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ON QUASI-HOMOMORPHISMS AND COMPOSITIONS OF AUTOMATA

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INTRODUCTION

In this note we shall deal with Mealy automata and Medvedev automata defined in a monoidal symmetric category with diagonal morphisms. We shall introduce a new notion of a quasi-homomorphism between such automata. This notion is a generalization of the usual notion of a homomorphism of automata and was introduced in some special cases by Nguyêñ Manh Trinh [7]. In the category of automata and quasi-homomorphisms some general compositions of automata (e.g. cascade products) can be described; it seems that the usual category of automata and homomorphisms has „too few" morphisms for this purpose.

This paper is based on some parts of the author's dissertation [6]. The author expresses his gratitude to Prof. A. Wiweger for valuable comments and suggestions.

By a *monoidal symmetric category with diagonal morphisms* we mean an 8-tuple $K = (K, \otimes, I, a, l, r, b, d)$, where K is a category, $\otimes : K \times K \to K$ is a bifunctor, I is a terminal object of K, a, l, r, b are natural isomorphisms

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z ,$$

$$l_X : I \otimes X \xrightarrow{\cong} X , \quad r_X : I \otimes X \xrightarrow{\cong} X , \quad b_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X ,$$

and d is a function which assigns to each object X of K a diagonal morphism $d_X : X \to X \otimes X$. These data are supposed to satisfy certain coherence conditions (cf. [2], [6]).

The symbol Set will denote the category of sets.

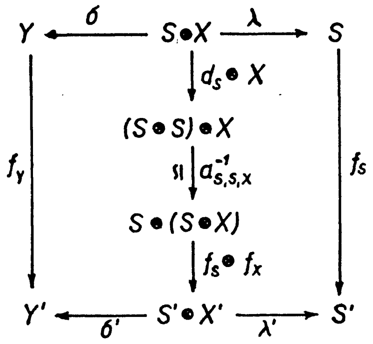
1. QUASI-HOMOMORPHISMS OF AUTOMATA

Let K be a monoidal symmetric category with diagonal morphisms.

A *Mealy automaton* (shortly : an automaton) *in the category* K is a 5-tuple $A = (X, S, Y, \delta, \lambda)$, where X, S and Y are (the input, state and output resp.) objects of K and $\delta : S \otimes X \to S$, $\lambda : S \otimes X \to Y$ are (the next-state, output) morphisms of K.

Let A and $A' = (X', S', Y', \delta', \lambda')$ be automata in K. A *quasi-homomorhhism* (shortly : a q-morphism) $f : A \to A'$ is a triple (f, A, A'), where $f = (f_X, f_S, f_Y)$ and $f_X : S \otimes X \to X'$, $f_S : S \to S'$, $f_Y : Y \to Y'$ are morphisms of K such that the diagram (1) is commutative.

(1)



Let $f : A \to A'$ and $g : A' \to A''$, $A'' = (X'', S'', Y'', \delta'', \lambda'')$, $g = (g_X, g_S, g_Y)$, be q-morphisms. The composition $h = g . f$ of f and g is defined by

(2) $h_X = (g . f)_X = g_X(f_S \otimes f_X)\, a^{-1}_{S,S,X}(d_S \otimes X)$,

(3) $h_S = (g . f)_S = g_S . f_S$,

and $h_Y = (g . f)_Y = g_Y . f_Y$.

It is easy to verify that the composition of two q-morphisms is a q-morphism and that the composition of q-morphisms is associative. Therefore automata in K and q-morphisms form a category. This category will be denoted by Qaut.

A Medvedev automaton (shortly : a semiautomaton) in K is a triple $A = (X, S, \delta)$, where X, S are objects of K and $\delta : S \otimes X \to S$ is a morphism of K.

A semiautomata q-morphism $f : A \to A'$, $A' = (X', S', \delta')$, is a triple (f, A, A'), where $f = (f_X, f_S)$ is a pair of morphisms of K, $f_X : S \otimes X \to X'$ and $f_S : S \to S'$, such that the right rectangle in (1) is commutative.

The composition of semiautomata q-morphisms is defined by (2) and (3). Semiautomata in K and their q-morphisms form a category QSaut.

There is a forgetful functor $\square$: Qaut $\to$ QSaut assigning to each automaton $A = (X, S, Y, \delta, \lambda)$ the semiautomaton $\square A = (X, S, \delta)$ and to each q-morphism (f, A, A'), $f = (f_X, f_S, f_Y)$ the semiautomata q-morphism $(\square f, \square A, \square A')$, $\square f = (f_X, f_S)$.

The functor $\square$ has a left adjoint functor F : QSaut $\to$ Qaut defined as follows: $F(A) = (X, S, S \otimes X, \delta, \mathrm{id}_{S \otimes X})$ for a semiautomaton $A = (X, S, \delta)$ and

$$F((f, A, A')) = (f_X, f_S, (f_S \otimes f_X)\, a^{-1}_{S,S,X}(d_S \otimes X), F(A), F(A'))$$

for a semiautomata q-morphism $f = (f_X, f_S)$.

The concept of a k-product is a generalization of a special kind of loop-free structures (cf. [5]).

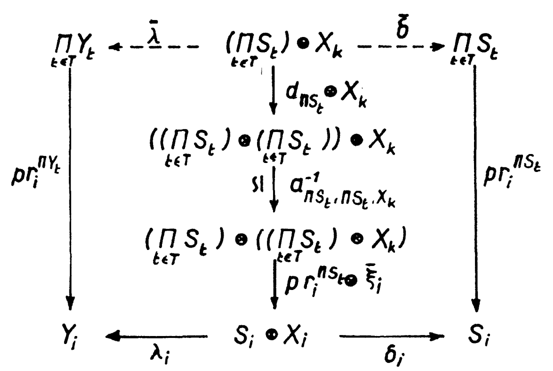
Let T be a fixed set of indices. We assume that the category K has selected products of all families indexed by T. If $(X_t)_{t \in T}$ is a family of objects of K then $\prod_{t \in T} X_t$ will denote the selected product of the family $(X_t)_{t \in T}$ and $\mathrm{pr}_i^{\Pi X_t} : \prod_{t \in T} X_t \to X_i$ will denote the selected projection on the i-th axis.

Let $k \in T$ be a fixed index. Let $(A_t)_{t \in T}$, $A_t = (X_t, S_t, Y_t, \delta_t, \lambda_t)$ be a family of automata and let $(\xi_t)_{t \in T}$, $\xi_t : S_k \otimes X_k \to X_t$ be a family of morphisms of K. We define

$$\bar{\xi}_t = \xi_t \,.\, (\mathrm{pr}_k^{\Pi S_t} \otimes X_k) \quad \text{for} \quad t \in T\,.$$

Then, by definition of a product, there exist unique morphisms $\bar{\delta} : (\prod_{t \in T} S_t) \otimes X_k \to \prod_{t \in T} S_t$, $\bar{\lambda} : (\prod_{t \in T} S_t) \otimes X_k \to \prod_{t \in T} Y_t$ in the commutative diagram (4).

(4)



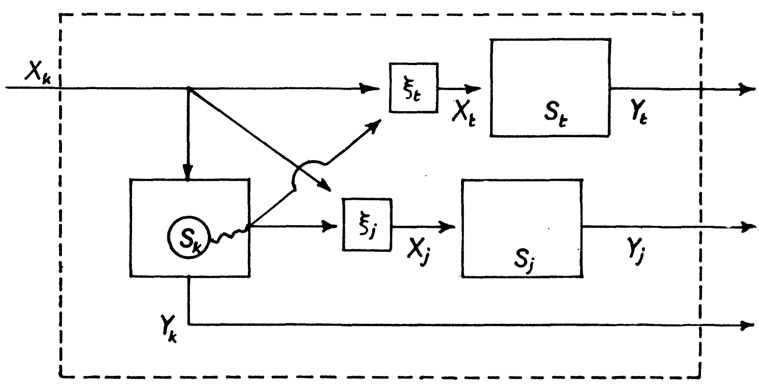


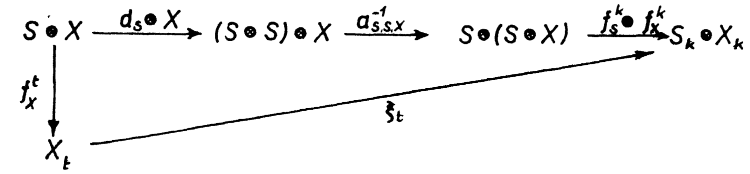
Figure 1. k-product of automata

The automaton $kA_t = (X_k, \prod_{t\in T} S_t, \prod_{t\in T} Y_t, \bar{\delta}, \bar{\lambda})$, where $\bar{\delta}$, $\bar{\lambda}$ are defined in the diagram (4) is called the *k-product of the family* $(A_t)_{t\in T}$ *with the connecting family of morphisms* $(\xi_t)_{t\in T}$.

In the case when K = Set and $\otimes$ is the cartesian product, the k-product is visualized by Figure 1.

The following theorem shows that kA_t has a categorical product-like property.

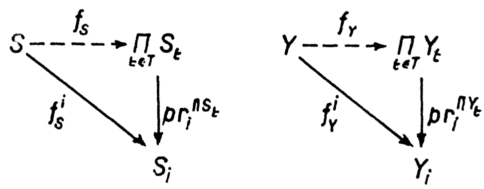
Theorem. *Let* $A = (X, S, Y, \delta, \lambda)$ *be an automaton in* K. *Let* $(f^t : A \to A_t)_{t\in T}$, $f^t = (f^t_X, f^t_S, f^t_Y)$, *be a family of q-morphisms satisfying the following condition: for every* $t \in T$ *the diagram*



is commutative.

Then there exists a q-morphism $f : A \to kA_t$ *such that* $f^i = p^i . f$, *where by* $p^i : kA_t \to A_i$, $i \in T$, *we denote the q-morphism* $p^i = (\xi_i, \mathrm{pr}_i^{\Pi S_t}, \mathrm{pr}_i^{\Pi Y_t})$.

Sketch of proof. It is clear that p^i, $i \in T$, are q-morphisms. One can check that $f = (f^k_X, f_S, f_Y)$ is a q-morphism in conclusion, where f_S and f_Y are defined by the commutative diagrams



for $i \in T$.

3. c-PRODUCTS

In Sections 3 and 4 we assume that T is a finite or countable set of indices and that the category K has selected products of all families indexed by T. In order to simplify the notation we assume that $T = \{1, 2, \ldots, n\}$ or T is the set of all natural numbers.

Let $(A_t)_{t\in T}$, $A_t = (X_t, S_t, Y_t, \delta_t, \lambda_t)$, be a family of automata in K. Let X be an object of K and let $\eta : (\prod_{t\in T} S_t) \otimes X \to X_1$, $\zeta_{t-1} : Y_{t-1} \otimes X \to X_t$, $t \geqq 2$, $t \in T$, be morphisms of K. We define

$$\mu_1 = \lambda_1 . (\mathrm{pr}_1^{\Pi S_t} \otimes \eta)\, a^{-1}_{\Pi S_t, \Pi S_t, X}(d_{\Pi S_t} \otimes X) : (\prod_{t\in T} S_t) \otimes X \to Y_1$$

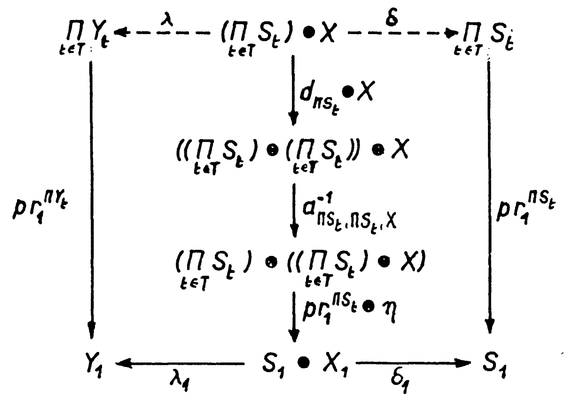
and

$$\mu_k = \lambda_k \,.\, (\mathrm{pr}_1^{\Pi S_t} \otimes \zeta_{k-1}) \,.\, (\prod S_t \otimes \mu_{k-1} \otimes X) \,.\, (\prod S_t \otimes a^{-1}_{\Pi S_t, X, X}) \,.$$
$$.\, a^{-1}_{\Pi S_t, \Pi S_t, X \otimes X} \,.\, (d_{\Pi S_t} \otimes d_X) : (\prod_{t \in T} S_t) \otimes X \to Y_k$$

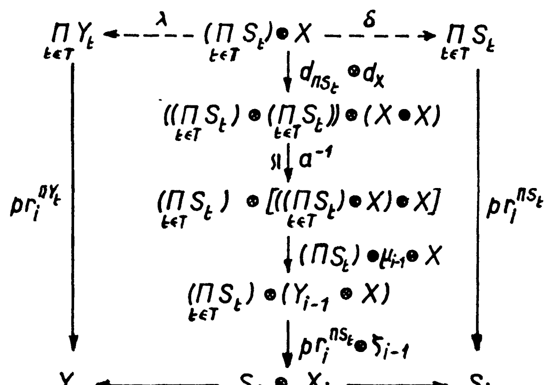
for $k \geqq 2$, $k \in T$.

Then, by definition of the product, there exist unique morphisms $\delta : (\prod_{t \in T} S_t) \otimes \otimes X \to \prod_{t \in T} S_t$, $\lambda : (\prod_{t \in T} S_t) \otimes X \to \prod_{t \in T} Y_t$ such that the diagrams (5) and (6) are commutative.

(5)



(6)



$(i \geqq 2,\ i \in T)$.

The automaton $cA_t = (X, \prod_{t \in T} S_t, \prod_{t \in T} Y_t, \delta, \lambda)$, where δ, λ are defined by diagrams (5) and (6) is called the *c-product of the family* (A_t) *with connecting morphisms* η *and* ζ_{t-1}, $t \geqq 2$, $t \in T$.

In the case when K = Set and $\otimes$ is the cartesian product, the automaton cA_t may be visualized by Figure 2.

In this case if $\eta = \eta' \,.\, \mathrm{pr}_2^{(\Pi S_t) \times X}$, $\eta' : X \to X_1$, and $T = \{1, 2\}$, then the c-product is the usual cascade in the sense of [1].

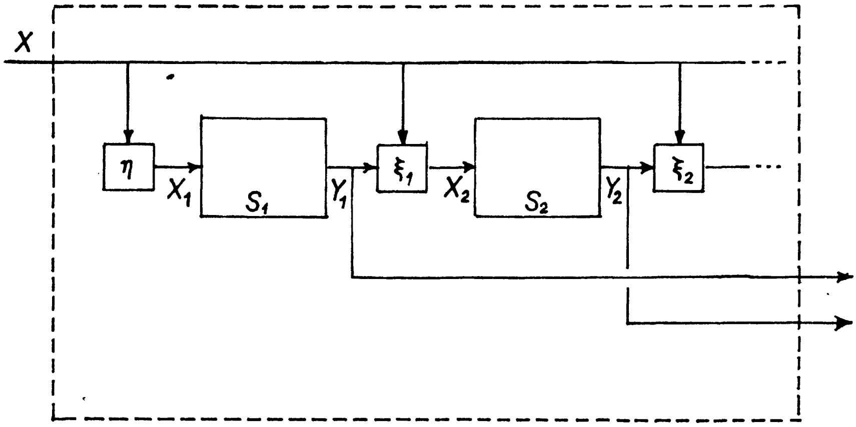


Figure 2. c-product of automata

There exist q-morphism from the c-product cA_t to each of its components. In fact, q-morphisms $f^i : cA_t \to A_i$, $i \in T$, may be defined by $f^1 = (f_X^1, f_S^1, f_Y^1) = (\eta, \mathrm{pr}_1^{\Pi S_t}, \mathrm{pr}_1^{\Pi Y_t})$ and for $i \in T$, $i \geqq 2$, $f^i = (f_X^i, f_S^i, f_Y^i)$, where

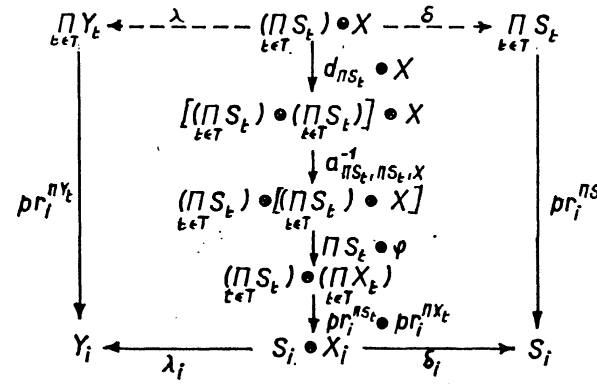
$$f_X^i = \zeta_{i-1} \cdot (\mu_{i-1} \otimes X) \cdot a_{\Pi S_t, X, X} \cdot \left(\prod S_t \otimes d_X\right), \quad f_S^i = \mathrm{pr}_i^{\Pi S_t}, f_Y^i = \mathrm{pr}_i^{\Pi Y_t}.$$

4. g-PRODUCTS

Let $(A_t)_{t \in T}$, $A_t = (X_t, S_t, Y_t, \delta_t, \lambda_t)$, be a family of automata in K. Let X be an object of K and let $\varphi : \prod_{t \in T} S_t \otimes X \to \prod_{t \in T} X_t$ be a morphism of K.

Then, by definition of the product, there exist unique morphisms $\delta : (\prod_{t \in T} S_t) \otimes \otimes X \to \prod_{t \in T} S_t$ and $\lambda : (\prod_{t \in T} S_t) \otimes X \to \prod_{t \in T} Y_t$ such that the diagram (7) is commutative.

(7)



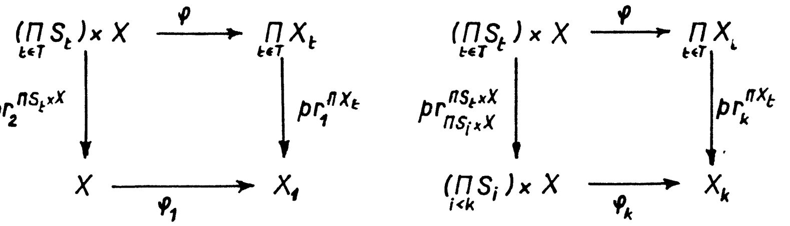
The automaton $gA_t = (X, \prod_{t \in T} S_t, \prod_{t \in T} Y_t, \delta, \lambda)$, where δ, λ are defined in diagram (7), is called the *g-product of the family $(A_t)_{t \in T}$ with the connecting morphism φ.*

In the case when K = Set, the g-product is the generalized product considered in [4].

There exist q-morphisms $p^k : gA_t \to A_k$, $k \in T$, from the g-product to each of its components. In fact, p^k may be defined by $p^k = (\mathrm{pr}_k^{\Pi X_t} \,.\, \varphi, \mathrm{pr}_k^{\Pi S_t}, \mathrm{pr}_k^{\Pi Y_t})$.

Now we shall assume that $\otimes$ is the categorical product $\times$. In this case we can show a relation between c-products and g-products (cf. [4] for the case when K = Set).

The g-product gA_t of the family $(A_t)_{t \in T}$ with the connecting morphism φ is called *g-α_0-product* if there exists a family of morphisms $(\varphi_k : (\prod_{i<k} S_i) \otimes X \to X_k)_{k \in T}$ such that the diagrams



$k \in T$, $k \geqq 2$, are commutative.

It may be shown that the c-product of a family of automata in K with $\eta = \eta' \,.\, \mathrm{pr}_2^{\Pi S_t \times X}$, where $\eta' : X \to X_1$ is a morphism of K, is a g-α_0-product.

References

[1] *Arbib, M. A.:* Theories of abstract automata. Prentice-Hall, Englewood Cliffs, New York 1969.

[2] *Budach, L., Hoehnke, H. J.:* Automaten und Funktoren. Akademie-Verlag, Berlin 1975.

[3] *Ehrig, H., Pfender, M.:* Kategorien und Automaten. Walter der Gruyter, Berlin—NewYork. 1972.

[4] *Gécseg, F.:* On products of abstract automata. Acta Sci. Math. (Szeged), *38* (1976), 21—43.

[5] *Hartmanis, J.:* Loop-free structure of sequential machines, Inform. Control *5* (1962), 25—43.

[6] *Cao Quyet Thang:* Banach automata and some problems of the universal theory of automata. Dissertation, Inst. Math. Polish Acad. Sci., Warsaw 1979.

[7] *Nguyen Manh Trinh:* Category of semiautomata with quasi-homomorphisms and its relation to the operation of composition of semiautomata (in Polish), Dissertation, Warsaw Univ., Warsaw 1976.

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