ for every $t > 0$ and $k \in \{1, 2, \ldots\}$,

(R_2) $\displaystyle\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{-i\tau\beta} \, \Phi_k(t, \beta) \, d\beta \right| d\tau \leqq 1$ for every $t > 0$ and $k \in \{1, 2, \ldots\}$,

(R_3) $\displaystyle\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-i\tau\beta} \, \Phi_k(t, \beta) \, d\beta \right) d\tau \xrightarrow[k \to \infty]{} 1$ for every $t > 0$,

(R_4) $\displaystyle\int_{\{\tau : |\tau - t|\} \geqq \delta} \left| \int_{-\infty}^{\infty} e^{-i\tau\beta} \, \Phi_k(t, \beta) \, d\beta \right| d\tau \xrightarrow[k \to \infty]{} 0$ for every $t > 0$ and $\delta > 0$,

(R_5) $\displaystyle\int_{0}^{\infty} e^{-\chi\tau} \, \tau \left(\int_{-\infty}^{\infty} |\Phi_k(\tau, \beta)| \, d\beta \right) d\tau < \infty$ for every $\chi > 0$ and $k \in \{1, 2, \ldots\}$,

(R_6) for every $\chi > 0$, $q \in \{1, 2, ..\}$ and $\varepsilon > 0$, there exists $N > 0$ such that

$$\int_{N}^{\infty} \left| \int_{0}^{\infty} e^{-\chi\tau} \tau^q \, \Phi_k(\tau, \beta) \, d\tau \right| d\beta \leqq \varepsilon, \quad \int_{-\infty}^{-N} \left| \int_{0}^{\infty} e^{-\chi\tau} \tau^q \, \Phi_k(\tau, \beta) \, d\tau \right| d\beta \leqq \varepsilon$$

for every $k \in \{1, 2, \ldots\}$.

5. Remark. In the whole paper, the properties $(R_1)-(R_6)$ of Laplace representative sequences stated in Section 4 will be referred to simply by their symbols without specifying the section.

6. Lemma. *Let Φ_k, $k \in \{1, 2, \ldots\}$, be a Laplace representative sequence. Then*

$$\Phi_k(t, \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau\beta} \left(\int_{-\infty}^{\infty} e^{-i\tau\eta} \, \Phi_k(t, \eta) \, d\eta \right) d\tau$$

for every $t > 0$, $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$.

Proof. Immediate consequence of the Fourier inversion theorem which is valid in virtue of (R_1) and (R_2).

7. Proposition. *Let Φ_k, $k \in \{1, 2, \ldots\}$, be a Laplace representative sequence. Then*

(a) $|\Phi_k(t, \beta)| \leqq \dfrac{1}{2\pi}$ *for every $t > 0$, $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$,*

(b) *the function $\Phi_k(t, \cdot)$ is continuous on $\mathbb{R}$ for every $t > 0$ and $k \in \{1, 2, \ldots\}$,*

(c) *the function Φ_k is measurable on $(0, \infty) \times \mathbb{R}$ for every $k \in \{1, 2, \ldots\}$.*

Proof. The statements (a) and (b) immediately follow from Lemma 6 and property (R_2). The statement (c) follows from (b) and property (R_5).

8. Lemma. *Let Φ_k, $k \in \{1, 2, \ldots\}$, be a Laplace representative sequence. Then* $e^{-it\beta}\,\Phi_k(t, \beta) \xrightarrow[k\to\infty]{} 1$ *for every $t > 0$ and $\beta \in \mathbb{R}$.*

Proof. For the sake of simplicity, let us denote

$$(1)\quad \varphi_k(t, \tau) = \int_{-\infty}^{\infty} e^{-i\tau\beta}\,\Phi_k(t, \beta)\,d\beta \text{ for } t > 0 \text{ and } \tau \in \mathbb{R}.$$

According to (R_2), (R_3) and (R_4) we can write with respect to the definition (1)

$$(2)\quad \int_{-\infty}^{\infty} |\varphi_k(t, \tau)|\,d\tau \leqq 1 \text{ for every } t > 0 \text{ and } k \in \{1, 2, \ldots\},$$

$$(3)\quad \int_{-\infty}^{\infty} \varphi_k(t, \tau)\,d\tau \xrightarrow[k\to\infty]{} 1 \text{ for every } t > 0,$$

$$(4)\quad \int_{\{\tau:\,|\tau - t|\} \geqq \delta} |\varphi_k(t, \tau)|\,d\tau \xrightarrow[k\to\infty]{} 0 \text{ for every } t > 0 \text{ and } \delta > 0.$$

It easily follows from the properties (2), (3) and (4) that

$$(5)\quad \int_{-\infty}^{\infty} e^{i\tau\beta}\,\varphi_k(t, \tau)\,d\tau \xrightarrow[k\to\infty]{} e^{it\beta} \text{ for every } t > 0 \text{ and } \beta \in \mathbb{R}.$$

Using Lemma 6 we obtain the required result from (1) and (5).

9. Lemma. *Let Φ_k, $k \in \{1, 2, \ldots\}$, be a Laplace representative sequence. Then*

$$\int_0^{\infty} e^{-\tau/s}\tau^{r-1}\,\Phi_k(\tau, \beta)\,d\tau \xrightarrow[k\to\infty]{} \frac{(r-1)!\,s^r}{(1 - is\beta)^r}$$

for every $s > 0$, $\beta \in \mathbb{R}$ and $r \in \{2, 3, \ldots\}$.

Proof. According to Proposition 7 and Lemma 8 we have

$$(1)\quad |\Phi_k(\tau, \beta)| \leqq \frac{1}{2\pi} \text{ for every } \tau > 0,\ \beta \in \mathbb{R} \text{ and } k \in \{1, 2, \ldots\},$$

$$(2)\quad e^{-i\tau\beta}\,\Phi_k(\tau, \beta) \xrightarrow[k\to\infty]{} 1 \text{ for every } \tau > 0 \text{ and } \beta \in \mathbb{R}.$$

Moreover, as is well-known,

$$(3)\quad \int_0^{\infty} e^{-(1/s - i\beta)\tau}\tau^{r-1}\,d\tau = \frac{(r-1)!}{\left(\frac{1}{s} - i\beta\right)^r} \text{ for every } s > 0,\ \beta \in \mathbb{R} \text{ and } r \in \{2, 3, \ldots\}.$$

Using (1)−(3) we obtain

$$\int_0^\infty e^{-\tau/s}\tau^{r-1}\, \Phi_k(\tau, \beta)\, d\tau = \int_0^\infty e^{-(1/s-i\beta)\tau}\tau^{r-1} e^{-i\beta\tau}\, \Phi_k(\tau, \beta)\, d\tau \xrightarrow[k\to\infty]{}$$

$$\int_0^\infty e^{-(1/s-i\beta)\tau}\tau^{r-1}\, d\tau = \frac{(r-1)!}{\left(\dfrac{1}{s} - i\beta\right)^r} = \frac{(r-1)!\, s^r}{(1 - is\beta)^r}$$

for every $s > 0$, $\beta \in R$ and $r \in \{1, 2, \ldots\}$.

But this was to prove.

10. Lemma. *Let Φ_k, $k \in \{1, 2, \ldots\}$, be a Laplace representative sequence. Then*

$$\int_0^\infty \int_{-\infty}^\infty e^{-\chi\tau}\tau^q |\Phi_k(\tau, \beta)|\, d\beta\, d\tau < \infty$$

for every $\chi > 0$, $q \in \{1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$.

Proof. Immediate consequence of (R_5).

11. Proposition. *Let $M \geqq 0$, $\omega \geqq 0$, $F \in (\mathrm{Re} > \omega) \to E$ and Φ_k, $k \in \{1, 2, \ldots\}$, be a Laplace representative sequence. If*

(α) *the function F is analytic in $(\mathrm{Re} > \omega)$,*

(β) $\|F(z)\| \leqq \dfrac{M}{\mathrm{Re}\, z - \omega}$ *for every* $\mathrm{Re}\, z > \omega$,

(γ) $\dfrac{1}{2\pi} \left\| \int_{-\infty}^\infty \Phi_k(t, \beta)\, F(\alpha + i\beta)\, d\beta \right\| \leqq M$ *for every $t > 0$, $\alpha > \omega$ and $k \in \{1, 2, \ldots\}$,*

then

$$\frac{1}{2\pi} \left\| \int_{-\infty}^\infty \frac{F(\alpha + i\beta)}{(1 - is\beta)^r}\, d\beta \right\| \leqq M$$

for every $s > 0$, $\alpha > \omega$ and $r \in \{2, 3, \ldots\}$.

Proof. With regard to (α) and (β) we obtain by means of Lemma 10 and of the Fubini theorem that

$$(1) \quad \int_0^\infty e^{-\tau/s}\tau^{r-1} \left(\int_{-\infty}^\infty \Phi_k(\tau, \beta)\, F(\alpha + i\beta)\, d\beta \right) d\tau =$$

$$= \int_{-\infty}^\infty \left(\int_0^\infty e^{-\tau/s}\tau^{r-1}\, \Phi_k(\tau, \beta)\, d\tau \right) F(\alpha + i\beta)\, d\beta$$

for every $s > 0$, $\alpha > \omega$ and $r \in \{2, 3, \ldots\}$.

By Lemma 9 we have

(2) $$\int_0^\infty e^{-\tau/s}\tau^{r-1}\,\Phi_k(\tau,\beta)\,d\tau \xrightarrow[k\to\infty]{} \frac{(r-1)!\,s^r}{(1-is\beta)^r}$$

for every $s>0$, $\beta\in\mathbb{R}$ and $r\in\{1,2,\ldots\}$.

Using Proposition 7 (sub (a)) we get

(3) $$\left|\int_0^\infty e^{-\tau/s}\tau^{r-1}\,\Phi_k(\tau,\beta)\,d\tau\right| \leqq \frac{1}{2\pi}\int_0^\infty e^{-\tau/s}\tau^{r-1}\,d\tau =$$

$$= \frac{1}{2\pi}(r-1)!\,s^r \text{ for every } s>0,\ \beta\in\mathbb{R},\ r\in\{1,2,\ldots\} \text{ and } k\in\{1,2,\ldots\}\,.$$

It follows from (R_6) that

(4) for every $s>0$, $r\in\{2,3,\ldots\}$ and $\varepsilon>0$ there exists $N>0$ such that

$$\int_N^\infty\left|\int_0^\infty e^{-\tau/s}\tau^{r-1}\,\Phi_k(\tau,\beta)\,d\tau\right|d\beta \leqq \varepsilon,$$

$$\int_{-\infty}^{-N}\left|\int_0^\infty e^{-\tau/s}\tau^{r-1}\,\Phi_k(\tau,\beta)\,d\tau\right|d\beta \leqq \varepsilon$$

for every $k\in\{1,2,\ldots\}$.

Using (α), (β) and (2), (3), (4) we obtain by the Lebesgue dominated convergence theorem that

(5) $$\int_{-\infty}^\infty\left(\int_0^\infty e^{-\tau/s}\tau^{r-1}\,\Phi_k(\tau,\beta)\,d\tau\right)F(\alpha+i\beta)\,d\beta \xrightarrow[k\to\infty]{} (r-1)!\,s^r\int_{-\infty}^\infty\frac{F(\alpha+i\beta)}{(1-is\beta)^r}\,d\beta$$

for every $s>0$, $\alpha>\omega$ and $r\in\{2,3,\ldots\}$.

On the other hand, it follows from (γ) that

(6) $$\left\|\int_0^\infty e^{-\tau/s}\tau^{r-1}\left(\int_{-\infty}^\infty \Phi_k(\tau,\beta)\,F(\alpha+i\beta)\right)d\beta\right\|\,d\tau \leqq 2\pi M\int_0^\infty e^{-\tau/s}\tau^{r-1} =$$

$$= 2\pi M(r-1)\,s^r \text{ for every } s>0,\ \alpha>\omega \text{ and } r\in\{1,2,\ldots\}.$$

The desired estimate is now an easy consequence of (1), (5) and (6).

12. Theorem (auxiliary). *Let $M\geqq 0$, $\omega\geqq 0$ and $F\in(\mathrm{Re}>\omega)\to E$. If the space E is reflexive, then the following statements* (A) *and* (B) *are equivalent*:

(A) (I) *the function F is analytic in* $(\mathrm{Re}>\omega)$,

(II) $\|F(z)\| \leqq \dfrac{M}{\mathrm{Re}\,z-\omega}$ *for every* $\mathrm{Re}\,z>\omega$,

(III) $\frac{1}{2\pi}\left\|\int_{-\infty}^{\infty}\frac{F(\alpha+i\beta)}{(1-is\beta)^r}\,d\beta\right\| \leqq M$ *for every* $s > 0$,

$\alpha > \omega$ *and* $r \in \{2, 3, \ldots\}$;

(B) *there exists a function* $f \in (0, \infty) \to E$ *such that*

(I) f *is measurable on* $(0, \infty)$,

(II) $\|f(t)\| \leqq M e^{\omega t}$ *for almost every* $t > 0$,

(III) $F(z) = \int_0^{\infty} e^{-z\tau} f(\tau)\,d\tau$ *for every* $\operatorname{Re} z > \omega$.

Proof. Use the same method as in the proof of [1], Theorem 7. The necessary modification of [1], Proposition 5 is in fact given in its proof.

13. Theorem. *Let* $M \geqq 0$, $\omega \geqq 0$, $F \in (\operatorname{Re} > \omega) \to E$ *and let* Φ_k, $k \in \{1, 2, \ldots\}$, *be a Laplace representative sequence. If the space E is reflexive, then the following statements* (A) *and* (B) *are equivalent*:

(A) (I) *the function F is analytic in* $(\operatorname{Re} > \omega)$,

(II) $\|F(z)\| \leqq \frac{M}{\operatorname{Re} z - \omega}$ *for every* $\operatorname{Re} z > \omega$,

(III) $\frac{1}{2\pi}\left\|\int_{-\infty}^{\infty}\Phi_k(t, \beta)\, F(\alpha + i\beta)\,d\beta\right\| \leqq M$ *for every* $t > 0$,

$\alpha > \omega$ *and* $k \in \{1, 2, \ldots\}$.

(B) *there exists a function* $f \in (0, \infty) \to E$ *such that*

(I) f *is measurable on* $(0, \infty)$,

(II) $\|f(t)\| \leqq M e^{\omega t}$ *for almost every* $t > 0$,

(III) $F(z) = \int_0^{\infty} e^{-z\tau} f(\tau)\,d\tau$ *for every* $\operatorname{Re} z > \omega$.

Proof. (A) $\Rightarrow$ (B): Immediate consequence of Proposition 11 and Theorem 12.

(B) $\Rightarrow$ (A): The properties (A) (I) and (II) are elementary easily provable properties of the Laplace transform.

Thus we have to prove (A) (III) only.

According to (R_1) we can apply the Fubini theorem in the following integral obtaining

$$\int_{-\infty}^{\infty}\Phi_k(t, \beta)\, F(\alpha + i\beta)\,d\beta = \int_{-\infty}^{\infty}\Phi_k(t, \beta)\left(\int_0^{\infty} e^{-(\alpha+i\beta)\tau} f(\tau)\,d\tau\right) d\beta =$$

$$= \int_0^{\infty} \left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tau\beta}\, \Phi_k(t, \beta)\, \mathrm{d}\beta \right) \mathrm{e}^{-\alpha\tau} f(\tau)\, \mathrm{d}\tau$$

for every $t > 0$, $\alpha > \omega$ and $k \in \{1, 2, \ldots\}$.

Using (R_2) we obtain from the preceding inequality that

$$\left\| \int_{-\infty}^{\infty} \Phi_k(t, \beta)\, F(\alpha + \mathrm{i}\beta)\, \mathrm{d}\beta \right\| \leqq \sup_{\tau > 0} \left(\mathrm{e}^{-\alpha\tau} \|f(\tau)\|\right) \int_0^{\infty} \left| \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tau\beta}\, \Phi_k(t, \beta)\, \mathrm{d}\beta \right| \mathrm{d}\tau \leqq$$

$$\leqq \sup_{\tau > 0} \left(\mathrm{e}^{-\alpha\tau} \|f(\tau)\|\right) \leqq \sup_{\tau > 0} \left(\mathrm{e}^{-\alpha\tau} M \mathrm{e}^{\omega\tau}\right) \leqq M$$

for every $t > 0$, $\alpha > \omega$ and $k \in \{1, 2, \ldots\}$, which proves (A) (III).

14. Remark. The implication (B) $\Rightarrow$ (A) of Theorem 13 was proved directly while the implication (A) $\Rightarrow$ (B) was reduced to the known Theorem 12 (precisely to its part (A) $\Rightarrow$ (B)). It is natural that it is also possible to proceed directly which will be done elsewhere.

15. Remark. We have seen in the course of the proof of Theorem 13 that the implication (B) $\Rightarrow$ (A) necessitates only the properties (R_1) and (R_2) of Laplace representative sequences.

16. Remark. In the light of Theorem 13 we can perform a certain evaluation of axioms $(\mathsf{R}_1)-(\mathsf{R}_6)$ of Laplace representative sequences.

It is clear that (R_1) is necessary even for the formulation of the condition (A) (III) in Theorem 13.

Let us now denote $\varphi_k(t, \tau) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tau\beta}\, \Phi_k(t, \beta)\, \mathrm{d}\beta$. The axioms $(\mathsf{R}_2)-(\mathsf{R}_4)$ say that the sequence $\varphi_k(t, \cdot)$ is essentially an approximation of the delta function at the point t. In the classical approach, such sequences are called singular integrals (see [6], Sec. 3.9 and [7], Chap. 10). If the functions φ_k depend only on the difference $t - \tau$, one speaks about approximate identity, which is a concept closely connected with the notion of Friedrich's mollifier. These properties $(\mathsf{R}_2)-(\mathsf{R}_4)$ seem also indispensable for proving Theorem 13 even if (R_2) is formulated in a somewhat restrictive form. The boundedness with a bound greater than 1 would be also sufficient but our restriction permits to retain (in a simple way) the constant M in Theorem 13 in both directions and, moreover, if the functions φ_k are nonnegative (as is usual), then the choice of the bound 1 is only a matter of normalization. Moreover, let us call the attention to the fact that axioms $(\mathsf{R}_2)-(\mathsf{R}_4)$ are sufficient to produce a restricted form of the inversion theorem (cf. Theorem 31 a Remark 32).

Axioms (R_5) and (R_6) are probably formulated for the first time and may seem a little involved. It is true that they were established to facilitate the proof of the implication (A) $\Rightarrow$ (B) in Theorem 13. At present, we are not able to say anything about their necessity and even independence of the remaining axioms but a careful analysis of the above mentioned proof (and other procedures) indicates that it is improbable that they could be omitted even if certain improvements are not excluded. A weakening of the property (R_6) at the cost of an additional assumption on the function F in Theorem 13 is sketched in the following Remark 17. In the special case of the so called summatory representative sequences (cf. Sec. 25), axioms (R_5) and (R_6) are dependent on $(\mathsf{R}_1)-(\mathsf{R}_4)$ as shown in Proposition 26.

17. Remark. Let us add the following condition in Theorem 13:

(A) (IV) $F(\alpha + i\beta) \xrightarrow[|\beta| \to \infty]{} 0$ for every $\alpha > \omega$.

This is admissible since, as is well-known, (A) (IV) follows from (B) (see e.g. [4], Prop. 5.1.2). But with the condition (A) (IV), Theorem 13 remains valid for a (at least formally) larger class of Laplace representative sequences, namely, we can replace the property (R_6) by

(R_6') for every $\chi > 0$ and $q \in \{1, 2, \ldots\}$, there exists a constant $c \geqq 0$ such that

$$\int_{-\infty}^{\infty} \left| \int_0^{\infty} e^{-\chi\tau} \tau^q \, \Phi_k(\tau, \beta) \, d\tau \right| d\beta \leqq c$$

for every $k \in \{1, 2, \ldots\}$.

Indeed, the reader himself easily establishes that the decisive Proposition 11 then remains valid if we add (A) (IV) to its assumptions.

18. Let us define

$$\operatorname{wid}_k(t, \beta) = \frac{1}{2\pi} \frac{1}{\left(1 - i \dfrac{t}{k+1} \beta\right)^{k+1}}$$

for $t > 0$, $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$.

Sometimes, the sequence wid_k, $k \in \{1, 2, \ldots\}$, will be called the *Widder representative sequence.*

19. Lemma. $\dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-it\beta}}{(1 - is\beta)^{k+1}} \, d\beta = \frac{1}{k!} \frac{1}{s^{k+1}} e^{-t/s} t^k$

for $t > 0$, $s > 0$ *and* $k \in \{1, 2, \ldots\}$,

$$= 0$$

for $t < 0$, $s > 0$ *and* $k \in \{1, 2, \ldots\}$.

Proof. Immediate consequence of the Fourier inversion theorem (after simple substitution) since

$$\int_0^{\infty} e^{i\beta\tau} \left(\frac{1}{k!} \frac{1}{s^{k+1}} e^{-\tau/s} \tau^k \right) d\tau = \frac{1}{k!} \frac{1}{s^{k+1}} \int_0^{\infty} e^{-(1/s - i\beta)\tau} \tau^k \, d\tau =$$

$$= \frac{1}{k!} \frac{1}{s^{k+1}} \frac{k!}{\left(\frac{1}{s} - \mathrm{i}\beta\right)^{k+1}} = \frac{1}{(1 - \mathrm{i}s\beta)^{k+1}}$$

for every $s > 0$, $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$.

20. Proposition. *The sequence* wid_k, $k \in \{1, 2, \ldots\}$, *is a Laplace representative sequence.*

Proof. The property (R_1) is immediate.

By Lemma 19 we can write

$$(1)\quad \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tau\beta} \, \mathrm{wid}_k(t, \beta) \, \mathrm{d}\beta = \frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \mathrm{e}^{-\tau(k+1)/t} \tau^k$$

for every $t > 0$, $\tau > 0$ and $k \in \{1, 2, \ldots\}$,

$$(2)\quad \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tau\beta} \, \mathrm{wid}_k(\tau, \beta) \, \mathrm{d}\beta = 0 \text{ for every } t > 0,\ \tau \leqq 0 \text{ and } k \in \{1, 2, \ldots\}.$$

Now properties (R_2) and (R_3) are immediate consequences of (1) and (2) since

$$\int_0^{\infty} \mathrm{e}^{-\tau(k+1)/t} \tau^k \, \mathrm{d}\tau = \frac{k!}{\left(\frac{k+1}{t}\right)^{k+1}} \quad \text{for} \quad t > 0 \quad \text{and} \quad k \in \{1, 2, \ldots\}.$$

Now we prove (R_4). To this aim let us fix $t > 0$ and $\delta > 0$. Without loss of generality we can suppose $\delta < t$. According to (1) and (2) we have to verify

$$(3)\quad \frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \left[\int_0^{t-\delta} \mathrm{e}^{-\tau(k+1)/t} \tau^k \, \mathrm{d}\tau + \int_{t+\delta}^{\infty} \mathrm{e}^{-\tau(k+1)/t} \tau^k \, \mathrm{d}\tau\right] \xrightarrow[k \to \infty]{} 0.$$

Since clearly $(1/\delta^2)(\tau - t)^2 \geqq 1$ for every $|\tau - t| \geqq \delta$ we easily get

$$\frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \left[\int_0^{t+\delta} \mathrm{e}^{-\tau(k+1)/t} \tau^k \, \mathrm{d}\tau + \int_{t-\delta}^{\infty} \mathrm{e}^{-\tau(k+1)/t} \tau^k \, \mathrm{d}\tau\right] =$$

$$= \frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \int_{\{\tau:\tau>0, |\tau-t| \geqq \delta\}} \mathrm{e}^{-\tau(k+1)/t} \tau^k \, \mathrm{d}\tau \leqq$$

$$\leqq \frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \int_{\{\tau:\tau>0, |\tau-t| \geqq \delta\}} \mathrm{e}^{-\tau(k+1)/t} \tau^k \frac{1}{\delta^2} (\tau - t)^2 \, \mathrm{d}\tau \leqq$$

$$\leqq \frac{1}{\delta^2} \frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \int_0^{\infty} \mathrm{e}^{-\tau(k+1)/t} \tau^k (\tau - t)^2 \, \mathrm{d}\tau =$$

$$= \frac{1}{\delta^2} \frac{1}{k!} \left(\frac{k+1}{t}\right)^{k+1} \int_0^{\infty} \mathrm{e}^{-\tau(k+1)/t} (\tau^{k+2} - 2t\tau^{k+1} + t^2 \tau^k) \, \mathrm{d}\tau =$$

$$= \frac{1}{\delta^2}\frac{1}{k!}\left(\frac{k+1}{t}\right)^{k+1}\left[\frac{(k+2)!}{\left(\frac{k+1}{t}\right)^{k+3}} - 2t\frac{(k+1)!}{\left(\frac{k+1}{t}\right)^{k+2}} + t^2\frac{k!}{\left(\frac{k+1}{t}\right)^{k+1}}\right] =$$

$$= \frac{t^2}{\delta^2}\left[\frac{k+2}{k+1} - 2 + 1\right] = \frac{t^2}{\delta^2}\frac{1}{k+1} \quad \text{for every } k \in \{1, 2, \dots\},$$

which proves (3).

Further, (R_5) follows from the following estimate for $q = 1$ (the full wording will be used later):

$$(4) \quad \int_0^\infty e^{-\chi\tau}\tau^q \left(\int_{-\infty}^\infty |\mathrm{wid}_k(\tau, \beta)| \, d\beta\right) d\tau =$$

$$= \int_0^\infty e^{-\chi\tau}\tau^q \left[\int_{-\infty}^\infty \frac{1}{\left(1 + \left(\frac{\tau}{k+1}\beta\right)^2\right)^{(k+1)/2}} \, d\beta\right] d\tau =$$

$$= (k+1)\int_0^\infty e^{-\chi\tau}\tau^{q-1} \, d\tau \int_{-\infty}^\infty \frac{1}{(1+\beta^2)^{(k+1)/2}} \, d\beta \leqq$$

$$\leqq \frac{(k+1)(q-1)!}{\chi^q}\int_{-\infty}^\infty \frac{1}{1+\beta^2} \, d\beta$$

for every $x > 0$, $q \in \{1, 2, \dots\}$ and $k \in \{1, 2, \dots\}$.

It remains to prove the most complicated property (R_6).

We begin with some auxiliary considerations.

It is easy to see that

$$(5) \quad \int_0^\infty e^{-\chi\tau}\tau^q \frac{1+\beta^2}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}} \, d\tau = \int_0^\infty e^{-\chi\tau}\tau^q \frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}} \, d\tau -$$

$$- \frac{d}{d\chi}\left[\int_0^\infty e^{-\chi\tau}\tau^{q-1} \frac{\beta^2}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}} \, d\tau\right]$$

for every $\chi > 0$, $\beta \in R$, $q \in \{1, 2, \dots\}$ and $k \in \{1, 2, \dots\}$.

The first term on the right hand side of (5) can be estimated as follows

$$(6) \quad \left|\int_0^\infty e^{-\chi\tau}\tau^q \frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}} \, d\tau\right| \leqq \int_0^\infty e^{-\chi\tau}\tau^q = \frac{q!}{\chi^{q+1}}$$

for every $\chi > 0$, $\beta \in R$, $q \in \{1, 2, \dots\}$ and $k \in \{1, 2, \dots\}$.

We need a similar estimate also for the second term on the right hand side of (5) but this will be essentially more laborious.

To this aim we first recall two elementary identities

$$(7)\quad \frac{d}{d\tau}\frac{1}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k-1}} = i\,\frac{k-1}{k+1}\,\frac{\beta}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k}},$$

$$\frac{d^2}{d\tau^2}\frac{1}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k-1}} = -\frac{k(k-1)}{(k+1)^2}\,\frac{\beta^2}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k+1}}$$

for every $\tau > 0$, $\beta \in \mathbb{R}$ and $k \in \{1, 2, \dots\}$.

Integrating by parts and using (7) we obtain

$$(8)\quad \int_0^\infty e^{-\chi\tau}\frac{\beta^2}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k+1}}\,d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}\left[\frac{d^2}{d\tau^2}\frac{1}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k-1}}\right]d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\,\chi\int_0^\infty e^{-\chi\tau}\left[\frac{d}{d\tau}\frac{1}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k-1}}\right]d\tau + i\,\frac{k+1}{k}\,\beta =$$

$$= -\frac{(k+1)^2}{k(k-1)}\,\chi^2\int_0^\infty e^{-\chi\tau}\frac{1}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k-1}}\,d\tau +$$

$$+\frac{(k+1)^2}{k(k-1)}\,\chi + i\,\frac{k-1}{k}\,\beta$$

for every $\chi > 0$, $\beta \in \mathbb{R}$ and $k \in \{2, 3, \dots\}$,

$$(9)\quad \int_0^\infty e^{-\chi\tau}\tau\frac{\beta^2}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k+1}}\,d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}\tau\left[\frac{d^2}{d\tau^2}\frac{1}{\left(1-i\dfrac{\tau}{k+1}\beta\right)^{k-1}}\right]d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi\tau - 1)\left[\frac{d}{d\tau}\frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k-1}}\right]d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi^2\tau - 2\chi)\frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k-1}}d\tau + \frac{(k+1)^2}{k(k-1)}$$

for every $\chi > 0$, $\beta \in \mathbb{R}$ and $k \in \{2, 3, \ldots\}$,

$$(10)\ \int_0^\infty e^{-\chi\tau}\tau^{q-1}\frac{\beta^2}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}}d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}\tau^{q-1}\left[\frac{d^2}{d\tau^2}\frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k-1}}\right]d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi\tau^{q-1} - (q-1)\,\tau^{q-2})\left[\frac{d}{d\tau}\frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k-1}}\right]d\tau =$$

$$= -\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi^2\tau^{q-1} - 2\chi(q-1)\,\tau^{q-2} +$$

$$+ (q-1)(q-2)\,\tau^{q-3})\frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k-1}}d\tau$$

for every $\chi > 0$, $\beta \in \mathbb{R}$, $q \in \{3, 4, \ldots\}$ and $k \in \{1, 2, \ldots\}$.

It follows from (8) that

$$(11)\ \frac{d}{d\chi}\left[\int_0^\infty e^{-\chi\tau}\frac{\beta^2}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}}d\tau\right] =$$

$$= \frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi^2\tau - 2\chi)\frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k-1}}d\tau + \frac{(k+1)^2}{k(k-1)}$$

for every $\chi > 0$, $\beta \in \mathbb{R}$ and $k \in \{2, 3, \ldots\}$.

Further, by (9) and (10)

$$(12)\quad \frac{\mathrm{d}}{\mathrm{d}\chi}\left[\int_0^\infty e^{-\chi\tau}\tau^{q-1}\frac{\beta^2}{\left(1-\mathrm{i}\dfrac{\tau}{k+1}\beta\right)^{k+1}}\,\mathrm{d}\tau\right]=$$

$$=\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi^2\tau^q-2\chi q\tau^{q-1}+q(q-1)\,\tau^{q-2})\frac{1}{\left(1-\mathrm{i}\dfrac{\tau}{k+1}\beta\right)^{k-1}}\,\mathrm{d}\tau$$

for every $\chi>0$, $\beta\in\mathbb{R}$, $q\in\{2,3,\ldots\}$ and $k\in\{2,3,\ldots\}$.

We obtain from (11) and (12) that

$$(13)\quad \left|\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\int_0^\infty e^{-\chi\tau}\frac{\beta^2}{\left(1-\mathrm{i}\dfrac{\tau}{k+1}\beta\right)^{k+1}}\,\mathrm{d}\tau\right]\right|\leqq$$

$$\leqq\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi^2\tau+2\chi)\,\mathrm{d}\tau+\frac{(k+1)^2}{k(k-1)}=$$

$$=\frac{(k+1)^2}{k(k-1)}\left(\chi^2\frac{1}{\chi^2}+2\chi\frac{1}{\chi}\right)+\frac{(k+1)^2}{k(k-1)}=4\frac{(k+1)^2}{k(k-1)}$$

for every $\chi>0$, $\beta\in\mathbb{R}$ and $k\in\{2,3,\ldots\}$,

$$(14)\quad \left|\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\int_0^\infty e^{-\chi\tau}\tau^{q-1}\frac{\beta^2}{\left(1-\mathrm{i}\dfrac{\tau}{k+1}\beta\right)^{k+1}}\,\mathrm{d}\tau\right]\right|\leqq$$

$$\leqq\frac{(k+1)^2}{k(k-1)}\int_0^\infty e^{-\chi\tau}(\chi^2\tau^q+2\chi q\tau^{q-1}+q(q-1)\,\tau^{q-2})\,\mathrm{d}\tau=$$

$$=\frac{(k+1)^2}{k(k-1)}\left(\chi^2\frac{q!}{\chi^{q+1}}+2\chi q\frac{(q-1)!}{\chi^q}+q(q-1)\frac{(q-2)!}{\chi^{q-1}}\right)=4\frac{(k+1)^2}{k(k-1)}\frac{q!}{\chi^{q-1}}$$

for every $\chi>0$, $\beta\in\mathbb{R}$, $q\in\{2,3,\ldots\}$ and $k\in\{2,3,\ldots\}$.

Summarizing (13) and (14) we get

$$(15)\quad \left|\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\int_0^\infty e^{-\chi\tau}\tau^{q-1}\frac{\beta^2}{\left(1-\mathrm{i}\dfrac{\tau}{k+1}\beta\right)^{k+1}}\,\mathrm{d}\tau\right]\right|\leqq$$

$$\leqq 4\frac{(k+1)^2}{k(k-1)}\frac{q!}{\chi^{q-1}}\leqq 4\,.\,6\frac{q!}{\chi^{q-1}}=24\frac{q!}{\chi^{q-1}}$$

for every $\chi>0$, $\beta\in\mathbb{R}$, $q\in\{1,2,\ldots\}$ and $k\in\{2,3,\ldots\}$.

It follows from (5), (6) and (15) that

$$(16)\quad \left|\int_0^\infty e^{-\chi\tau}\tau^q \operatorname{wid}_k(\tau,\beta)\,d\tau\right| = \frac{1}{2\pi}\left|\int_0^\infty e^{-\chi\tau}\tau^q \frac{1}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}}\,d\tau\right| =$$

$$= \frac{1}{2\pi}\frac{1}{1+\beta^2}\left|\int_0^\infty e^{-\chi\tau}\tau^q \frac{1+\beta^2}{\left(1 - i\frac{\tau}{k+1}\beta\right)^{k+1}}\,d\tau\right| \leqq$$

$$\leqq \frac{1}{2\pi}\frac{1}{1+\beta^2}\left(\frac{q!}{\chi^{q+1}} + 24\frac{q!}{\chi^{q-1}}\right)$$

for every $\chi > 0$, $\beta \in \mathbb{R}$, $q \in \{1, 2, \ldots\}$ and $k \in \{2, 3, \ldots\}$.

Now the property (R_6) immediately follows from (4) (for $k = 1$) and from (16).

21. Remark. Proposition 20 shows that Theorem 12 is a formal consequence of Theorem 13 since the difference between the inequalities (A) (III) in Theorem 12 and (A) (III) with $\Phi_k = \operatorname{wid}_k$, $k \in \{1, 2, \ldots\}$, in Theorem 13 is only formal.

22. Let us define $\operatorname{roo}_k(t, \beta) = \dfrac{1}{2\pi}\dfrac{\sqrt{k}\,e^{2k}}{\sqrt{(k + it\beta)}}\, e^{-2\sqrt{k}\sqrt{(k+it\beta)}}$ for $t > 0$, $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$.

The sequence roo_k, $k \in \{1, 2, \ldots\}$, will be called the *Rooney representative sequence.*

23. Lemma. $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-i\tau\beta}\frac{\sqrt{k}\,e^{2k}}{\sqrt{(k + it\beta)}}\, e^{-2\sqrt{k}\sqrt{(k+it\beta)}}\,d\beta = e^{2k}\sqrt{\frac{k}{\pi t}}\, e^{-k(\tau/t + t/\tau)}\frac{1}{\sqrt{\tau}}$
for every $t > 0$, $\tau > 0$ *and* $k \in \{1, 2, \ldots\}$,

$= 0$ *for every* $t < 0$, $\tau \leqq 0$ *and* $k \in \{1, 2, \ldots\}$.

Proof. Use formulas [9] 4.5 (3) and (27) and Fourier inversion theorem.

24. Proposition. *The sequence* roo_k, $k \in \{1, 2, \ldots\}$, *is a Laplace representative sequence.*

Proof. The property (R_1) is obvious.

The properties $(R_2)-(R_4)$ follow from Lemma 23 by using Laplace's asymptotic evaluation treated in [10], Ch. VII, Section 2. Cf. the proof of Theorem 1 in [11].

Now we have to prove (R_5) and (R_6). But this can be done in a very similar way as in the proof of Proposition 20 and thus we omit the details.

25. A Laplace representative sequence Φ_k, $k \in \{1, 2, \ldots\}$, is called a *summatory representative sequence* if the function $e^{-it\beta}\,\Phi_k(t, \beta)$ is independent of $t > 0$ for every $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$.

26. Proposition. *Let $\Phi_k \in (0, \infty) \times \mathbb{R} \to \mathbb{C}$ for every $k \in \{1, 2, \ldots\}$. The sequence Φ_k, $k \in \{1, 2, \ldots\}$, is a summatory representative sequence if and only if*

(a) *the function $e^{-it\beta} \Phi_k(t, \beta)$ is independent of $t > 0$ for every $\beta \in \mathbb{R}$ and $k \in \in \{1, 2, \ldots\}$,*

(b) *the conditions $(R_1)-(R_4)$ are fulfilled.*

Proof. We only have to prove that the conditions (R_5) and (R_6) hold if the assumptions (a) and (b) are fulfilled.

Let us denote $\Psi_k(\beta) = e^{it\beta} \Phi_k(t, \beta)$ for $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$, which is admissible according to (a).

It follows from (R_1) which is valid according to (b) that

$$(1)\quad \int_{-\infty}^{\infty} |\Psi_k(\beta)|\, d\beta = \int_{-\infty}^{\infty} |e^{-it\beta} \Phi_k(t, \beta)|\, d\beta < \infty \text{ for every } k \in \{1, 2, \ldots\}.$$

Using (1) we obtain

$$(2)\quad \int_0^{\infty}\int_{-\infty}^{\infty} e^{-\chi\tau}|\Phi_k(\tau, \beta)|\, d\beta\, d\tau = \int_0^{\infty} e^{-\chi\tau}\tau\, d\tau \int_{-\infty}^{\infty} |\Psi_k(\beta)|\, d\beta = \frac{1}{\chi}\int_{-\infty}^{\infty} |\Psi_k(\beta)|\, d\beta$$

for every $\chi > 0$ and $k \in \{1, 2, \ldots\}$.

But (1) and (2) prove (R_5).

Further, we can write

$$(3)\quad \int_0^{\infty} e^{-\chi\tau}\tau^q\, \Phi_k(\tau, \beta)\, d\tau = \int_0^{\infty} e^{-\chi\tau}\tau^q\, e^{i\tau\beta}\, \Psi_k(\beta)\, d\tau =$$

$$= \int_0^{\infty} e^{-(\chi - i\beta)\tau}\tau^q\, d\tau\, \Psi_k(\beta) = \frac{q!}{(\chi - i\beta)^{q+1}}\, \Psi_k(\beta)$$

for every $\chi > 0$, $\beta \in \mathbb{R}$, $q \in \{1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$.

Using the Fourier inversion theorem we obtain from (R_1) and (R_2), which are applicable according to (b), that

$$(4)\quad |\Psi_k(\beta)| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\beta}\left(\int_{-\infty}^{\infty} e^{-i\tau\eta}\, \Psi_k(\eta)\, d\eta\right) d\tau\right| \leqq$$

$$\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{-i\tau\eta}\, \Psi_k(\eta)\, d\eta\right| d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{-i\tau\eta}e^{-it\eta}\, \Phi_k(t, \eta)\, d\eta\right| d\tau =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{-i\sigma\eta}\, \Phi_k(t, \eta)\, d\eta\right| d\sigma \leqq \frac{1}{2\pi}$$

for every $\beta \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$.

Now (3) and (4) give

$$(5)\quad \left|\int_0^\infty e^{-\chi\tau}\tau^q\,\Phi_k(\tau,\beta)\,d\tau\right| \leqq \frac{1}{2\pi}\,\frac{q!}{(\chi^2+\beta^2)^{(q+1)/2}} \leqq \frac{1}{2\pi}\,\frac{q!}{\chi^{q+1}}\,\frac{1}{\chi^2+\beta^2}$$

for every $\chi > 0$, $\beta \in R$, $q \in \{1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$.

But (5) implies (R_6).
The proof is complete.

27. Remark. An easy inspection shows that the conditions $(R_1)-(R_4)$ from the preceding Proposition 26 are almost identical with (A_1), (C), (D), (E) of [3], p. 224 and 225. The only difference is in (R_2) and (C). As to this difference, see Remark 16.

28. Examples of summatory representative sequences:

(C) **Cesàro representative sequence:**

$$\mathrm{ces}_k(t,\beta) = \frac{1}{2\pi}\,e^{it\beta}\left(1-\frac{|\beta|}{k}\right) \quad \text{for} \quad t>0\,,\quad |\beta| \leqq k \quad \text{and} \quad k\in\{1,2,\ldots\}\,,$$

$$= 0 \quad \text{for} \quad t>0\,,\quad \beta>k\,,\quad k\in\{1,2,\ldots\}\,,$$

(A) **Abel representative sequence:**

$$\mathrm{abe}_k(t,\beta) = \frac{1}{2\pi}\,e^{it\beta}e^{-|\beta|/k} \quad \text{for} \quad t>0\,,\quad \beta\in R \quad \text{and} \quad k\in\{1,2,\ldots\}\,,$$

(G) **Gauss representative sequence:**

$$\mathrm{gau}_k(t,\beta) = \frac{1}{2\pi}\,e^{it\beta}e^{-\beta^2/k^2} \quad \text{for} \quad t>0\,,\quad \beta\in R \quad \text{and} \quad k\in\{1,2,\ldots\}\,,$$

(N^+) **Nemo right representative sequence:**

$$\mathrm{nem}_k^+(t,\beta) = \frac{1}{2\pi}\,e^{it\beta}\,\frac{k^2}{(k-i\beta)^2} \quad \text{for} \quad t>0\,,\quad \beta\in R \quad \text{and} \quad k\in\{1,2,\ldots\}\,,$$

(N^-) **Nemo left representative sequence:**

$$\mathrm{nem}_k^-(t,\beta) = \frac{1}{2\pi}\,e^{it\beta}\,\frac{k^2}{(k+i\beta)^2} \quad \text{for} \quad t>0\,,\quad \beta\in R \quad \text{and} \quad k\in\{1,2,\ldots\}\,.$$

29. Lemma. *We have*

$$(c)\quad \frac{1}{2\pi}\int_{-k}^{k} e^{-i(\tau-t)\beta}\left(1-\frac{|\beta|}{k}\right)d\beta = \frac{k}{2\pi}\left(\frac{\sin\frac{1}{2}k(t-\tau)}{\frac{1}{2}k(t-\tau)}\right)^2,$$

(a) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-i(\tau-t)\beta}e^{-|\beta|/k}\,d\beta = \frac{k}{\pi}\,\frac{1}{1+(k(t-\tau))^2}$,

(g) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-i(\tau-t)\beta}e^{-\beta^2/k^2}\,d\beta = \frac{k}{2\sqrt{\pi}}\,e^{-(k(t-\tau))^2/4}$,

(n^+) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-i(\tau-t)\beta}\,\frac{k^2}{(k-i\beta)^2}\,d\beta = k^2(\tau-t)\,e^{-k(\tau-t)}$, $\tau > t$,

$= 0$, $\tau \leqq t$,

(n^-) $\dfrac{1}{2\pi}\,e^{-i(\tau-t)\beta}\,\dfrac{k^2}{(k+i\beta)^2}\,d\beta = k^2(t-\tau)\,e^{-k(t-\tau)}$, $\tau < t$,

$= 0$, $\tau \geqq t$,

for every $t > 0$, $\tau \in R$ *and* $k \in \{1, 2, \ldots\}$.

Proof. The identities (c), (a) and (g) are frequent, see e.g. [4], p. 407.

In the case of (n^+) and (n^-) it is easy to calculate the Fourier transforms of the right hand sides of these identities and then use the Fourier inversion theorem.

30. Proposition. *The sequences* ces_k, abe_k, gau_k, nem_k^+, nem_k^-, $k \in \{1, 2, \ldots\}$ *are Laplace representative sequences.*

Proof. Use Proposition 26 and Lemma 29.

31. Corollary. *Let* $M \geqq 0$, $\omega \geqq 0$ *and* $F \in (\mathrm{Re} > \omega) \to E$. *If the space* E *is reflexive, then the following statements* (A) *and* (B) *are equivalent:*

(A) (I) *the function* F *is analytic in* $(\mathrm{Re} > \omega)$,

(II) $\|F(z)\| \leqq \dfrac{M}{\mathrm{Re}\, z - \omega}$ *for every* $\mathrm{Re}\, z > \omega$,

(III) *one of the following inequalities holds:*

(W) $\dfrac{1}{2\pi}\left\|\displaystyle\int_{-\infty}^{\infty} \frac{F(\alpha+i\beta)}{[1-i(t/(k+1))\beta]^{k+1}}\,d\beta\right\| \leqq M$,

(R) $\dfrac{1}{2\pi}\left\|\displaystyle\int_{-\infty}^{\infty} \frac{\sqrt{k}\,e^{2k}}{\sqrt{(k+it\beta)}}\,e^{-2\sqrt{k}\sqrt{(k+it\beta)}}\,F(\alpha+i\beta)\,d\beta\right\| \leqq M$,

(C) $\dfrac{1}{2\pi}\left\|\displaystyle\int_{-k}^{k} e^{it\beta}\left(1-\frac{|\beta|}{k}\right)F(\alpha+i\beta)\,d\beta\right\| \leqq M$,

(A) $\dfrac{1}{2\pi}\left\|\displaystyle\int_{-\infty}^{\infty} e^{it\beta}e^{-|\beta|/k}\,F(\alpha+i\beta)\,d\beta\right\| \leqq M$,

(G) $\frac{1}{2\pi}\left\|\int_{-\infty}^{\infty} e^{it\beta}e^{-\beta^2/k^2}F(\alpha+i\beta)\,d\beta\right\| \leqq M,$

(N⁺) $\frac{1}{2\pi}\left\|\int_{-\infty}^{\infty} e^{it\beta}\frac{k^2}{(k-i\beta)^2}F(\alpha+i\beta)\,d\beta\right\| \leqq M,$

(N⁻) $\frac{1}{2\pi}\left\|\int_{-\infty}^{\infty} e^{it\beta}\frac{k^2}{(k+i\beta)^2}F(\alpha+i\beta)\,d\beta\right\| \leqq M$

for every $t > 0$, $\alpha > \omega$ *and* $k \in \{1, 2, \ldots\}$,

(B) *as in* Theorem 13.

Proof. Use Theorem 13 and Propositions 20, 24 and 30.

32. Theorem. *Let* $\omega \geqq 0, f \in (0, \infty) \to E$ *and* $\Phi_k \in (0, \infty) \times R \to C$, $k \in \{1, 2, \ldots\}$. *If*

(α) *the function* f *is measurable on* $(0, \infty)$,
(β) *there is a constant* $M \geqq 0$ *such that* $\|f(t)\| \leqq Me^{\omega t}$ *for almost every* $t > 0$,
(γ) *the sequence* Φ_k, $k \in \{1, 2, \ldots\}$, *satisfies the conditions* (R₁)–(R₄),

then

$$e^{\alpha t}\int_{-\infty}^{\infty}\Phi_k(t,\beta)\left(\int_0^{\infty}e^{-(\alpha+i\beta)\tau}f(\tau)\,d\tau\right)d\beta \xrightarrow[\substack{k\to\infty\\ k\in\{1,2,\ldots\}}]{} f(t)$$

for every continuity point $t > 0$ *of the function* f *and every* $\alpha > \omega$.

Proof. According to Fubini's theorem we get from (R₁) and (α), (β) that

(1) $$\int_{-\infty}^{\infty}\Phi_k(t,\beta)\left(\int_0^{\infty}e^{-(\alpha+i\beta)\tau}f(\tau)\,d\tau\right)d\beta = \int_0^{\infty}\left(\int_{-\infty}^{\infty}e^{-i\tau\beta}\,\Phi_k(t,\beta)\,d\beta\right)e^{-\alpha\tau}f(\tau)\,d\tau$$

for every $t > 0$, $\alpha > \omega$ and $k \in \{1, 2, \ldots\}$.

Let us now write

(2) $\phi_k(t,\tau) = \int_{-\infty}^{\infty} e^{-i\tau\beta}\,\Phi_k(t,\beta)\,d\beta$ for $t > 0$, $\tau \in R$ and $k \in \{1, 2, \ldots\}$.

According to (R₂), (R₃) and (R₄) we obtain from (2) that

(3) $\int_{-\infty}^{\infty}|\phi_k(t,\tau)|\,d\tau \leqq 1$ for every $t > 0$ and $k \in \{1, 2, \ldots\}$,

(4) $\int_{-\infty}^{\infty}\phi_k(t,\tau)\,d\tau \xrightarrow[k\to\infty]{} 1$ for every $t > 0$,

(5) $\int_{\{\tau:|\tau-t|\geqq\delta\}}|\phi_k(t,\tau)|\,d\tau \xrightarrow[k\to\infty]{} 0$ for every $t > 0$ and $\delta > 0$.

Let us now consider the integral $\int_0^\infty \phi_k(t, \tau)\, e^{-\alpha\tau} f(\tau)\, d\tau$ for some $\alpha > \omega$ and $t > 0$, at which the function f is continuous. Splitting the integral in question into two parts, on a small neighborhood of t and on the rest, and using (3), (4) and (5) we get after a little computation that

(6) $\int_0^\infty \phi_k(t, \tau)\, e^{-\alpha\tau} f(\tau)\, d\tau \xrightarrow[k\to\infty]{} e^{-\alpha t} f(t)$ for every continuity point $t > 0$ of f and for every $\alpha > \omega$.

The desired result follows from (1), (2) and (6).

33. Corollary. *Let* $\omega \geqq 0$ *and* $f \in (0, \infty) \to E$. *If the assumptions* (α) *and* (β) *of* Theorem 32 *hold, then*

(W) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha t}}{[1 - i(t/(k+1))\beta]^{k+1}} \left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t),$$

(R) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha t}\sqrt{k}\, e^{2k}}{\sqrt{(k + it\beta)}}\, e^{-2\sqrt{k}\sqrt{(k+it\beta)}} \left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t),$$

(C) $$\frac{1}{2\pi}\int_{-k}^{k} e^{(\alpha+i\beta)t}\left(1 - \frac{|\beta|}{k}\right)\left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t),$$

(A) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t} e^{-|\beta|/k}\left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t),$$

(G) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t} e^{-\beta^2/k^2}\left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t),$$

(N⁺) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t} \frac{k^2}{(k - i\beta)^2}\left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t),$$

(N⁻) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t} \frac{k^2}{(k + i\beta)^2}\left(\int_0^\infty e^{-(\alpha+i\beta)\tau} f(\tau)\, d\tau\right) d\beta \xrightarrow[k\in\{1,2,\ldots\}]{k\to\infty} f(t)$$

for every continuity point $t > 0$ *of the function* f *and for every* $\alpha > \omega$.

Proof. Immediate consequence of Theorem 32 and Propositions 20, 24 and 30.

34. Theorem. *Let* $\omega \geqq 0$, $f \in (0, \infty) \to E$, $\Phi_k \in (0, \infty) \times R \to C$, $k \in \{1, 2, \ldots\}$, *and* $\vartheta \in C$. *If*
(α), (β), (γ) *as in* Theorem 32,

(δ) $\int_{-\infty}^{t}\left(\int_{-\infty}^{\infty} e^{-i\tau\beta}\, \Phi_k(t, \beta)\, d\beta\right) d\tau \xrightarrow[k\to\infty]{} \vartheta$ *for every* $t > 0$, *then*

$$e^{\alpha t}\int_{-\infty}^{\infty}\Phi_k(t,\beta)\left(\int_0^{\infty}e^{-(\alpha+i\beta)\tau}f(\tau)\,d\tau\right)d\beta \xrightarrow[k\in\{1,2,\dots\}]{k\to\infty} \lim_{\substack{s\to t\\ s<t}}(\vartheta f(s)) + \lim_{\substack{s\to t\\ s>t}}((1-\vartheta)f(s))$$

for every $t > 0$ *such that* $\lim_{\substack{s\to t\\ s<t}}(\vartheta f(s))$ *and* $\lim_{\substack{s\to t\\ s>t}}((1-\vartheta)f(s))$ *exist and for every* $\alpha > \omega$.

Proof. We proceed similarly as in the proof of Theorem 32. It is only necessary to split the integral $\int_0^{\infty}\phi_k(t,\tau)\,e^{-\alpha\tau}f(\tau)\,d\tau$ into two parts: $\int_0^t$ and $\int_t^{\infty}$ and utilize the additional assumption (δ).

35. Proposition. *We have for every* $t > 0$:

(a) $$\int_{-\infty}^{t}\left(\int_{-\infty}^{\infty}e^{-i\tau\beta}\,\mathrm{wid}_k(t,\beta)\,d\beta\right)d\tau \xrightarrow[k\in\{1,2,\dots\}]{k\to\infty} \tfrac{1}{2}$$

and the same relation for roo_k, ces_k, abe_k *and* gau_k,

(b) $$\int_{-\infty}^{t}\left(\int_{-\infty}^{\infty}e^{-i\tau\beta}\,\mathrm{nem}_k^{+}(t,\beta)\,d\beta\right)d\tau \xrightarrow[k\in\{1,2,\dots\}]{k\to\infty} 0,$$

(c) $$\int_{-\infty}^{t}\left(\int_{-\infty}^{\infty}e^{-i\tau\beta}\,\mathrm{nem}_k^{-}(t,\beta)\,d\beta\right)d\tau \xrightarrow[k\in\{1,2,\dots\}]{k\to\infty} 1.$$

Proof. An easy consequence of Lemmas 19, 23 and 29.

36. Corollary. *Let* $\omega \geqq 0$ *and* $f \in (0,\infty) \to E$. *If the assumptions* (α) *and* (β) *of* Theorem 32 *hold, then the statements of* Corollary 33 *are true with the following extensions*:

(a) *the relations* (W), (R), (C), (A) and (G) *hold with* $\frac{1}{2}(f(t_+) + f(t_-))$ *instead of* $f(t)$ *for every* $t > 0$ *at which* $f(t_+)$ *and* $f(t_-)$ *exist, and for every* $\alpha > \omega$,

(b) *the relation* (N^+) *holds with* $f(t_+)$ *instead of* $f(t)$ *for every* $t > 0$ *at which* $f(t_+)$ *exists and for every* $\alpha > \omega$,

(c) *the relation* (N^-) *holds with* $f(t_-)$ *instead of* $f(t)$ *for every* $t > 0$ *at which* $f(t_-)$ *exists and for every* $\alpha > \omega$.

Proof. Use Theorem 34 and Proposition 35.

37. Remark. The preceding Theorem 32 (or 34) is a restricted form of the so called inversion theorem. The restriction consists in the requirement of continuity (or of the existence of certain limits) of the given function at the point of approximation. A more complete statement, the approximation almost everywhere or at the Lebesgue points requires a strengthening of the property (R_2). Such conditions are known from the closely related theory of the so called singular integrals, see [6], Section 3.9 and [7], Ch. 10. The most complete answer to this problem was given by D. K. Faddeev in [8].

38. Remark. Notice (cf. Remark 15) that the descriptive part of Theorem 13, i.e. the implication (B) $\Rightarrow$ (A), requires only the properties (R_1) and (R_2), but the existence part, (A) $\Rightarrow$ (B), needs the whole set of properties $(R_1)-(R_6)$. Theorem 32 on the restricted inversion stands in the middle and is based on the properties $(R_1)-(R_4)$.

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