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EXPRESSING RATIONALS AS A SUM OF A SMALL NUMBER OF UNIT FRACTIONS

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I. INTRODUCTION

For a given rational number a/b, we wish to consider the solvability of the equation

$$\frac{a}{b} = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \tag{1}$$

where the x_i are integers (not necessarily positive). For a fixed positive integer a, let $L = L(a)$ be the smallest value of n for which (1) is solvable for all sufficiently large integers b.

Even if the x_i are required to be positive, it is clear that $L \leqq a$. Although a well-known conjecture of SCHINZEL [2] is that $L = 3$ for all $a \geqq 3$, no one has succeeded in finding an improvement on the trivial estimate for even one value of a.

A similar result is conjectured for the case under discussion, where the x_i may be negative. This problem has proved to be somewhat easier, and it is known that $L = 3$ for $3 \leqq a \leqq 35$. This result can be used to show that

$$L \leqq 3\left(\left[\frac{a}{35}\right]\right) + 1\,,$$

where [ ] is the greatest integer function. Also, some minor improvements of this estimate are fairly easy to obtain.

The principal objective of this paper is to obtain a significantly better upper bound for L; namely one of order $\log a$.

II. APPROXIMATIONS USING SMALL NUMERATORS

Before proving the estimate mentioned above, we will need some preliminary results. These results, concerning Farey fractions and approximations using small numerators, are also interesting in their own right.

Most problems in rational approximation involve the existence of a good approximation c/d to some number, where d is small. We will be interested in approximating, or more precisely, in decomposing a given rational, using fractions with small numerators.

Let a/b be a reduced rational number, $0 < a/b < 1$. We will approximate a/b using a sequence of fractions $a_0/b_0 = a/b$, a_1/b_1, $a_2/b_2 \dots$, to be defined below.

Let $\mathfrak{F}_n$ denote the Farey series of order n. We will also use the following special notation. Write the triple $T = (i\,j\,k)$ to mean $a_i/b_i < a_j/b_j < a_k/b_k$ are three consecutive elements of $\mathfrak{F}_{b_j}$. Note that this implies $a_i + a_k = a_j$ and $b_i + b_k = b_j$, Also, write the five-tuple $F = (i\,j\,k : x\,y)$ to mean $T = (i\,j\,k)$ and $a = xa_i + ya_k$. $b = xb_i + yb_k$.

We now define the sequence $\{a_i/b_i\}$ by specifying successive triples T_m. Let $T_1 = (1\,0\,2)$, so that $a_1/b_1 < a_0/b_0 < a_2/b_2$ are consecutive elements of $\mathfrak{F}_{b_0} = \mathfrak{F}_b$. Note that $F_1 = (1\,0\,2\colon\ 1\,1)$.

Now, given a triple $T_{m-2} = (i\,j\,k)$ we define T_{m-1} by choosing a_m/b_m such that

$$T_{m-1} = \begin{cases} (i\,k\,m) & \text{if} \quad b_i < b_k\,, \\ (m\,i\,k) & \text{if} \quad b_i > b_k\,. \end{cases}$$

Note that the only case in which $b_i = b_k$ is when $(i\,j\,k)$ represents $\frac{0}{1}\,\frac{1}{2}\,\frac{1}{1}$, at which point the sequence must terminate anyway. The fraction a_m/b_m is the next term in the sequence.

Lemma 1. *If* $F_{m-2} = (i\,j\,k\colon\ x\,y)$ *then* $F_{m-1} = (m\,i\,k\colon\ x\,x + y)$ *or* $F_{m-1} = (i\,k\,m\colon\ x + y\,y)$.

Proof. By definition of the sequence $\{a_i/b_i\}$ we know that either $T_{m-1} = (m\,i\,k)$ or $T_{m-1} = (i\,k\,m)$. In the former case $a = xa_i + ya_k = x(a_m + a_k) + ya_k = xa_m + (x + y)\,a_k$. In the later case $a = xa_i + ya_k = xa_i + y(a_i + a_m) = (x + y)\,a_i + ya_m$. Similar calculations hold for b.

Lemma 2. *If* $F_n = (i\,j\,k\colon\ x\,y)$ *then* $ab_i - ba_i = y$ *and* $ab_k - ba_k = -x$.

Proof. Use induction on n. $F_1 = (1\,0\,2\colon 1\,1)$ and $ab_1 - ba_1 = 1$, $ab_2 - ba_2 = -1$ by the well-known property of $\mathfrak{F}_b$. [1, Theorem 28]

Now suppose the result is true for $F_{m-2} = (i\,j\,k\colon x\,y)$. If $F_{m-1} = (m\,i\,k\colon x\,x + y)$ then $ab_m - ba_m = a(b_i - b_k) - b(a_i - a_k) = (ab_i - ba_i) - (ab_k - ba_k) = y + x$ by the induction hypothesis. ($ab_k - ba_k = -x$ following immediately from the induction hypothesis.) A similar calculation holds if $F_{m-1} = (i\,k\,m\colon x + y\,y)$. Note that the conclusion of Lemma 2 can be witten as:

$$\frac{a}{b} = \frac{a_i}{b_i} + \frac{y}{bb_i}, \quad \frac{a}{b} = \frac{a_k}{b_k} - \frac{x}{bb_k}.$$

In the above procedure it is clear that $x + y$ is monotonically increasing, and the sequence does not terminate until $x + y = b > a$. Thus, for any real number λ, $1 < \lambda < a$, we eventually encounter an F_m where $x < \lambda$, $y < \lambda$, and $x + y \geqq \lambda$. When this occurs we must have either $a_i < a/\lambda$ or $a_k < a/\lambda$, for the following reasons. Assume $a_i \geqq a/\lambda$ and $a_k \geqq a/\lambda$. Then $a = xa_i + ya_k \geqq (x + y)\, a/\lambda \geqq a$ with strict inequality (and hence a contradiction) if $a_i > a/\lambda$ or $a_k > a/\lambda$ or $x + y > \lambda$. Also, if $a_i = a_k = a/\lambda$ and $x + y = \lambda$ then by Lemma 2, $\lambda = x + y = ba_k - ab_k + ab_i - ba_i = a(b_i - b_k)$. So $b_i - b_k = \lambda/a$ and thus λ/a and a/λ are both integers which implies $a = \lambda$ contradicting the fact that $\lambda < a$.

Theorem 1. *For $0 < a/b < 1$ and every integer $n > 1$, there exist integers x_i, z_i such that*

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{x_i}{z_i} \quad \text{and} \quad |x_i| < a^{1/n}.$$

Proof. Let $\lambda = a^{1/n}$. By the above remarks

$$\frac{a}{b} = \frac{x_1}{z_1} + \frac{A_1}{B_1} \quad \text{where} \quad |x_1| < a^{1/n} \quad \text{and} \quad A_1 < a/\lambda = a^{(n-1)/n}.$$

Similarly,

$$\frac{A_1}{B_1} = \frac{x_2}{z_2} + \frac{A_2}{B_2} \quad \text{where} \quad |x_2| < a^{1/n} \quad \text{and} \quad A_2 < A_1/\lambda < a/\lambda^2 = a^{(n-2)/n}.$$

Proceeding in this manner, we obtain in general

$$\frac{a}{b} = \frac{x_1}{z_1} + \ldots + x_r z_r + \frac{A_r}{B_r} \quad \text{where} \quad |x_i| < a^{1/n} \quad \text{and}$$

$$A_r < A_{r-1}/\lambda < \ldots < a/\lambda^r < a^{(n-r)/n}.$$

Letting $r = n - 1$, we obtain the desired result.

Professor M. J. Knight, in a private communication, has noted that Theorem 1 can also be proved using geometry of numbers.

III. AN ESTIMATE FOR L

We are now ready to state and prove our principal result.

Theorem 2. *For a given positive integer a and all integers b sufficiently large, the equation (1) is solvable in integers x_i, where $n \leqq 3 \log a/\log 36 + 3$.*

Proof. Following the same procedure as in the proof of Theorem 1, with $\lambda = 36$, we obtain

$$\frac{a}{b} = \sum_{i=1}^{s} \frac{y_i}{z_i} + \frac{A_s}{B_s} \quad \text{where} \quad |y_i| < 36 \quad \text{and} \quad A_s < a/36^s.$$

Choose s so that $36^s \leqq a < 36^{s+1}$. Then

$$\frac{a}{b} = \sum_{i=1}^{s+1} \frac{y_i}{z_i} \quad \text{where} \quad |y_i| < 36 .$$

By [3, Theorem 4] each

$$\frac{y_i}{z_i} = \frac{1}{x_{i_1}} + \frac{1}{x_{i_2}} + \frac{1}{x_{i_3}}$$

so $L \leqq 3(s + 1)$. The condition that b is sufficiently large guarantees that the z_i are sufficiently large also. We now note that $s \leqq \log a/\log 36$, which completes the proof of the theorem. We also note that for large values of a, $n \leqq 3 \log a/\log 36 + 3 <$ $< \log a$ where log denotes the natural logarithm.

IV. CONCLUDING REMARKS

The bound on L given in Theorem 2 is still a long way from the conjectured result, so an improved estimate would be of interest. The same conjecture suggests that the result in Theorem 1 is probably not the best possible when $n \geqq 3$. However, we do have the following result when $n = 2$.

Theorem. 3 *For $0 < a/b < 1$ there exist integers x_1, x_2, z_1, z_2 such that*

$$\frac{a}{b} = \frac{x_1}{z_1} + \frac{x_2}{z_2} \quad \textit{and} \quad |x_i| < \sqrt{a} .$$

Moreover, the bound on the $|x_i|$ is the best possible.

Proof. By Theorem 1, we need only prove the last statement.

Let $a = (n + 1)^2 - 1 = n^2 + 2n$ and let b be a prime such that $b \equiv n + 1 \pmod{a}$. Infinitely many such primes exist since $(n + 1, a) = 1$.

By Theorem 1

$$\frac{a}{b} = \frac{x_1}{z_1} + \frac{x_2}{z_2} \quad \text{where} \quad |x_i| \leqq n . \tag{2}$$

Now, assume (2) holds where $|x_i| \leqq n - 1$. Then by Theorem 1′ of [4] there exist $d_1, d_2 \mid b$ such that $x_1 d_1 + x_2 d_2 = ka$ for some integer $k \neq 0$. Since b is prime, its only divisors are ± 1, $\pm b$. If $|d_1 d_2| = 1$ or b^2 then $x_1 d_1 + x_2 d_2 = ka$ is possible only for $k = 0$, which is the excluded case.

Thus it suffices to show that $x_1 + bx_2 \equiv x_1 + (n + 1) x_2 \not\equiv 0 \pmod{n^2 + 2n}$ for all $|x_i| \leqq n - 1$ except $x_1 = x_2 = 0$. But this follows immediately from the fact that

$$2 \leqq |x_1 + (n + 1) x_2| \leqq n^2 + n - 2 .$$

It would still be quite interesting to know if Theorem 1 can be improved for $n \geqq 3$.

References

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