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ON EXPONENTIAL GROWTH OF SOLUTIONS OF ABSTRACT INITIAL VALUE PROBLEMS

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In this paper, we study certain properties of the following standard initial value problem in a Banach space E:

[*] $$u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0$$

$$u(0_+) = u'(0_+) = \ldots = u^{(n-2)}(0_+) = 0\,, \quad u^{(n-1)}(0_+) = x$$

where $t > 0$, $x \in E$, $u(t) \in E$ for $t > 0$ and where $A_1, A_2, \ldots, A_n$ are closed linear operators from E into E (see [1]).

The main result may be roughly described in the following way:

We suppose that the solutions u of [*] exist for x from a dense subset of E and that they depend continuously on x, uniformly for t from bounded subintervals of $(0, \infty)$.

Under these assumptions we prove that there exists a constant ω (depending only on $A_1, A_2, \ldots, A_n$ and not on u) so that the functions $\mathrm{e}^{-\omega t}u(t)$ depend continuously on x uniformly on the whole interval $(0, \infty)$.

Such a situation is well known for the case $n = 1$ as the theorem on exponential boundedness of strongly continuous operator semigroups (see [2]) because the operator A_1 may be always considered as the generator of such a semigroup.

Analogously for the case $n = 2$ with $A_1 = 0$ we can interpret A_2 as the generator of a strongly continuous cosine operator function which also can be proved to be exponentially bounded (see [3]).

In both the above cases, the exponential boundedness is a consequence of certain functional equations satisfied by semigroups and cosine functions of operators.

In our general case we cannot proceed in such a way because we do not know any functional equations for solutions of the standard initial value problem [*] for the operators $A_1, A_2, \ldots, A_n$. Our proof is based on different ideas, in particular on the method of local or finite Laplace transform (see [4] and [5]).

1. PRELIMINARIES

1.1. We shall use the following notation: (1) R – the real number field, (2) R^+ – the set of all positive real numbers, (3) (a, b) – the set of all real numbers between a and b if $a, b \in R$, $a < b$, (4) (ω, ∞) – the set of all real numbers greater than ω if $\omega \in \mathrm{R}$, (5) $M_1 \to M_2$ – the set of all mappings of the whole set M_1 into the set M_2.

1.2. In the whole paper, E will be a Banach space over R with the norm $\|\cdot\|$.

1.3. We shall denote: (1) $\mathsf{L}^+(E)$ – the set of all linear operators from E into E, (2) $\mathsf{L}(E)$ – the Banach space of all closed everywhere defined operators from $\mathsf{L}^+(E)$, equipped with the usual norm, (3) $\mathsf{D}(A)$ – the domain of the operator $A \in \mathsf{L}^+(E)$, (4) $\mathsf{R}(A)$ – the range of the operator $A \in \mathsf{L}^+(E)$, (5) I – the identical operator from $\mathsf{L}(E)$.

1.4. Let Λ be an open interval and $f \in \Lambda \to E$. The notions of differentiability and of derivatives of the function f are considered in the usual sense (see [2]). The notions of integrability and of integral are used in the simplest form as absolutely convergent Riemann integrals (see [2]) of vector-valued functions.

1.5. Let $f \in \mathrm{R}^+ \to E$. If the function f is continuous on R^+ and bounded on $(0, 1)$, then we shall use the following notations:

$$\lfloor 0 \int_0^t f(\tau)\,\mathrm{d}\tau = f(t) \quad \text{for every} \quad t \in \mathrm{R}^+,$$

$$\lfloor r \int_0^t f(\tau)\,\mathrm{d}\tau = \frac{1}{(r-1)!}\int_0^t (t-\tau)^{r-1} f(\tau)\,\mathrm{d}\tau \quad \text{for every} \quad t \in \mathrm{R}^+$$

and $r \in \{1, 2, \ldots\}$.

2. AUXILIARY RESULTS

2.1. Lemma. *Let $A \in \mathsf{L}^+(E)$, let Λ be an open bounded interval and $f \in \Lambda \to E$. If* (α) *the operator A is closed,* (β) *the function f is continuous and bounded on Λ,* (γ) *$f(t) \in \mathsf{D}(A)$ for every $t \in \Lambda$,* (δ) *the function Af is continuous and bounded on Λ, then* (a) $\int_\Lambda f(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A)$, (b) $A \int_\Lambda f(\tau)\,\mathrm{d}\tau = \int_\Lambda A f(\tau)\,\mathrm{d}\tau$.

2.2. Lemma. *Let $f \in \mathrm{R}^+ \to E$. If the function f is continuous on R^+ and bounded on $(0, 1)$, then for every $s \in \{0, 1, \ldots\}$, the function*

$$\lfloor s \int_0^t f(\tau)\,\mathrm{d}\tau$$

has the same property.

2.3. Lemma. *Let $f \in \mathrm{R}^+ \to E$. If the function f is continuous on R^+ and bounded on $(0, 1)$, then*

$$\underline{|s_1+s_2}\int_0^t f(\tau)\,\mathrm{d}\tau = \underline{|s_1}\int_0^t \left(\underline{|s_2}\int_0^\tau f(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau \quad \textit{for every } t \in \mathrm{R}^+ \textit{ and } s_1, s_2 \in \{0, 1, \ldots\}.$$

2.4. Lemma. *Let $f \in \mathrm{R}^+ \to E$. If the function f is continuous on R^+ and bounded on $(0, 1)$, then for every $s \in \{0, 1, \ldots\}$*

(a) *the function* $\underline{|s}\int_0^t f(\tau)\,\mathrm{d}\tau$ *is s-times differentiable on* R^+,

(b) $\dfrac{\mathrm{d}^r}{\mathrm{d}t^r}\left(\underline{|s}\int_0^t f(\tau)\,\mathrm{d}\tau\right) = \underline{|s-r}\int_0^t f(\tau)\,\mathrm{d}\tau$ *for every* $t \in \mathrm{R}^+$ *and* $r \in \{0, 1, \ldots, s\}$.

2.5. Lemma. *Let $f \in \mathrm{R}^+ \to E$ and $q \in \{1, 2, \ldots\}$. If (α) the function f is q-times differentiable on R^+, $(\beta)\, f^{(q)}$ is continuous on R^+ and bounded on $(0, 1)$, $(\gamma)\, f(0_+) = f'(0_+) = \ldots = f^{(q-2)}(0_+) = 0$, then*

(a) $f^{(q-1)}(0_+)$ *exists,*

(b) $\underline{|l}\int_0^t f^{(q-i)}(\tau)\,\mathrm{d}\tau = \underline{|l+i-1}\int_0^t f^{(q-1)}(\tau)\,\mathrm{d}\tau$ *for every* $t \in \mathrm{R}^+$,

$l \in \{0, 1, \ldots\}$ *and* $i \in \{1, 2, \ldots, q\}$,

(c) $\underline{|l+1}\int_0^t f^{(q)}(\tau)\,\mathrm{d}\tau = \underline{|l}\int_0^t f^{(q-1)}(\tau)\,\mathrm{d}\tau - \dfrac{t^l}{l!} f^{(q-1)}(0_+)$ *for every* $t \in \mathrm{R}^+$ *and*

$l \in \{0, 1, \ldots\}$.

2.6. Lemma. *Let $f \in \mathrm{R}^+ \to E$ and $r \in \{0, 1, \ldots\}$. If (α) the function f is r-times continuously differentiable and all these derivatives are bounded on R^+, $(\beta)\, f(0_+) = f(0_+) = \ldots = f^{(r-1)}(0_+)$, then for every $\lambda \in \mathrm{R}^+$*

$$\lambda^r \int_0^\infty \mathrm{e}^{-\lambda\tau} f(\tau)\,\mathrm{d}\tau = \int_0^\infty \mathrm{e}^{-\lambda\tau} f^{(r)}(\tau)\,\mathrm{d}\tau\,.$$

2.7. Lemma. *Let $\varphi \in \mathrm{R}^+ \to \mathrm{R}$, $f \in \mathrm{R}^+ \to E$ and $r \in \{0, 1, \ldots\}$. If (α) the function φ is r-times continuously differentiable on R^+, (β) the function f is continuous on R^+,*

(γ) $\displaystyle\int_0^\infty |\varphi(\tau)|\,\|f(\tau)\|\,\mathrm{d}\tau < \infty, \int_0^\infty |\varphi^{(r)}(\tau)| \left\|\underline{|r}\int_0^\tau f(\sigma)\,\mathrm{d}\sigma\right\|\mathrm{d}\tau < \infty,$

(δ) $\varphi^{(j)}(t)\,\underline{|j+1}\int_0^t f(\tau)\,\mathrm{d}\tau \to 0$ *for* $t \to 0_+$ *and* $t \to \infty$ *and for* $j \in \{0, 1, \ldots, r-1\}$, *then*

$$\int_0^\infty \varphi(\tau) f(\tau)\, \mathrm{d}\tau = (-1)^r \int_0^\infty \varphi^{(r)}(\tau) \left(\underline{|r} \int_0^\tau f(\sigma)\, \mathrm{d}\sigma \right) \mathrm{d}\tau .$$

2.8. Sublemma. *Let Λ be an open interval, $\varphi \in \Lambda \to \mathrm{R}$ and $\Phi \in \Lambda \to \mathsf{L}(E)$. If*

(α) $\varphi(\lambda) \neq 0$ *and* $\Phi(\lambda)$ *is one-to-one,* $(\Phi(\lambda))^{-1} \in \mathsf{L}(E)$ *for every* $\lambda \in \Lambda$,

(β) $\|(\Phi(\lambda))^{-1}\| \leqq (\varphi(\lambda))^{-1}$ *for every* $\lambda \in \Lambda$,

(γ) *the functions* φ, Φ *are infinitely differentiable on* Λ,

(δ) $\|\Phi^{(r)}(\lambda)\| \leqq (-1)^{r+1} \varphi^{(r)}(\lambda)$ *for every* $\lambda \in \Lambda$ *and* $r \in \{1, 2, \ldots\}$,

then

(a) *the functions* $(\varphi(\lambda))^{-1}$ *and* $(\Phi(\lambda))^{-1}$ *are infinitely differentiable on* Λ,

(b) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\Phi(\lambda))^{-1} \right\| \leqq (-1)^p \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\varphi(\lambda))^{-1}$ *for every* $\lambda \in \Lambda$ *and* $p \in \{0, 1, \ldots\}$.

Proof. First, we easily obtain from (α) and (γ) that

(1) the functions $(\varphi(\lambda))^{-1}$, $(\Phi(\lambda))^{-1}$ are differentiable on Λ,

(2) $\dfrac{\mathrm{d}}{\mathrm{d}\lambda} (\varphi(\lambda))^{-1} = -(\varphi(\lambda))^{-2} \varphi'(\lambda)$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} (\Phi(\lambda))^{-1} = -(\Phi(\lambda))^{-1} \Phi'(\lambda) (\Phi(\lambda))^{-1} \text{ for every } \lambda \in \Lambda.$$

We now proceed by induction on $p \in \{0, 1, \ldots\}$.

The case $p = 0$ is the assumption (β).

Let now $p \in \{0, 1, \ldots\}$ be arbitrary and let us suppose that the assertions (a) and (b) are true for every $0, 1, \ldots, p$.

It follows from the assumptions (α)–(γ) and from the induction hypothesis by means of the Leibniz theorem that

(3) the functions $(\varphi(\lambda))^{-2} \varphi'(\lambda)$ and $(\Phi(\lambda))^{-1} \Phi'(\lambda) (\Phi(\lambda))^{-1}$ are p-times differentiable on Λ.

Further, it follows from (1), (2), (3) that

(4) the functions $(\varphi(\lambda))^{-1}$ and $(\Phi(\lambda))^{-1}$ are $(p + 1)$-times differentiable on Λ,

(5) $\dfrac{\mathrm{d}^{p+1}}{\mathrm{d}\lambda^{p+1}} (\varphi(\lambda))^{-1} = \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \left(\dfrac{\mathrm{d}}{\mathrm{d}\lambda} (\varphi(\lambda))^{-1} \right) = - \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} ((\varphi(\lambda))^{-2} \varphi'(\lambda))$,

$$\frac{\mathrm{d}^{p+1}}{\mathrm{d}\lambda^{p+1}} (\Phi(\lambda))^{-1} = \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} (\Phi(\lambda))^{-1} \right) = - \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} ((\Phi(\lambda))^{-1} \Phi'(\lambda) (\Phi(\lambda))^{-1})$$

for every $\lambda \in \Lambda$.

Using now (δ) and (5) we get by means of the Leibniz theorem

$$(6)\quad \left\|\frac{\mathrm{d}^{p+1}}{\mathrm{d}\lambda^{p+1}}(\Phi(\lambda))^{-1}\right\| = \left\|\frac{\mathrm{d}^{p}}{\mathrm{d}\lambda^{p}}((\Phi(\lambda))^{-1}\,\Phi'(\lambda)\,(\Phi(\lambda))^{-1})\right\| =$$

$$= \left\|\sum_{k=0}^{p}\binom{p}{k}\left[\sum_{l=0}^{k}\binom{k}{l}\left(\frac{\mathrm{d}^{l}}{\mathrm{d}\lambda^{l}}(\Phi(\lambda))^{-1}\right)\Phi^{(k-l+1)}(\lambda)\right]\left(\frac{\mathrm{d}^{p-k}}{\mathrm{d}\lambda^{p-k}}(\Phi(\lambda))^{-1}\right)\right\| \leqq$$

$$\leqq \sum_{k=0}^{p}\binom{p}{k}\left[\sum_{l=0}^{k}\binom{l}{k}(-1)^{l}\frac{\mathrm{d}^{l}}{\mathrm{d}\lambda^{l}}(\varphi(\lambda))^{-1}\,(-1)^{k-l}\,\varphi^{k-l+1}(\lambda)\right].$$

$$.\,(-1)^{p-k}\frac{\mathrm{d}^{p-k}}{\mathrm{d}\lambda^{p-k}}(\varphi(\lambda))^{-1} =$$

$$= (-1)^{p}\sum_{k=0}^{p}\binom{p}{k}\left[\sum_{l=0}^{k}\binom{k}{l}\left(\frac{\mathrm{d}^{l}}{\mathrm{d}\lambda^{l}}(\varphi(\lambda))^{-1}\right)\varphi^{(k-l+1)}(\lambda)\right]\frac{\mathrm{d}^{p-k}}{\mathrm{d}\lambda^{p-k}}(\varphi(\lambda))^{-1} =$$

$$= (-1)^{p}\frac{\mathrm{d}^{p}}{\mathrm{d}\lambda^{p}}((\varphi(\lambda))^{-1}\,\varphi'(\lambda)\,(\varphi(\lambda))^{-1}) = (-1)^{p}\frac{\mathrm{d}^{p}}{\mathrm{d}\lambda^{p}}((\varphi(\lambda))^{-2}\,\varphi'(\lambda)) =$$

$$= (-1)^{p+1}\frac{\mathrm{d}^{p+1}}{\mathrm{d}\lambda^{p+1}}(\varphi(\lambda))^{-1}\ \text{for every}\ \lambda \in \Lambda.$$

The proof is complete because (4) and (6) confirm the induction hypothesis.

2.9. Lemma. *Let Λ be an open interval, $\varphi, \psi \in \Lambda \to \mathrm{R}$ and $\Phi, \Psi \in \Lambda \to \mathsf{L}(E)$. If*

(α) *the functions $\varphi, \psi, \Phi, \Psi$ are infinitely differentiable on Λ,*

(β) $\|\Phi^{(p)}(\lambda)\| \leqq (-1)^{p}\,\varphi^{(p)}(\lambda)$ *for every $\lambda \in \Lambda$ and $p \in \{0, 1, \ldots\}$,*

(γ) $1 + \psi(\lambda) \neq 0$ *and $I + \Psi(\lambda)$ is one-to-one, $(I + \Psi(\lambda))^{-1} \in \mathsf{L}(E)$ for every $\lambda \in \Lambda$,*

(δ) $\|(I + \Psi(\lambda))^{-1}\| \leqq (1 + \psi(\lambda))^{-1}$ *for every $\lambda \in \Lambda$,*

(ε) $\|\Psi^{(r)}(\lambda)\| \leqq (-1)^{r+1}\,\psi^{(r)}(\lambda)$ *for every $\lambda \in \Lambda$ and $r \in \{1, 2, \ldots\}$,*

then

(a) *the function $\Phi(\lambda)(I + \Psi(\lambda))^{-1}$ is infinitely differentiable on Λ,*

(b) $\left\|\frac{\mathrm{d}^{p}}{\mathrm{d}\lambda^{p}}(\Phi(\lambda)(I + \Psi(\lambda))^{-1})\right\| \leqq (-1)^{p}\frac{\mathrm{d}^{p}}{\mathrm{d}\lambda^{p}}(\varphi(\lambda)(1 + \psi(\lambda))^{-1})$ *for every $\lambda \in \Lambda$*

and $p \in \{0, 1, \ldots\}$.

Proof. Since the functions $1 + \psi$ and $I + \Psi$ satisfy the hypotheses of Sublemma 2.8, the assertions of the present lemma follow from 2.8 and from the Leibniz theorem.

2.10. Lemma. *Let $f \in \mathrm{R}^+ \to E$, let ω be a constant and $F \in (\omega, \infty) \to E$. If*

(α) *the function f is measurable and $\mathrm{e}^{-\omega t} f(t)$ is bounded on R^+,*

(β) $F(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda\tau} f(\tau)\, \mathrm{d}\tau$ *for every $\lambda > \omega$,*

then

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \to_{p\to\infty, p>\omega t} f(t)$$

for every $t \in \mathrm{R}^+$ which is a point of continuity of the function f.

Proof. The same as the proof of [1], Proposition 4.10, because the assumption of continuity can be clearly weakened to measurability.

2.11. Lemma. *Let M, ω be two nonnegative constants and $F \in (\omega, \infty) \to E$. If*

(α) *the function F is infinitely differentiable on (ω, ∞),*

(β) $\|F^{(p)}(\lambda)\| \leqq \dfrac{Mp!}{(\lambda-\omega)^{p+1}}$ *for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$,*

then

$$\left\|\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\frac{1}{\lambda^s}F(\lambda)\right\| \leqq M(-1)^p \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\left(\frac{1}{\lambda^s(\lambda-\omega)}\right) \text{ for every } \lambda > \omega,\ p \in \{0, 1, \ldots\} \text{ and}$$
$s \in \{0, 1, \ldots\}$.

Proof. Easy by means of the Leibniz theorem.

2.12. Lemma. *Let ω be a nonnegative constant and $F \in (\omega, \infty) \to E$. If*

(α) *the function F is infinitely differentiable on (ω, ∞),*

(β) *there exist $M \geqq 0$ and $s \in \{0, 1, \ldots\}$ such that*

$$\|F^{(p)}(\lambda)\| \leqq M(-1)^p \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\left(\frac{1}{\lambda^s(\lambda-\omega)}\right) \text{ for every } \lambda > \omega \text{ and } p \in \{0, 1, \ldots\},$$

then

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1}\left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\mathrm{e}^{-\lambda T}F(\lambda)\right]_{\lambda=(p+1)/t} \to_{p\to\infty, p>\omega t} 0 \text{ for every } 0 < t < T.$$

Proof. First, we easily see that

$$(1)\quad \left\|\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}(\mathrm{e}^{-\lambda T}F(\lambda))\right\| = \left\|\sum_{k=0}^{p}\binom{p}{k}(-T)^k\,\mathrm{e}^{-\lambda T}F^{(p-k)}(\lambda)\right\| \leqq$$

$$\leqq M(-1)^p \sum_{k=0}^{p}\binom{p}{k}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(\mathrm{e}^{-\lambda T})\frac{\mathrm{d}^{p-k}}{\mathrm{d}\lambda^{p-k}}\left(\frac{1}{\lambda^s(\lambda-\omega)}\right) = M(-1)^p\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\frac{\mathrm{e}^{-\lambda T}}{\lambda^s(\lambda-\omega)}$$

for every $T > 0$, $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

On the other hand, we obtain from Lemma 2.7 for $\varphi(t) = \mathrm{e}^{-\lambda t}$, $f(t) = \mathrm{e}^{\omega t}$, $t \in \mathrm{R}^+$, and for $r = s$ that

$$(2)\quad \frac{1}{\lambda^s(\lambda - \omega)} = \frac{1}{\lambda^s}\int_0^\infty \mathrm{e}^{-\lambda\tau}\mathrm{e}^{\omega\tau}\,\mathrm{d}\tau = \int_0^\infty \mathrm{e}^{-\lambda\tau}\left(\underline{|s}\int_0^\tau \mathrm{e}^{\omega\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\tau \text{ for every } \lambda > \omega.$$

By (2), we can write

$$(3)\quad \frac{\mathrm{e}^{-\lambda T}}{\lambda^s(\lambda - \omega)} = \mathrm{e}^{-\lambda T}\int_0^\infty \mathrm{e}^{-\lambda\tau}\left(\underline{|s}\int_0^\tau \mathrm{e}^{\omega\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\tau = \int_0^\infty \mathrm{e}^{-\lambda(\tau+T)}\left(\underline{|s}\int_0^\tau \mathrm{e}^{\omega\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\tau =$$

$$= \int_T^\infty \mathrm{e}^{-\lambda\tau}\left(\underline{|s}\int_0^{\tau - T} \mathrm{e}^{\omega\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\tau \text{ for every } T > 0 \text{ and } \lambda > \omega.$$

By Lemma 2.10, taking here $f(t) = 0$ for $0 < t < T$ and

$f(t) = \underline{|s}\displaystyle\int_0^{t-T} \mathrm{e}^{\omega\sigma}\,\mathrm{d}\sigma$ for $t \geqq T$, we obtain from (3) that

$$(4)\quad \frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1}\left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\left(\frac{\mathrm{e}^{-\lambda T}}{\lambda^s(\lambda - \omega)}\right)\right]_{\lambda = (p+1)/t} \to_{p\to\infty, p>\omega t} 0 \text{ for every } 0 < t < T.$$

The assertion is an immediate consequence of (1) and (4).

2.13. Lemma. *Let $f \in (0, T) \to E$, $T > 0$ and $s \in \{1, 2, \ldots\}$. If the function f is s-times continuously differentiable on $(0, T)$, then*

$$\text{(a)}\quad \frac{1}{h^s}\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}f(t + kh) \to_{h\to 0+} f^{(s)}(t) \textit{ for every } t \in (0, T),$$

$$\text{(b)}\quad \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}f(t + kh) = \int_0^h\int_0^h \ldots \int_0^h f^{(s)}(t + \tau_1 + \tau_2 + \ldots + \tau_s)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \ldots \mathrm{d}\tau_s$$

for every $t \in (0, T)$ and $h > 0$ such that $sh < T - t$.

Proof. It is easy to see that (a) follows from (b). Hence we prove only (b). We proceed by induction on s. The case $s = 1$ of the identity (b) is evident. Now suppose that (b) is true for an $s \in \{1, 2, \ldots\}$ and prove it for $s + 1$. Using the known identity

$$\binom{s+1}{k} = \binom{s}{k} + \binom{s}{k-1}$$

for every $k \in \{1, 2, \ldots, s\}$, we easily obtain

$$(1)\quad \sum_{k=0}^{s+1}\binom{s+1}{k}(-1)^{s+1-k}f(t + kh) = \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}(f(t + (k+1)h) -$$

$- f(t + kh))$ for every $t \in (0, T)$ and $h > 0$ such that $(s + 1) h < T - t$.

Using the induction hypothesis, we can write by (1) that

(2) $$\sum_{k=0}^{s+1} \binom{s+1}{k} (-1)^{s+1-k} f(t + kh) = \int_0^h \int_0^h \dots \int_0^h (f^{(s)}(t + h + \tau_1 + \tau_2 \dots + \tau_s) -$$

$$- f^{(s)}(t + \tau_1 + \tau_2 + \dots + \tau_s)) \, d\tau_1 \, d\tau_2 \dots d\tau_s$$ for every $t \in (0, T)$ and $h > 0$ such that $(s + 1) h < T - t$.

From (2) we easily obtain that

(3) $$\sum_{k=0}^{s+1} \binom{s+1}{k} (-1)^{s+1-k} f(t + kh) =$$

$$= \int_0^h \int_0^h \dots \int_0^h f^{(s+1)}(t + \tau_1 + \tau_2 + \dots + \tau_s + \tau_{s+1}) \, d\tau_1 \, d\tau_2 \dots d\tau_s \, d\tau_{s+1}$$

for every $t \in (0, T)$ and $h > 0$ such that $(s + 1) h < T - t$.

But (3) proves the induction step and hence completes the proof.

2.14. Lemma. *Let $f \in R^+ \to E$, let M, ω be two nonnegative constants and $s \in \in \{0, 1, \dots\}$. If*

(α) *the function f is s-times continuously differentiable on R^+,*

(β) $\int_0^\infty e^{-\lambda\tau} f(\tau) \, d\tau$ *exists for every $\lambda > \omega$,*

(γ) $$\left\| \frac{d^p}{d\lambda^p} \left(\lambda^s \int_0^\infty e^{-\lambda\tau} f(\tau) \, d\tau \right) \right\| \leqq \frac{Mp!}{(\lambda - \omega)^{p+1}}$$ *for every $\lambda > \omega$ and $p \in \{0, 1, \dots\}$,*

then $\|f^{(s)}(t)\| \leqq M e^{\omega t}$ for every $t \in R^+$.

Proof. Let us first write

(1) $$F(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau) \, d\tau \text{ for } \lambda > \omega.$$

It is easy to see that

(2) the function F is infinitely differentiable on (ω, ∞).

Further, we shall write

(3) $G_r(\lambda) = \lambda^r F(\lambda)$ for $\lambda > \omega$ and $r \in \{0, 1, \dots\}$.

For the sake of simplicity we further introduce the following notation:

(4) $f_q(t) = \dfrac{(-1)^q}{q!}\left(\dfrac{q+1}{t}\right)^{q+1} F^{(q)}\left(\dfrac{q+1}{t}\right)$ for $t \in \mathrm{R}^+$ and $q \in \{0, 1, \ldots\}$

such that $q > \omega t$.

We now prove some auxiliary results.

It follows from (1), (α) and (β) in virtue of Lemma 2.10 that

(5) $f_q(t) \to_{q\to\infty} f(t)$ for every $t \in \mathrm{R}^+$.

It is clear from (2) and (4) that

(6) for every $q \in \{0, 1, \ldots\}$, the function f_q is infinitely differentiable on $\{t : q > \omega t\}$.

Now we shall prove that

(7) $f_q^{(r)}(t) = \dfrac{(-1)^{q+r}}{q!\,(q+1)^r}\left(\dfrac{q+1}{t}\right)^{q+r+1} G_r^{(q+r)}\left(\dfrac{q+1}{t}\right)$ for every $t \in \mathrm{R}^+$ and

$q \in \{0, 1, \ldots\}$ such that $q > \omega t$ and for every $r \in \{0, 1, \ldots\}$.

We proceed by proving (7) by induction on r.

The case $r = 0$ is clear from definitions (3) and (4).

Suppose now that (7) is valid for some $r \in \{0, 1, \ldots\}$. Since, by (3), clearly $G_{r+1}^{(q+r+1)}(\lambda) = (q + r + 1)\, G_r^{(q+r)}(\lambda) + \lambda G_r^{(q+r+1)}(\lambda)$ for every $\lambda > \omega$ and $q, r \in \{0, 1, \ldots\}$, we obtain from (7) that

$$f_q^{(r+1)}(t) = \frac{(-1)^{q+r}}{q!\,(q+1)^r}(q+r+1)\left(\frac{q+1}{t}\right)^{q+r}\left(-\frac{q+1}{t^2}\right) G_r^{(q+r)}\left(\frac{q+1}{t}\right) +$$

$$+ \frac{(-1)^{q+r}}{q!\,(q+1)^r}\left(\frac{q+1}{t}\right)^{q+r+1}\left(-\frac{q+1}{t^2}\right) G_r^{(q+r+1)}\left(\frac{q+1}{t}\right) =$$

$$= \frac{(-1)^{q+r+1}}{q!\,(q+1)^{r+1}}\left(\frac{q+1}{t}\right)^{q+r+2}\left[(q+r+1)\, G_r^{(q+r)}\left(\frac{q+1}{t}\right) +\right.$$

$$\left.+ \frac{q+1}{t} G_r^{(q+r+1)}\left(\frac{q+1}{t}\right)\right] = \frac{(-1)^{q+r+1}}{q!\,(q+1)^{r+1}}\left(\frac{q+1}{t}\right)^{q+r+2} G_{r+1}^{(q+r+1)}\left(\frac{q+1}{t}\right)$$

for every $t \in \mathrm{R}^+$ and $q \in \{0, 1, \ldots\}$ such that $q > \omega t$. But this last identity justifies the induction step and proves (7).

After these auxiliary results, we proceed to the proof proper of our Lemma 2.14.

In view of (1) and (3), the assumption (γ) can be written in the form

(8) $\|G_s^{(p)}(\lambda)\| \leqq \dfrac{Mp!}{(\lambda - \omega)^{p+1}}$ for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

It follows from (7) and (8) that

$$(9)\quad \|f_q^{(s)}(t)\| = \frac{1}{q!\,(q+1)^s}\left(\frac{q+1}{t}\right)^{q+s+1}\left\|G_s^{(q+s)}\left(\frac{q+1}{t}\right)\right\| \leqq$$

$$\leqq \frac{M(q+s)!}{q!\,(q+1)^s}\left(\frac{\frac{q+1}{t}}{\frac{q+1}{t}-\omega}\right)^{q+s+1} = M\,\frac{(q+s)!}{q!\,(q+1)^s}\left(\frac{1}{1-\frac{\omega t}{q+1}}\right)^{q+s+1}$$

for every $t \in \mathrm{R}^+$ and $q \in \{0, 1, \ldots\}$ such that $q > \omega t$.

Using Lemma 2.13 (b) we get easily from (6) and (9) that

$$(10)\quad \left\|\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k} f_q(t+kh)\right\| \leqq$$

$$\leqq \int_0^h\int_0^h\ldots\int_0^h \|f_q^{(s)}(t+\tau_1+\tau_2+\ldots+\tau_s)\|\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_s \leqq$$

$$\leqq M\,\frac{(q+s)!}{q!\,(q+1)^s}\int_0^h\int_0^h\ldots\int_0^h\left(\frac{1}{1-\frac{\omega(t+\tau_1+\tau_2+\ldots+\tau_s)}{q+1}}\right)^{q+s+1}\,.$$

$$.\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_s$$

for every $t \in \mathrm{R}^+$ and $q \in \{0, 1, \ldots\}$ such that $q > \omega t$ and every $h > 0$ such that $q > \omega(t + sh)$.

Letting $q \to \infty$ in (10), we get from (5) by virtue of Lemmas 2.17 and 2.19 that

$$(11)\quad \left\|\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k} f(t+kh)\right\| \leqq M\int_0^h\int_0^h\ldots\int_0^h \mathrm{e}^{\omega(t+\tau_1+\tau_2+\ldots+\tau_s)}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_s \leqq$$

$$\leqq Mh^s\,\mathrm{e}^{\omega(t+sh)} \text{ for every } t \in \mathrm{R}^+ \text{ and } h > 0.$$

By virtue of Lemma 2.13 (a) we immediately get the desired result from (11) for $h \to 0_+$.

2.15. Lemma. *Let $f \in \mathrm{R}^+ \to E$. If the function f is continuous on R^+ and bounded on $(0, 1)$, then*

$$\lambda^s\int_0^T \mathrm{e}^{-\lambda\tau}\left(\frac{1}{\lfloor s}\int_0^\tau f(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau = \int_0^T \mathrm{e}^{-\lambda\tau} f(\tau)\,\mathrm{d}\tau - \mathrm{e}^{-\lambda\tau}\sum_{r=0}^{s-1}\lambda^r\frac{1}{\lfloor r+1}\int_0^T f(\tau)\,\mathrm{d}\tau$$

for every $T > 0$, $\lambda \in \mathrm{R}$ and $s \in \{1, 2, \ldots\}$.

Proof. By induction on s.

2.16. Lemma.

$$\lambda^{l+1}\int_0^T \mathrm{e}^{-\lambda\tau}\frac{\tau^l}{l!}\,\mathrm{d}\tau = 1 - \mathrm{e}^{-\lambda\tau}\sum_{k=0}^{l}\frac{(\lambda T)^k}{k!} \text{ for every } T>0,\ \lambda\in\mathrm{R} \text{ and } l\in\{0,1,\ldots\}.$$

Proof. By induction on l.

2.17. Lemma. *For every* $t\in\mathrm{R}^+$, $\omega\geqq 0$ *and* $s\in\{0,1,\ldots\}$,

$$\left(\frac{\dfrac{p+1}{t}}{\dfrac{p+1}{t}-\omega}\right)^{p+s+1} \to_{p\to\infty,\,p>\omega t} \mathrm{e}^{\omega t}\,.$$

2.18. Lemma. *For every* $t\in\mathrm{R}^+$, $\omega\geqq 0$ *and* $s\in\{0,1,\ldots\}$,

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1}\left[\!\left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\left(\frac{1}{\lambda^s(\lambda-\omega)}\right)\right]\!\right]_{\lambda=(p+1)/t} \to_{p\to\infty,\,p>\omega t} \lfloor\underline{s}\int_0^t \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau\,.$$

Proof. It is easy to prove by induction on s that for $\lambda>\omega$ and $s\in\{0,1,\ldots\}$,

$$\frac{1}{\lambda^s(\lambda-\omega)} = \int_0^\infty \mathrm{e}^{-\lambda\tau}\left(\lfloor\underline{s}\int_0^\tau \mathrm{e}^{\omega\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\tau\,.$$

Now it suffices to apply Lemma 2.10.

2.19. Lemma. *For every* $t\in\mathrm{R}^+$, $\omega\geqq 0$, $p\in\{0,1,\ldots\}$ *such that* $p>2\omega t$ *and* $s\in\{0,1,\ldots\}$,

$$\left(\frac{\dfrac{p+1}{t}}{\dfrac{p+1}{t}-\omega}\right)^{p+s+1} \leqq 2^s\mathrm{e}^{2\omega t}\,.$$

2.20. Lemma. *For every* $t>0$, $\omega\geqq 0$ *and* $s\in\{0,1,\ldots\}$,

$$\lfloor\underline{s}\int_0^t \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \leqq \mathrm{e}^{\omega t}\frac{t^s}{s!}\,.$$

3. MAIN RESULTS

3.1. Let $A_1, A_2, \ldots, A_n\in\mathsf{L}^+(E)$, $n\in\{1,2,\ldots\}$, and $u\in\mathrm{R}^+\to E$. The function u will be called a ***standard solution for the operators*** $A_1, A_2, \ldots, A_n$ if

(1) u is n-times differentiable on R^+,

(2) $u^{(n-i)}(t) \in \mathsf{D}(A_i)$ for every $t \in \mathrm{R}^+$ and $i \in \{1, 2, \ldots, n\}$,

(3) the functions $A_i u^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,

(4) $u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0$ for every $t \in \mathrm{R}^+$,

(5) $u(0_+) = u'(0_+) = \ldots = u^{(n-2)}(0_+) = 0$.

3.2. Proposition. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $u \in \mathrm{R}^+ \to E$. *If the function* u *is a standard solution for the operators* $A_1, A_2, \ldots, A_n$, *then*

(a) $u^{(n-1)}(0_+)$ *exists,*

(b) *the functions* $u, u', \ldots, u^{(n)}$ *are continuous on* R^+ *and bounded on* $(0, 1)$.

3.3. Proposition. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $u \in \mathrm{R}^+ \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the function* u *is a standard solution for the operators* $A_1, A_2, \ldots, A_n$,

then for every $t \in \mathrm{R}^+$, $i \in \{1, 2, \ldots, n\}$ *and* $l \in \{0, 1, \ldots\}$, *the following statements hold:*

(a) $\underline{\lfloor l}\int_0^t u^{(n-i)}(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_i)$ *and* $A_i \underline{\lfloor l}\int_0^t u^{(n-i)}(\tau)\,\mathrm{d}\tau = \underline{\lfloor l}\int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau$,

(b) $\underline{\lfloor l+i-1}\int_0^t u^{(n-1)}(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_i)$ *and* $A_i \underline{\lfloor l+i-1}\int_0^t u^{(n-1)}(\tau)\,\mathrm{d}\tau = \underline{\lfloor l}\int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau$,

(c) $\underline{\lfloor l}\int_0^t u(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_i)$ *and* $A_i \underline{\lfloor l}\int_0^t u(\tau)\,\mathrm{d}\tau = \underline{\lfloor n-i+l}\int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau$.

Proof. Use 2.1–2.5 and 3.2.

3.4. Proposition. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, *and let* $K \in \mathrm{R}^+ \to \mathrm{R}$ *be a nonnegative function. If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *there exists a dense subset* $Z \subseteq E$ *such that for every* $x \in Z$, *we can find a standard solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *for which* $u^{(n-1)}(0_+) = x$,

(γ) *for every standard solution* u *for the operators* $A_1, A_2, \ldots, A_n$, *for every* $0 < t \leqq T$ *and for every* $i \in \{1, 2, \ldots, n\}$, *we have*

$$\left\| \int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau \right\| \leqq K(T) \|u^{(n-1)}(0_+)\| ,$$

then there exists a function $W \in \mathrm{R}^+ \times E \to E$ *such that*

(a) *for every* $x \in E$, *the function* $W(\cdot, x)$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(b) $\int_0^t (t - \tau)^{i-1} W(\tau, x)\, \mathrm{d}\tau \in \mathrm{D}(A_i)$ for every $x \in E$, $t \in \mathrm{R}^+$ *and* $i \in \{1, 2, \ldots, n\}$,

(c) *the function* $A_i \int_0^t (t - \tau)^{i-1} W(\tau, x)\, \mathrm{d}\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$ *for every* $x \in E$ *and* $i \in \{1, 2, \ldots, n\}$,

(d) $W(t, x) + \sum_{i=1}^{n} \frac{1}{(i - 1)!} A_i \int_0^t (t - \tau)^{i-1} W(\tau, x)\, \mathrm{d}\tau = x$ *for every* $x \in E$ *and* $t \in \mathrm{R}^+$,

(e) *for every* $t \in \mathrm{R}^+$, *the function* $W(t, \cdot)$ *is a linear mapping,*

(f) $\left\| \frac{1}{(i - 1)!} A_i \int_0^t (t - \tau)^{i-1} W(\tau, x)\, \mathrm{d}\tau \right\| \leqq K(T)\, \|x\|$ *for every* $x \in E$, $0 < t \leqq T$ *and* $i \in \{1, 2, \ldots, n\}$,

(g) $W(t, u^{(n-1)}(0_+)) = u^{(n-1)}(t)$ *for every standard solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *and for every* $t \in \mathrm{R}^+$.

Proof. First, we easily see from (γ) that

(1) for every $x \in E$, there exists at most one standard solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$.

Further, let us denote

(2) D – the set of all $x \in E$ for which there exists a standard solution u for the operators $A_1, A_2, \ldots, A_n$ satisfying $u^{(n-1)}(0_+) = x$.

Now we define a function $W_0 \in \mathrm{R}^+ \times D \to E$ in the following way:

(3) $W_0(t, x) = u^{(n-1)}(t)$ for every $t \in \mathrm{R}^+$ where u is, according to (1) and (2), the unique standard solution u for the operators $A_1, A_2, \ldots, A_n$ satisfying $u^{(n-1)}(0_+) = x$.

Since, by (α), Lemmas 3.2 and 3.3 are applicable we easily obtain from (3) that

(4) the function W_0, defined by (3), possesses the properties (a)–(g) if we write there W_0 instead of W and D instead of E.

Further, it follows from (4) (the properties (d) and (f)) that there exists a non-negative function $K_0 \in \mathrm{R}^+ \to \mathrm{R}$ such that

(5) $\|W_0(t, x)\| \leqq K_0(T)\, \|x\|$ for every $0 < t \leqq T$.

On the other hand, by the assumption (β)

(6) the set D, defined by (2), is dense in E.

Now, it is easy to conclude the proof.

We first obtain from (5) and (6) that there exists a unique continuous extension $W \in \mathrm{R}^+ \times E \to E$ of the function $W_0 \in \mathrm{R}^+ \times D \to E$. By (4), it suffices to extend also the validity of the properties (a)–(f) from D to E which is easily done by means of the assumption (α). The property (g) is automatically satisfied by W as a consequence of the definition (2) of the set D.

3.5. Proposition. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, and $W \in \mathrm{R}^+ \times E \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *for every standard solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *such that* $u^{(n-1)}(0_+) = 0$, *we have* $u(t) = 0$ *for every* $t \in \mathrm{R}^+$,

(γ) *the conditions* 3.4 (a)–(d) *are fulfilled,*

then for every $x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ *and for every* $t \in \mathrm{R}^+$,

$$W(t, x) + \sum_{i=1}^{n} \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1}\, W(\tau, A_i x)\, \mathrm{d}\tau = x.$$

Proof. For the sake of simplicity of formulas we shall write $A_0 = I$.

By means of Lemmas 2.1–2.3 and of Proposition 3.3, we easily obtain from (α) and (γ) that

(1) for every $x \in E$ and $l \in \{0, 1, \ldots\}$, the function

$\lfloor\!\underline{\,l\,}\int_0^t W(\tau, x)\, \mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(2) $\lfloor\!\underline{\,j+l\,}\int_0^t W(\tau, x)\, \mathrm{d}\tau \in \mathsf{D}(A_j)$ for every $x \in E$, $t \in \mathrm{R}^+$, $j \in \{0, 1, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$,

(3) for every $x \in E$, $j \in \{0, 1, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$, the function

$A_j \lfloor\!\underline{\,j+l\,}\int_0^t W(\tau, x)\, \mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(4) $\displaystyle\sum_{j=0}^{n} A_j \lfloor\!\underline{\,j+l\,}\int_0^t W(\tau, x)\, \mathrm{d}\tau = \frac{t^l}{l!}\, x$ for every $x \in E$, $t \in \mathrm{R}^+$ and $l \in \{0, 1, \ldots\}$.

Let us now fix an $x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ and let us write

(5) $w(t) = \displaystyle\sum_{k=0}^{n} \lfloor\!\underline{\,k\,}\int_0^t W(\tau, A_k x)\, \mathrm{d}\tau - x$ for $t \in \mathrm{R}^+$.

We conclude:

(6) the function w is continuous on R^+ and bounded on $(0, 1)$,

(7) $\lfloor\underline{j}\int_0^t w(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_j)$ for every $t \in \mathrm{R}^+$ and $j \in \{0, 1, \dots, n\}$,

(8) the functions $A_j \lfloor\underline{j}\int_0^t w(\tau)\,\mathrm{d}\tau$ are continuous on R^+ and bounded on $(0, 1)$ for every $j \in \{0, 1, \dots, n\}$,

(9)
$$\sum_{j=0}^{n} A_j \lfloor\underline{j}\int_0^t w(\tau)\,\mathrm{d}\tau = \sum_{j=0}^{n} A_j \lfloor\underline{j}\int_0^t \left(\sum_{k=0}^{n} \lfloor\underline{k}\int_0^\tau W(\sigma, A_k x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau - \sum_{j=0}^{n} \frac{t^j}{j!} A_j x =$$
$$= \sum_{j=0}^{n}\sum_{k=0}^{n} A_j \lfloor\underline{j+k}\int_0^t W(\tau, A_k x)\,\mathrm{d}\tau - \sum_{j=0}^{n} \frac{t^j}{j!} A_j x =$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{n} A_j \lfloor\underline{j+k}\int_0^t W(\tau, A_k x)\,\mathrm{d}\tau - \sum_{j=0}^{n} \frac{t^j}{j!} A_j r =$$
$$= \sum_{k=0}^{n} \frac{t^k}{k!} A_k x - \sum_{j=0}^{n} \frac{t^j}{j!} A_j x = 0 \text{ for every } t \in \mathrm{R}^+.$$

Let us now denote

(10) $w_0(t) = \lfloor\underline{n}\int_0^t w(\tau)\,\mathrm{d}\tau$ for $t \in \mathrm{R}^+$.

It immediately follows from (6)–(10) by virtue of Lemma 2.4 that w_0 is a standard solution for the operators $A_1, A_2, \dots, A_n$ such that $w_0^{(n-1)}(0_+) = 0$. Thus by (β), $w_0(t) = 0$ for every $t \in \mathrm{R}^+$. But according to Lemma 2.4 we see at once that also $w(t) = 0$ for every $t \in \mathrm{R}^+$. This fact together with (5) implies the assertion since $x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \dots \cap \mathsf{D}(A_n)$ was chosen arbitrary.

3.6. Proposition. *Let $A_1, A_2, \dots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \dots\}$, and let M, ω be two nonnegative constants. If*

(α) *the operators $A_1, A_2, \dots, A_n$ are closed,*

(β) *for every $\lambda > \omega$, the operator $\lambda^n I + \lambda^{n-1} A_1 + \dots + A_n$ is one-to-one and $(\lambda^n I + \lambda^{n-1} A_1 + \dots + A_n)^{-1} \in \mathsf{L}(E)$,*

(γ) *the functions $A_i(\lambda^n I + \lambda^{n-1} A_1 + \dots + A_n)^{-1}$ are infinitely differentiable on (ω, ∞) for every $i \in \{1, 2, \dots, n\}$,*

(δ) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \lambda^{n-i-1} A_i (\lambda^n I + \lambda^{n-1} A_1 + \dots + A_n)^{-1} \right\| \leq \dfrac{Mp!}{(\lambda - \omega)^{p+1}}$ *for every $\lambda > \omega$, $i \in \{1, 2, \dots, n\}$ and $p \in \{0, 1, \dots\}$,* *then for every standard solution u for the operators $A_1, A_2, \ldots, A_n$, for every $t \in \mathrm{R}^+$ and for every $i \in \{1, 2, \ldots, n\}$,*

$$\left\| \int_0^t A_i \, u^{(n-i)}(\tau) \, \mathrm{d}\tau \right\| \leqq M e^{\omega t} \| u^{(n-1)}(0_+) \| .$$

Proof. For the sake of simplicity we shall write

(1) $R(\lambda) = (\lambda^n I + \lambda^{n-1} A_1 + \ldots + A_n)^{-1}$ for $\lambda > \omega$.

Then by assumption (δ)

(2) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \lambda^{n-i-1} A_i R(\lambda) \right\| \leqq \dfrac{M p!}{(\lambda - \omega)^{p+1}}$ for every $\lambda > \omega$, $i \in \{1,2, \ldots, n\}$ and

$p \in \{0, 1, \ldots\}$.

Since clearly $\lambda^{n-1} R(\lambda) = \dfrac{1}{\lambda} I - \sum\limits_{i=1}^{n} \lambda^{n-i-1} A_i R(\lambda)$ for every $\lambda > \omega$, we get at once from (2) that

(3) $\left\| \dfrac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\lambda^{n-1} R(\lambda)) \right\| \leqq \dfrac{(1 + nM) \, p!}{(\lambda - \omega)^{p+1}}$ for every $\lambda > \omega$ and $p \in \{0, 1, \ldots\}$.

Now let u be a fixed standard solution for the operators $A_1, A_2, \ldots, A_n$ which will be examined up to the end of the proof.

We get from the properties 3.1 (a)–(e) and from the assumption (α) by means of Lemmas 2.2, 2.4, 2.5, Proposition 3.2 and Proposition 3.3 that

(4) the function $\lfloor^{l} \!\! \int_0^t u(\tau) \, \mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$ for every $l \in \{0, 1, \ldots\}$,

(5) $\lfloor^{l} \!\! \int_0^t u(\tau) \, \mathrm{d}\tau \in \mathsf{D}(A_i)$ for every $t \in \mathrm{R}^+$ and $l \in \{0, 1, \ldots\}$,

(6) the function $A_i \lfloor^{l} \!\! \int_0^t u(\tau) \, \mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$,

(7) $u(t) + \sum\limits_{i=1}^{n} A_i \lfloor^{i} \!\! \int_0^t u(\tau) \, \mathrm{d}\tau = \dfrac{t^{n-1}}{(n-1)!} u^{(n-1)}(0_+)$ for every $t \in \mathrm{R}^+$.

In virtue of Lemma 2.1, we deduce immediately from (α) and (4)–(6) that

(8) $\displaystyle\int_0^T e^{-\lambda\tau} \left(\lfloor^{l} \!\! \int_0^\tau u(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}\tau \in \mathsf{D}(A_i)$ and

$$A_i \int_0^T e^{-\lambda\tau} \left(\underline{|l} \int_0^\tau u(\sigma)\, d\sigma \right) d\tau = \int_0^T e^{-\lambda\tau} \left(A_i \underline{|l} \int_0^\tau u(\sigma)\, d\sigma \right) d\tau \text{ for every } T > 0,$$

$\lambda \in \mathrm{R}$, $i \in \{1, 2, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$.

Multiplying (7) by $e^{-\lambda\tau}$, integrating from 0 to T and using (8) we can write

$$(9) \quad \int_0^T e^{-\lambda\tau} u(\tau)\, d\tau + \sum_{i=1}^{n} A_i \int_0^T e^{-\lambda\tau} \left(\underline{|i} \int_0^\tau u(\sigma)\, d\sigma \right) d\tau = \int_0^T e^{-\lambda\tau} \frac{\tau^{n-1}}{(n-1)!}\, d\tau\, u^{(n-1)}(0_+)$$

for every $T > 0$ and $\lambda \in \mathrm{R}$.

Multiplying (9) by λ^n and using Lemmas 2.15 and 2.16 we get

$$(10) \quad \lambda^n \int_0^T e^{-\lambda\tau} u(\tau)\, d\tau +$$

$$+ \sum_{i=1}^{n} \lambda^{n-i} A_i \left[\int_0^T e^{-\lambda\tau} u(\tau)\, d\tau - e^{-\lambda T} \sum_{k=0}^{i-1} \lambda^k \underline{|k+1} \int_0^T u(\tau)\, d\tau \right] =$$

$$= \left[1 - e^{-\lambda T} \sum_{l=0}^{n-1} \frac{(\lambda T)^l}{l!} \right] u^{(n-1)}(0_+) \text{ for every } T > 0 \text{ and } \lambda \in \mathrm{R}.$$

By means of (8) we obtain from (10) after rearranging the terms

$$(11) \quad (\lambda^n I + \lambda^{n-1} A_1 + \ldots + A_n) \int_0^T e^{-\lambda\tau} u(\tau)\, d\tau =$$

$$= u^{(n-1)}(0_+) - e^{-\lambda T} \sum_{l=0}^{n-1} \frac{(\lambda T)^l}{l!} u^{(n-1)}(0_+) +$$

$$+ e^{-\lambda T} \sum_{i=1}^{n} \sum_{k=0}^{i-1} \lambda^{n-i+k} A_i \underline{|k+1} \int_0^T u(\tau)\, d\tau \text{ for every } T > 0 \text{ and } \lambda \in \mathrm{R}.$$

Owing to (β) we can rewrite (11) in view of (1) in the form

$$(12) \quad \int_0^T e^{-\lambda\tau} u(\tau)\, d\tau = R(\lambda)\, u^{(n-1)}(0_+) - \sum_{l=0}^{n-1} \frac{T^l}{l!} e^{-\lambda T} \lambda^l R(\lambda)\, u^{(n-1)}(0_+) +$$

$$+ \sum_{j=1}^{n} \sum_{k=0}^{j-1} e^{-\lambda T} \lambda^{n-j+k} R(\lambda)\, A_j \underline{|k+1} \int_0^T u(\tau)\, d\tau \text{ for every } T > 0 \text{ and } \lambda > \omega.$$

By Lemma 2.15 we get from (12) that

$$(13) \quad \int_0^T e^{-\lambda\tau} \left(\underline{|i} \int_0^\tau u(\sigma)\, d\sigma \right) d\tau = \lambda^{-i} R(\lambda)\, u^{(n-1)}(0_+) -$$

$$-\sum_{l=0}^{n-1} \frac{T^l}{l!} \mathrm{e}^{-\lambda T} \lambda^{l-i} R(\lambda)\, u^{(n-1)}(0_+) +$$

$$+\sum_{j=1}^{n} \sum_{k=0}^{j-1} \mathrm{e}^{-\lambda T} \lambda^{n-j+k-i} R(\lambda)\, A_j \lfloor\underline{k+1} \int_0^T u(\tau)\, \mathrm{d}\tau -$$

$$-\sum_{r=0}^{i-1} \mathrm{e}^{-\lambda T} \lambda^{r-i} \lfloor\underline{r+1} \int_0^T u(\tau)\, \mathrm{d}\tau \text{ for every } T > 0,\ \lambda > \omega \text{ and } i \in \{1, 2, \ldots, n\}.$$

It follows from (5), (8) and (13) that

$$(14)\quad \int_0^T \mathrm{e}^{-\lambda\tau} \left(A_i \lfloor\underline{i} \int_0^\tau u(\sigma)\, \mathrm{d}\sigma \right) \mathrm{d}\tau = \lambda^{-i} A_i R(\lambda)\, u^{(n-1)}(0_+) -$$

$$-\sum_{l=0}^{n-1} \frac{T^l}{l!} \mathrm{e}^{-\lambda T} \lambda^{l-i} A_i R(\lambda)\, u^{(n-1)}(0_+) +$$

$$+\sum_{j-1}^{n} \sum_{k=0}^{j-1} \mathrm{e}^{-\lambda T} \lambda^{n-j+k-i} A_i R(\lambda)\, A_j \lfloor\underline{k+1} \int_0^T u(\tau)\, \mathrm{d}\tau -$$

$$-\sum_{r=0}^{i-1} \mathrm{e}^{-\lambda T} \lambda^{r-i} A_i \lfloor\underline{r+1} \int_0^T u(\tau)\, \mathrm{d}\tau \text{ for every } T > 0,\ \lambda > \omega \text{ and } i \in \{1, 2, \ldots, n\}.$$

In virtue of Lemmas 2.10, 2.11 and 2.12 we get from (2), (3), (12) and (14) that

$$(15)\quad u(t) = \lim_{p \to \infty} \frac{(-1)^p}{p!} \left(\frac{p+1}{t} \right)^{p+1} \left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} R(\lambda) \right]_{\lambda = (p+1)/t} u^{(n-1)}(0_+) \text{ for every } t \in \mathrm{R}^+,$$

$$(16)\quad A_i \lfloor\underline{i} \int_0^t u(\tau)\, \mathrm{d}\tau = \lim_{p \to \infty} \frac{(-1)^p}{p!} \left(\frac{p+1}{t} \right)^{p+1} \left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\lambda^{-i} A_i R(\lambda)) \right]_{\lambda = (p+1)/t} u^{(n-1)}(0_+)$$

for every $t \in \mathrm{R}^+$ and $i \in \{1, 2, \ldots, n\}$.

Writing

$$R(\lambda) = \frac{1}{\lambda^{n-1}} (\lambda^{n-1} R(\lambda))$$

and

$$\lambda^{-i} A_i R(\lambda) = \frac{1}{\lambda^{n-1}} (\lambda^{n-i-1} A_i R(\lambda))$$

for $\lambda > \omega$ and $i \in \{1, 2, \ldots, n\}$ we see from (2), (3), (15) and (16) by means of Lemmas 2.11, 2.18 and 2.20 with $s = n - 1$ that

$$(17)\quad \|u(t)\| \leqq (1 + nM)\, \mathrm{e}^{\omega t} \frac{t^{n-1}}{(n-1)!} \text{ for every } t \in \mathrm{R}^+,$$

(18) $\left\| A_i \,{}^{\llcorner i}\!\!\int_0^t u(\tau)\,\mathrm{d}\tau \right\| \leqq M\mathrm{e}^{\omega t} \dfrac{t^{n-1}}{(n-1)!}$ for every $t \in \mathrm{R}^+$ and $i \in \{1, 2, \ldots, n\}$.

In virtue of Lemma 2.1 we easily deduce from (8) that

(19) $\displaystyle\int_0^\infty \mathrm{e}^{-\lambda\tau} \left({}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}\tau \in \mathsf{D}(A_i)$ and

$$A_i \int_0^\infty \mathrm{e}^{-\lambda\tau} \left({}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}\tau = \int_0^\infty \mathrm{e}^{-\lambda\tau} \left(A_i \,{}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}\tau \text{ for every } \lambda > \omega \text{ and}$$

$i \in \{1, 2, \ldots, n\}$.

It follows from (7) and (19) that

(20) $\displaystyle(\lambda^n I + \lambda^{n-1} A_1 + \ldots + A_n) \int_0^\infty \mathrm{e}^{-\lambda\tau} u(\tau)\,\mathrm{d}\tau =$

$$= \lambda^n \left(I + \frac{1}{\lambda} A_1 + \ldots + \frac{1}{\lambda^n} A_n \right) \int_0^\infty \mathrm{e}^{-\lambda\tau} u(\tau)\,\mathrm{d}\tau =$$

$$= \lambda^n \left[\int_0^\infty \mathrm{e}^{-\lambda\tau} u(\tau)\,\mathrm{d}\tau + \sum_{i=1}^n A_i \int_0^\infty \mathrm{e}^{-\lambda\tau} \left({}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}\tau \right] =$$

$$= \lambda^n \int_0^\infty \mathrm{e}^{-\lambda\tau} \left[u(\tau) + \sum_{i=1}^n A_i \,{}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right] \mathrm{d}\tau =$$

$$= \lambda^n \int_0^\infty \mathrm{e}^{-\lambda\tau} \frac{\tau^{n-1}}{(n-1)!} \,\mathrm{d}\tau\, u^{(n-1)}(0_+) = u^{(n-1)}(0_+) \text{ for every } \lambda > \omega.$$

In view of (1) we can rewrite (20) in the form

(21) $\displaystyle\int_0^\infty \mathrm{e}^{-\lambda\tau} u(\tau)\,\mathrm{d}\tau = R(\lambda)\, u^{(n-1)}(0_+)$ for every $\lambda > \omega$.

Now we easily obtain from (18), (19) and (21):

(22) $\displaystyle\int_0^\infty \mathrm{e}^{-\lambda\tau} \left(A_i \,{}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}\tau = A_i \int_0^\infty \mathrm{e}^{-\lambda\tau} \left({}^{\llcorner i}\!\!\int_0^\tau u(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}\tau =$

$$= \lambda^{-i} A_i \int_0^\infty \mathrm{e}^{-\lambda\tau} u(\tau)\,\mathrm{d}\tau = \lambda^{-i} A_i R(\lambda)\, u^{(n-1)}(0_+)$$

for every $\lambda > \omega$ and $i \in \{1, 2, \ldots, n\}$.

On the other hand, we obtain by Proposition 3.3 that

$$(23)\quad A_i \lfloor\!\underline{i}\int_0^t u(\tau)\,\mathrm{d}\tau \doteq A_i \lfloor\!\underline{n}\int_0^t u^{(n-i)}(\tau)\,\mathrm{d}\tau =$$

$$= \lfloor\!\underline{n}\int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau \text{ for every } t \in \mathrm{R}^+ \text{ and } i \in \{1, 2, \ldots, n\}.$$

According to Lemma 2.4 we see from (23) that

(24) the function $A_i \lfloor\!\underline{i}\int_0^t u(\tau)\,\mathrm{d}\tau$ is $(n-1)$-times continuously differentiable on R^+,

$$(25)\quad \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left(A_i \lfloor\!\underline{i}\int_0^t u(\tau)\,\mathrm{d}\tau\right) = \int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau \text{ for every } t \in \mathrm{R}^+ \text{ and } i \in \{1, 2, \ldots, n\}.$$

By (2), (22) and (24), Lemma 2.14 is clearly applicable with

$$f(t) = A_i \lfloor\!\underline{i}\int_0^t u(\tau)\mathrm{d}\tau \text{ and } s = n - 1$$

and so we immediately get the desired result from the identity (25).

The proof is complete.

3.7. Remark. The case $n = 1$ in the above Proposition was proved by Ju. J. Ljubič in [5], p. 30. The present generalisation for arbitrary $n \in \{1, 2, \ldots\}$ is based on the same idea, but brings about many technical difficulties.

3.8. Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathrm{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If*

(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*

(β) *there exists a dense subset $Z \subseteq E$ such that for every $x \in Z$, we can find a standard solution u for the operators $A_1, A_2, \ldots, A_n$ so that $u)^{(n-1)}(0_+) = x$,*

(γ) *for every $T > 0$, there exists a $K \geqq 0$ such that for every standard solution u for the operators $A_1, A_2, \ldots, A_n$, for every $0 < t \leqq T$ and for every $i \in \{1, 2, \ldots, n\}$, the following inequality holds:*

$$\left\| \int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau \right\| \leqq K \|u^{(n-1)}(0_+)\| ,$$

then there exist nonnegative constants M, ω such that for every standard solution u for the operators $A_1, A_2, \ldots, A_n$, for every $t \in \mathrm{R}^+$ and for every $i \in \{1, 2, \ldots, n\}$,

$$\left\| \int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau \right\| \leqq M\mathrm{e}^{\omega t} \|u^{(n-1)}(0_+)\| .$$

Note 1. The assumption (γ) can be formulated equivalently as follows:

(γ') *for every* $T > 0$, *there exists a* $K \geqq 0$ *such that for every standard solution* u *for the operators* $A_1, A_2, \ldots, A_n$, *for every* $0 < t \leqq T$ *and for every* $i \in \{1, 2, \ldots \ldots, n-1\}$, *the following inequalities hold*:

$$\|u^{(n-1)}(t)\| \leqq K\|u^{(n-1)}(0_+)\| ,$$

$$\left\|\int_0^t A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau\right\| \leqq K\|u^{(n-1)}(0_+)\| .$$

Note 2. The conclusion of our theorem moreover implies:

$$\|u^{(n-1)}(t)\| \leqq (1 + nM)\,\mathrm{e}^{\omega t}\|u^{(n-1)}(0_+)\| \quad \text{for every} \quad t \in \mathrm{R}^+ .$$

Proof. For the sake of simplicity of formulas we shall write in the whole proof $A_0 = I$.

Further, we choose a fixed nonnegative function $K \in \mathrm{R}^+ \to \mathrm{R}$ such that the assumption (γ) holds for every $T > 0$ with the constant $K = K(T)$.

By Proposition 3.4 we can fix a function $W \in \mathrm{R}^+ \times E \to E$ such that

(1) the function W possesses the properties 3.4 (a)–(f).

It follows easily from (1) by using properties 3.4 (d) and (f) that there exists a nonnegative function $K_0 \in \mathrm{R}^+ \to \mathrm{R}$ such that

(2) $$\left\|A_j\, {}^{\lfloor j}\!\int_0^t W(\tau, x)\,\mathrm{d}\tau\right\| \leqq K_0(T)\,\|x\| \quad \text{for every } x \in E,\ 0 < t \leqq T \text{ and}$$

$j \in \{0, 1, ,\ldots, n\}$.

Now we prepare some auxiliary results and introduce some notations.

In virtue of Lemmas 2.1–2.3 we easily get from (1) (properties 3.4 (a)–(d)) that

(3) $${}^{\lfloor j+s}\!\int_0^t W(\tau, x)\,\mathrm{d}\tau \in \mathsf{D}(A_j),$$

$$A_j\, {}^{\lfloor j+s}\!\int_0^t W(\tau, x)\,\mathrm{d}\tau = {}^{\lfloor s}\!\int_0^t \left(A_j\, {}^{\lfloor j}\!\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau \quad \text{for every } x \in E,\ t \in \mathrm{R}^+,$$

$j \in \{0, 1, \ldots, n\}$ and $s \in \{0, 1, \ldots\}$.

Further, in virtue of Lemma 2.1 we get from (3) that

(4) $$\int_0^\infty \varphi(\tau)\left({}^{\lfloor j+s}\!\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau \in \mathsf{D}(A_j),$$

$$A_j \int_0^\infty \varphi(\tau)\left({}^{\lfloor j+s}\!\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau = \int_0^\infty \varphi(\tau)\left(A_j\, {}^{\lfloor j+s}\!\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau$$

for every continuous bounded function $\varphi \in R^+ \to R$ vanishing outside of a bounded subset of R^+, for every $x \in E$, $j \in \{0, 1, \dots, n\}$ and $s \in \{0, 1, \dots\}$.

Further, we choose a fixed function $\vartheta \in R^+ \to R$ such that

(5) ϑ is $(n + 1)$-times continuously differentiable on R^+,

(6) $0 \leqq \vartheta(t) \leqq 1$ for every $t \in R^+$, $\vartheta(t) = 1$ for $0 < t \leqq 1$ and $\vartheta(t) = 0$ for $t \geqq 2$.

Let us further denote

(7) $L = \operatorname*{Max}_{j \in \{0,1,\dots,n+1\}} \left(\sup_{t \in R^+} |\vartheta^{(j)}(t)| \right).$

Integrating by parts we get from (5) and (6) that

(8) $$\int_0^\infty e^{-\lambda\tau} \vartheta(\tau) \, d\tau = \frac{1}{\lambda}\left(1 + \int_0^\infty e^{-\lambda\tau} \vartheta'(\tau) \, d\tau\right) \text{ for every } \lambda > 0.$$

Further, it follows from (1), (5) and (6) in virtue of Lemma 2.7 that

(9) $$\int_0^\infty e^{-\lambda\tau} \vartheta(\tau) \left(\underline{|s} \int_0^\tau W(\sigma, x) \, d\sigma \right) d\tau =$$
$$= (-1)^r \sum_{k=0}^{r} \binom{r}{k} (-\lambda)^k \int_0^\infty e^{-\lambda\tau} \vartheta^{(r-k)}(\tau) \left(\underline{|s+r} \int_0^\tau W(\sigma, x) \, d\sigma \right) d\tau \text{ for every } x \in E,$$
$\lambda \in R$ and $r, s \in \{0, 1, \dots\}$.

After these preparatory technicalities we return to the proof proper.

The first step will be the proof that the operator $\sum_{j=0}^{n} \lambda^{n-j} A_j$ is one-to-one for sufficiently large λ.

First, according to (1), we can apply Proposition 3.5 and write

(10) $$\sum_{j=0}^{n} \underline{|j} \int_0^t W(\tau, A_j x) \, d\tau = x \text{ for every } x \in D(A_1) \cap D(A_2) \cap \dots \cap D(A_n) \text{ and}$$
$t \in R^+$.

Multiplying (10) by $e^{-\lambda\tau} \vartheta(\tau)$ and integrating over $(0, \infty)$ we get

(11) $$\sum_{j=0}^{n} \int_0^\infty e^{-\lambda\tau} \vartheta(\tau) \left(\underline{|j} \int_0^\tau W(\sigma, A_j x) \, d\sigma \right) d\tau =$$
$$= \int_0^\infty e^{-\lambda\tau} \vartheta(\tau) \, d\tau \, x \text{ for every } x \in D(A_1) \cap D(A_2) \cap \dots D(A_n) \text{ and } \lambda \in R.$$

It follows from (8), (9) with $s = j$ and $r = n - j$ and from (11) that

$$(12)\quad \int_0^\infty e^{-\lambda\tau}\,\vartheta(\tau)\left(\lfloor\underline{n}\int_0^\tau W\Big(\sigma, \sum_{j=0}^{n}\lambda^{n-j}A_j x\Big)\,d\sigma\right)d\tau\; +$$

$$+\sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^k\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\lfloor\underline{n}\int_0^\tau W(\sigma, A_j x)\,d\sigma\right)d\tau =$$

$$= \frac{1}{\lambda}\left(1 + \int_0^\infty e^{-\lambda\tau}\,\vartheta'(\tau)\,d\tau\right)x \text{ for every } x \in \mathsf{D}(A_1)\cap\mathsf{D}(A_2)\cap\ldots\cap\mathsf{D}(A_n)$$

and $\lambda > 0$.

Multiplying (12) by λ and rearranging the terms we get

$$(13)\quad \lambda\int_0^\infty e^{-\lambda\tau}\,\vartheta(\tau)\left(\lfloor\underline{n}\int_0^\tau W\Big(\sigma, \sum_{j=0}^{n}\lambda^{n-j}A_j x\Big)\,d\sigma\right)d\tau =$$

$$= x - \Bigg[-\int_0^\infty e^{-\lambda\tau}\,\vartheta'(\tau)\,d\tau\, x -$$

$$-\sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\lfloor\underline{n}\int_0^\tau W(\sigma, A_j x)\,d\sigma\right)d\tau\Bigg]$$

for every $x \in \mathsf{D}(A_1)\cap\mathsf{D}(A_2)\cap\ldots\cap\mathsf{D}(A_n)$ and $\lambda > 0$.

Immediately from (13) we get the following identity:

$$(14)\quad x = -\int_0^\infty e^{-\lambda\tau}\,\vartheta'(\tau)\,d\tau\, x -$$

$$-\sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\lfloor\underline{n}\int_0^\tau W(\sigma, A_j x)\,d\sigma\right)d\tau$$

for every $x \in \mathsf{D}(A_1)\cap\mathsf{D}(A_2)\cap\ldots\cap\mathsf{D}(A_n)$ and $\lambda > 0$ such that

$$\sum_{j=0}^{n}\lambda^{n-j}A_j x = 0.$$

It follows from (1), (3) and (4) that

$$(15)\quad A_r\Bigg(\sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\,.$$

$$.\left(\lfloor\underline{n}\int_0^\tau W(\sigma, A_j x)\,d\sigma\right)d\tau\Bigg) = \sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\,.$$

$$.\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\lfloor\underline{n-r}\int_0^\tau\left(A_r\,\lfloor\underline{r}\int_0^\sigma W(\varrho, A_j x)\,d\varrho\right)d\sigma\right)d\tau \text{ for every}$$

$x \in \mathsf{D}(A_1)\cap\mathsf{D}(A_2)\cap\ldots\cap\mathsf{D}(A_n)$, $\lambda \in \mathrm{R}$ and $r \in \{0, 1, \ldots, n-1\}$.

The properties (2), (5), (6), (7) and (15) imply

$$(16)\quad \left\| -\int_0^\infty e^{-\lambda\tau}\,\vartheta'(\tau)\,d\tau\, A_r x - A_r\left(\sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\cdot\right.\right.$$

$$\left.\left.\cdot\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\underline{n}\int_0^\tau W(\sigma, A_j x)\,d\sigma\right)d\tau\right)\right\| \leqq \int_1^2 e^{-\lambda\tau}\,d\tau\, L\|A_r x\| +$$

$$+\sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1}\binom{n-j}{k}\lambda^{k+1}\int_1^2 e^{-\lambda\tau}\,d\tau\, L\frac{2^{n-r}}{(n-r)!}K_0(2)\,\|A_j x\| \leqq \frac{1}{\lambda}L\|A_r x\| +$$

$$+\sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1}\binom{n-j}{k}\lambda^{k+1}\frac{e^{-\lambda}}{\lambda}L\frac{2^{n-r}}{(n-r)!}K_0(2)\,\|A_j x\| \leqq$$

$$\leqq \frac{1}{\lambda}L\|A_r x\| + (1+\lambda)^n e^{-\lambda}L 2^n K_0(2)\sum_{j=0}^{n-1}\|A_j x\| \leqq$$

$$\leqq \left[L\frac{1}{\lambda} + 2^n L K_0(2)(1+\lambda)^n e^{-\lambda}\right]\sum_{j=0}^{n-1}\|A_j x\| \quad \text{for every}$$

$x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$, $\lambda > 0$ and $r \in \{0, 1, \ldots, n-1\}$.

It is now easy to see from (16) that there exists a constant ω_0 such that

(17) $\omega_0 \geqq 0$,

$$(18)\quad \left\| -\int_0^\infty e^{-\lambda\tau}\,\vartheta'(\tau)\,d\tau\, A_r x - A_r\left(\sum_{j=0}^{n-1}(-1)^{n-j}\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\cdot\right.\right.$$

$$\left.\left.\cdot\int_0^\infty e^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\underline{n}\int_0^\tau W(\sigma, A_j x)\,d\sigma\right)d\tau\right)\right\| \leqq \frac{1}{2n}\sum_{j=0}^{n-1}\|A_j x\| \quad \text{for every}$$

$x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$, $\lambda > \omega_0$ and $r \in \{0, 1, \ldots, n-1\}$.

It follows easily from (14), (17) and (18) that

(19) $\sum_{r=0}^{n-1}\|A_r x\| \leqq \frac{1}{2}\sum_{j=0}^{n-1}\|A_j x\|$ for every $x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ and

$\lambda > \omega_0$ such that $\sum_{j=0}^{n}\lambda^{n-j}A_j x = 0$.

On the other hand, let us observe that, because $A_0 = I$,

(20) $\sum_{l=0}^{n-1}\|A_l x\| = 0$ if and only if $x = 0$.

But as an immediate consequence of (19) and (20) we have:

(21) the operator $\sum_{j=0}^{n} \lambda^{n-j} A_j$ is one-to-one for every $\lambda > \omega_0$.

Now we proceed to the next step of our proof, namely, we shall try to find some analytical expressions for the operator $(\sum_{j=0}^{n} \lambda^{n-j} A_j)^{-1}$ for sufficiently large $\lambda > \omega_0$.

According to (1) we can write

(22) $\sum_{j=0}^{n} A_j \underline{\lfloor j} \int_0^t W(\tau, x)\, d\tau = x$ for every $x \in E$ and $t \in R^+$.

Multiplying (22) by $e^{-\lambda\tau} \vartheta(\tau)$, integrating over $(0, \infty)$ as permitted by (5) and (6) and using (4) we get at once

(23) $\sum_{j=0}^{n} A_j \left(\int_0^\infty e^{-\lambda\tau} \vartheta(\tau) \left(\underline{\lfloor j} \int_0^\tau W(\sigma, x)\, d\sigma \right) d\tau \right) = \int_0^\infty e^{-\lambda\tau} \vartheta(\tau)\, d\tau\, x$ for every

$x \in E$ and $\lambda \in R$.

It follows from (4) and (9) with $s = j$ and $r = n - j$ and from (8) that (23) can be rewritten in the form

(24) $\sum_{j=0}^{n} \lambda^{n-j} A_j \left(\int_0^\infty e^{-\lambda\tau} \vartheta(\tau) \left(\underline{\lfloor n} \int_0^\tau W(\sigma, x)\, d\sigma \right) d\tau \right) +$

$+ \sum_{j=0}^{n-1} (-1)^{n-j} A_j \left(\sum_{k=0}^{n-j-1} \binom{n-j}{k} (-\lambda)^k \int_0^\infty e^{-\lambda\tau} \vartheta^{(n-j-k)}(\tau)\, . \right.$

$\left. . \left(\underline{\lfloor n} \int_0^\tau W(\sigma, x)\, d\sigma \right) d\tau \right) = \frac{1}{\lambda} \left(1 + \int_0^\infty e^{-\lambda\tau} \vartheta'(\tau)\, d\tau \right) x$ for every $x \in E$ and $\lambda > 0$.

Multiplying (24) by λ and rearranging the terms we get

(25) $\sum_{j=0}^{n} \lambda^{n-j} A_j \left(\int_0^\infty \lambda e^{-\lambda\tau} \vartheta(\tau) \left(\underline{\lfloor n} \int_0^\tau W(\sigma, x)\, d\sigma \right) d\tau \right) = x - \left[- \int_0^\infty e^{-\lambda\tau} \vartheta'(\tau)\, d\tau\, x - \right.$

$- \sum_{j=0}^{n-1} (-1)^{n-j} A_j \left(\sum_{k=0}^{n-j-1} \binom{n-j}{k} (-\lambda)^{k+1} \, . \right.$

$\left. \left. . \int_0^\infty e^{-\lambda\tau} \vartheta^{(n-j-k)}(\tau) \left(\underline{\lfloor n} \int_0^\tau W(\sigma, x)\, d\sigma \right) d\tau \right) \right]$ for every $x \in E$ and $\lambda > 0$.

For the sake of simplicity we define

(26) $G(\lambda)\, x = \int_0^\infty \lambda e^{-\lambda\tau} \vartheta(\tau) \left(\underline{\lfloor n} \int_0^\tau W(\sigma, x)\, d\sigma \right) d\tau$ for $x \in E$ and $\lambda \in R$,

$$(27)\ H(\lambda)\,x = -\int_0^\infty \mathrm{e}^{-\lambda\tau}\,\vartheta'(\tau)\,\mathrm{d}\tau\, x - \sum_{j=0}^{n-1}(-1)^{n-j}A_j\left(\sum_{k=0}^{n-j-1}\binom{n-j}{k}(-\lambda)^{k+1}\,.\right.$$

$$\left.\cdot\int_0^\infty \mathrm{e}^{-\lambda\tau}\,\vartheta^{(n-j-k)}(\tau)\left(\underline{\vert n}\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right) \text{ for } x \in E \text{ and } \lambda \in \mathrm{R}.$$

According to (26) and (27) we can rewrite (25) in the form

$$(28)\ \sum_{j=0}^{n}\lambda^{n-j}A_j\,G(\lambda) = I - H(\lambda) \text{ for every } \lambda \in \mathrm{R}.$$

Now, we shall establish some necessary estimates for G and H.

We begin with the function G.

In virtue of Lemmas 2.4 and 2.6 we get easily from (1), (3), (4) and (26) that

$$(29)\ \lambda^{n-i-1}A_i\,G(\lambda)\,x = A_i\left(\int_0^\infty \lambda^{n-i}\mathrm{e}^{-\lambda\tau}\,\vartheta(\tau)\left(\underline{\vert n}\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right) =$$

$$= A_i\left(\int_0^\infty \mathrm{e}^{-\lambda\tau}\left(\frac{\mathrm{d}^{n-i}}{\mathrm{d}^{n-i}}\left(\vartheta(\tau)\,\underline{\vert n}\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\right)\mathrm{d}\tau\right) =$$

$$= A_i\left(\int_0^\infty \mathrm{e}^{-\lambda\tau}\sum_{k=0}^{n-i}\binom{n-i}{k}\vartheta^{(n-i-k)}(\tau)\left(\underline{\vert n-k}\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right) =$$

$$= \sum_{k=0}^{n-i}\binom{n-i}{k}\int_0^\infty \mathrm{e}^{-\lambda\tau}\,\vartheta^{(n-i-k)}(\tau)\left(A_i\,\underline{\vert n-k}\int_0^\tau W(\sigma, x)\,\mathrm{d}\sigma\right)\mathrm{d}\tau =$$

$$= \sum_{k=0}^{n-i}\binom{n-i}{k}\int_0^\infty \mathrm{e}^{-\lambda\tau}\,\vartheta^{(n-i-k)}(\tau)\left(\underline{\vert n-k-i}\int_0^\tau A_i\,\underline{\vert i}\int_0^\sigma W(\varrho, x)\,\mathrm{d}\varrho\,\mathrm{d}\sigma\right)\mathrm{d}\tau$$

for every $x \in E$, $\lambda \in \mathrm{R}$ and $i \in \{1, 2, \ldots, n\}$.

It follows from (29) that

$$(30)\ \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\left(\lambda^{n-i-1}A_i\,G(\lambda)\,x\right) =$$

$$= \sum_{k=0}^{n-i}\binom{n-i}{k}\int_0^\infty \mathrm{e}^{-\lambda\tau}(-\tau)^p\,\vartheta^{(n-i-k)}(\tau)\left(\underline{\vert n-k-i}\int_0^\tau A_i\left(\underline{\vert i}\int_0^\sigma W(\varrho, x)\,\mathrm{d}\varrho\right)\mathrm{d}\sigma\right)\mathrm{d}\tau$$

for every $x \in E$, $\lambda \in \mathrm{R}$, $i \in \{1, 2, \ldots, n\}$ and $p \in \{0, 1, \ldots\}$.

Using (2), (5), (6) and (7) we get easily the following estimate from (30):

$$(31)\ \left\|\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\left(\lambda^{n-i-1}A_i\,G(\lambda)\,x\right)\right\| \leqq \sum_{k=0}^{n-i}\binom{n-i}{k}\int_0^\infty \mathrm{e}^{-\lambda\tau}\tau^p\,\mathrm{d}\tau\, L\,\frac{2^{n-k-i}}{(n-k-i)!}\,K_0(2)\|x\| \leqq$$

$$\leqq \frac{p!}{\lambda^{p+1}} 2^{2n} L K_0(2) \|x\| \text{ for every } x \in E,\ \lambda > 0,\ i \in \{1, 2, \dots, n\} \text{ and}$$

$p \in \{0, 1, \dots\}$.

The estimate (31) may be written, as is easily seen, in the form

$$(32)\quad \left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\lambda^{n-i-1} A_i\, G(\lambda)) \right\| \leqq \frac{p!}{\lambda^{p+1}} 2^{2n} L K_0(2)$$

for every $\lambda > 0$, $i \in \{1, 2, \dots, n\}$ and $p \in \{0, 1, \dots\}$.

Let us now denote

$$(33)\quad H_0(\lambda)\, x = - \int_0^\infty \mathrm{e}^{-\lambda\tau}\, \vartheta'(\tau)\, \mathrm{d}\tau\, x \text{ for } x \in E \text{ and } \lambda \in \mathrm{R},$$

$$(34)\quad H_1(\lambda)\, x = \sum_{j=0}^{n-1} (-1)^{n-j} A_j \left(\sum_{k=0}^{n-j-1} \binom{n-j}{k} (-\lambda)^{k+1} \cdot \right.$$

$$\left. \cdot \int_0^\infty \mathrm{e}^{-\lambda\tau}\, \vartheta^{(n-j-k)}(\tau) \left(\underline{|n} \int_0^\tau W(\sigma, x)\, \mathrm{d}\sigma \right) \mathrm{d}\tau \right) \text{ for every } x \in E \text{ and } \lambda \in \mathrm{R}.$$

By (27), (33) and (34),

$$(35)\quad H(\lambda) = H_0(\lambda) - H_1(\lambda) \text{ for every } \lambda \in \mathrm{R}.$$

Now we easily obtain with regard to (5)–(7) and (33) that

$$(36)\quad \left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} H_0(\lambda)\, x \right\| \leqq \int_0^\infty \mathrm{e}^{-\lambda\tau} \tau^p\, \mathrm{d}\tau\, L \|x\| \leqq \frac{p!}{\lambda^{p+1}} L \|x\| \text{ for every } x \in E \text{ and } \lambda > 0.$$

On the other hand, in virtue of Lemmas 2.4 and 2.6 it follows from (1), (3)–(7) and (34) that

$$(37)\quad H_1(\lambda) x = \sum_{j=0}^{n-1} (-1)^{n-j} A_j \left(\sum_{k=0}^{n-j-1} (-1)^{k+1} \binom{n-j}{k} \int_0^\infty \mathrm{e}^{-\lambda\tau} \cdot \right.$$

$$\left. \cdot \left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}} \left(\vartheta^{(n-j-k)}(\tau) \underline{|n} \int_0^\tau W(\sigma, x)\, \mathrm{d}\sigma \right) \right) \mathrm{d}\tau \right) =$$

$$= \sum_{j=0}^{n-1} (-1)^{n-j} A_j \left(\sum_{k=0}^{n-j-1} (-1)^{k+1} \binom{n-j}{k} \int_0^\infty \mathrm{e}^{-\lambda\tau} \cdot \right.$$

$$\left. \cdot \left(\sum_{l=0}^{k+1} \binom{k+1}{l} \vartheta^{(n-j-l+1)}(\tau) \underline{|n-l} \int_0^\tau W(\sigma, x)\, \mathrm{d}\sigma \right) \mathrm{d}\tau \right) =$$

$$= \sum_{j=0}^{n-1} (-1)^{n-j} \sum_{k=0}^{n-j-1} (-1)^{k+1} \binom{n-j}{k} \sum_{l=0}^{k+1} \binom{k+1}{l} \int_0^\infty \mathrm{e}^{-\lambda\tau} \cdot$$

$$. \vartheta^{(n-j-l+1)}(\tau) \left(A_j \lfloor\underline{n-l} \int_0^\tau W(\sigma, x) \, \mathrm{d}\sigma \right) \mathrm{d}\tau =$$

$$= \sum_{j=0}^{n-1} (-1)^{n-j} \sum_{k=0}^{n-j-1} (-1)^{k+1} \binom{n-j}{k} \sum_{l=0}^{k+1} \binom{k+1}{l} \int_0^\infty \mathrm{e}^{-\lambda\tau} \, .$$

$$. \vartheta^{(n-j-l+1)}(\tau) \left(\lfloor\underline{n-j-l} \int_0^\tau A_j \left(\lfloor\underline{j} \int_0^\sigma W(\varrho, x) \, \mathrm{d}\varrho \right) \mathrm{d}\sigma \right) \mathrm{d}\tau$$

for every $x \in E$ and $\lambda \in \mathrm{R}$.

It follows at once from (37) that

$$(38) \quad \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} H_1(\lambda) \, x = \sum_{j=0}^{n-1} (-1)^{n-j} \sum_{k=0}^{n-j-1} (-1)^{k+1} \binom{n-j}{k} \sum_{l=0}^{k+1} \binom{k+1}{l} \int_0^\infty \mathrm{e}^{-\lambda\tau} (-\tau)^p \, .$$

$$. \vartheta^{(n-j-l+1)}(\tau) \left(\lfloor\underline{n-j-l} \int_0^\tau \left(A_j \lfloor\underline{j} \int_0^\sigma W(\varrho, x) \, \mathrm{d}\varrho \right) \mathrm{d}\sigma \right) \mathrm{d}\tau$$

for every $x \in E$, $\lambda \in \mathrm{R}$ and $p \in \{0, 1, \ldots\}$.

According to (2), (5), (6), (7) and (38) we obtain

$$(39) \quad \left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} H_1(\lambda) \, x \right\| \leqq$$

$$\leqq \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} \binom{n-j}{k} \sum_{l=0}^{k+1} \binom{k+1}{l} \int_0^\infty \mathrm{e}^{-\lambda\tau} \tau^p \, \mathrm{d}\tau \, L \frac{2^{n-l-j}}{(n-l-j)!} K_0(2) \, \|x\| \leqq$$

$$\leqq \frac{p!}{\lambda^{p+1}} \, n \, 2^{3n} L K_0(2) \, \|x\| \leqq \frac{p!}{\lambda^{p+1}} \, 2^{4n} \, L K_0(2) \, \|x\|$$

for every $x \in E$, $\lambda > 0$ and $p \in \{0, 1, \ldots\}$.

It follows from (35), (36) and (39) that

$$(40) \quad \left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} H(\lambda) \, x \right\| \leqq \frac{p!}{\lambda^{p+1}} L(1 + 2^{4n} K_0(2)) \, \|x\|$$

for every $x \in E$, $\lambda > 0$ and $p \in \{0, 1, \ldots\}$.

But (40) implies

$$(41) \quad \left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} H(\lambda) \right\| \leqq \frac{p!}{\lambda^{p+1}} L(1 + 2^{4n} K_0(2)) \text{ for every } \lambda > 0 \text{ and } p \in \{0, 1, \ldots\}.$$

Let us now denote

$$(42) \quad \omega_1 = \max\left(\omega_0, [1 + L(1 + 2^{4n} K_0(2))]^{-1}\right).$$

It is clear from (42) that

(43) $\omega_1 \geqq \omega_0$.

Now by (41) and (42), we have

(44) $\|H(\lambda)\| < 1$ for every $\lambda > \omega_1$.

Summing up (21), (28), (43) and (44) we obtain the following basic relation:

(45) $(\sum_{j=0}^{n} \lambda^{n-j} A_j)^{-1} = G(\lambda)(I - H(\lambda))^{-1}$ for every $\lambda > \omega_1$.

Let us now write

(46) $N = L(1 + 2^{4n} K_0(2))$,

(47) $\omega = \max(\omega_1, N)$,

(48) $M = 2^{2n} L K_0(2)$,

(49) $R(\lambda) = (\sum_{j=0}^{n} \lambda^{n-j} A_j)^{-1}$ for $\lambda > \omega_1$.

It is clear from (46)–(48) that

(50) $\omega \geqq 0$, $M \geqq 0$.

Taking now $\varphi(\lambda) = M/\lambda$, $\psi(\lambda) = -N/\lambda$, $\Phi(\lambda) = \lambda^{n-i-1} A_i G(\lambda)$, $\Psi(\lambda) = -H(\lambda)$ for $\lambda > \omega$ in Lemma 2.9 we get from (32) and (41)–(49) that

(51) $$\left\| \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\lambda^{n-i-1} A_i R(\lambda)) \right\| \leqq$$

$$\leqq (-1)^p \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \left(\frac{M}{\lambda} \frac{1}{1 - \frac{N}{\lambda}} \right) = (-1)^p \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} \left(\frac{M}{\lambda - N} \right) =$$

$$= M \frac{p!}{(\lambda - N)^{p+1}} \leqq M \frac{p!}{(\lambda - \omega)^{p+1}} \text{ for every } \lambda > \omega,$$

$i \in \{1, 2, \ldots, n\}$ and $p \in \{0, 1, \ldots\}$.

We see from (α), (21), (42), (47), (49), (50) and (51) that all the assumptions of Proposition 3.6 are fulfilled and consequently, its application gives the desired result.

The proof is complete.

3.9. Remark. The preceding theorem shows that any system of operators $A_1, A_2, \ldots$ $\ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, satisfying the assumptions (α), (β) and (γ) is correct of class zero in the terminology of [1] or strictly correct in the terminology of some other papers of the author. The assumptions (β) and (γ) describe the property which is usually called (in more or less rough form) well-posedness of the problem in question.

The assertion of Theorem 3.8 is well-known for $n = 1$ and $n = 2$, $A_1 = 0$, where the functional equations characterizing semigroups and cosine functions of operators are used to get the necessary estimates (cf. [2] and [3]). But similar functional equations are not known in the general case considered here, and they almost surely do not exist. Therefore, the preceding proof had to be based on completely different ideas. One of them, namely, how to express the generalised resolvent of the system $A_1, A_2, \ldots$ $\ldots, A_n$ by means of a form of local (finite) Laplace transform of solutions, is due to Chazarain [4].

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