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# FUNDAMENTAL SOLUTIONS OF THE DIFFERENTIAL OPERATOR $(-1)^n D_1^n D_2^n + a(iD_1)^n + b(iD_2)^n + c$

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This paper is concerned with fundamental solutions of the partial differential operator

$$(1) \qquad (-1)^n D_1^n D_2^n + a(iD_1)^n + b(iD_2)^n + c\,,$$

where $D_1 = -i(\partial/\partial x_1)$, $D_2 = -i(\partial/\partial x_2)$, in the space of generalized functions [1], [2]. Conditions of existence of temperate fundamental solutions are derived for the operator with constant coefficients and for arbitrary $n$.

## INTRODUCTION

Let $G$ be an open convex set in the real two dimensional space $R_2$. We denote by $C^k(G)$, $0 \leqq k < \infty$ the set of all functions defined in $G$ whose partial derivatives of order $\leqq k$ all exist and are continuous. We define $C^\infty(G) = \bigcap_{k=0}^{\infty} C^k(G)$. The set of all functions $\varphi \in C^\infty(G)$ with compact support in $G$ is denoted by $C_0^\infty(G)$. A distribution $u$ in $G$ is a continuous linear functional on $C_0^\infty(G)$. The set of all distributions in $G$ is denoted by $\mathscr{D}'(G)$, the space of all distributions with compact support in $G$ by $\mathscr{E}'(G)$.

Further, we denote by $\mathscr{S}(R_2)$ the set of all functions $\varphi \in C^\infty(R_2)$ such that

$$\sup_x |x^\beta D^\alpha \varphi(x)| < \infty$$

for all multiindices $\alpha$ and $\beta$, where $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}$, $\alpha_1 + \alpha_2 = |\alpha|$. A continuous linear functional $u$ on $\mathscr{S}$ is called a *temperate distribution*. The set of all temperate distributions is denoted by $\mathscr{S}'$.

The Fourier transform $F[f]$ of a function $f \in L_1(R_2)$ is defined by

$$F[f](\xi) = \iint_{R_2} e^{-i(x_1\xi_1 + x_2\xi_2)} f(x_1, x_2)\, dx_1\, dx_2\,.$$

If $u \in \mathcal{S}'$, the Fourier transform $F[u]$ is defined by

$$F[u](\varphi) = u(F[\varphi]), \quad \varphi \in \mathcal{S}.$$

A distribution $E \in \mathcal{D}'(R_2)$ is called a *fundamental solution of the differential operator* $P(D)$ if $P(D)E = \delta$, where $\delta$ is the Dirac distribution.

We denote by $P(\xi)$ the polynomial which we obtain by replacing the $D_j$ in the operator $P(D)$ by $\xi_j$. We say that an operator $Q(D)$ is *weaker* than $P(D)$, when

$$\tilde{Q}(\xi)/\tilde{P}(\xi) < C \quad \text{if } \xi \text{ is real},$$

where $\tilde{P}(\xi) = [\sum_{|\alpha| \geqq 0} |P^{(\alpha)}(\xi)|^2]^{1/2}$, $\tilde{Q}(\xi) = [\sum_{|\alpha| \geqq 0} |Q^{(\alpha)}(\xi)|^2]^{1/2}$,

$$P^{(\alpha)}(\xi) = \partial^{|\alpha|}P/\partial\xi_1^{\alpha_1}\,\partial\xi_2^{\alpha_2}, \quad \tilde{Q}^{(\alpha)}(\xi) = \partial^{|\alpha|}Q/\partial\xi_1^{\alpha_1}\,\partial\xi_2^{\alpha_2}, \quad \alpha_1 + \alpha_2 = |\alpha|.$$

Let $\mathcal{K}$ be the set of positive functions $k$ defined in $R_2$ for which there exist positive constants $C$ and $N$ such that

$$k(\xi + \eta) \leqq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in R_2.$$

If $k \in \mathcal{K}$ and $0 \leqq p \leqq \infty$, we denote by $\mathcal{B}_{p,k}$ the set of all distributions $u \in \mathcal{S}'$ such that $F[u]$ is a function and

$$\|u\|_{p,k} = \left(\frac{1}{(2\pi)}\int |k(\xi)\, F[u](\xi)|^p \,d\xi\right)^{1/p} < \infty.$$

In the case $p = \infty$ we shall interpret $\|u\|_{p,k}$ as ess. sup $|k(\xi)\, F[u](\xi)|$.

## 1. THE OPERATOR $(-1)^n D_1^n D_2^n + a(iD_1)^n + b(iD_2)^n + c$ WITH CONSTANT COEFFICIENTS

Let us consider the differential operator (1) with constant coefficients and $abc > 0$. Then the following theorem holds.

**Theorem 1.** *Let $n = 2m + 1$, $m = 0, 1, 2, \ldots$. Then there exists only one temperate fundamental solution E of the differential operator (1) which is in the space $\mathcal{B}_{\infty,\tilde{P}}$. This fundamental solution is proper in the sense of the denition of L. Hörmander.*

Proof. The differential operator (1) can be written in the form

$$P(D) = -D_1^n D_2^n \pm iaD_1^n \pm ibD_2^n + c, \tag{2}$$

where the sign + holds if $m$ is even and − if $m$ is odd.

The corresponding polynomial is

$$P(\xi) = -\xi_1^n \xi_2^n \pm ia\xi_1^n \pm ib\xi_2^n + c. \tag{3}$$

$P(\xi) \neq 0$, if $\xi$ is real.

Since $|P(\xi)| = [(-\xi_1^n\xi_2^n + c)^2 + (a\xi_1^n + b\xi_2^n)^2]^{1/2} \neq 0$ if $\xi$ is real and $1/|P(\xi)| \to 0$ if $\xi \to \infty$, there always exist $K > 0$ and $N > 0$ such that $1/|P(\xi)| = 1/[\xi_1^{2n}\xi_2^{2n} + a^2\xi_1^{2n} + b^2\xi_2^{2n} + c^2 + 2(ab - c)\,\xi_1^n\xi_2^n]^{1/2} \leqq k(1 + |\xi|^2)^N$. Thus it follows from results of L. Hörmander ([3], p. 36) that there exists one and only one temperate fundamental solution $E$ of the differential operator $P(D)$ for which we have

$$F[E] = 1/(-\xi_1^n\xi_2^n \pm ia\xi_1^n \pm ib\xi_2^n + c)\,.$$

Since

$$|\tilde{P}(\xi)\,F[E]\,(\xi)| = \frac{[\sum_{|\alpha|=0}^{n} |\partial^{|\alpha|}(-\xi_1^n\xi_2^n \pm ia\xi_1^n \pm ib\xi_2^n + c)/\partial\xi_1^{\alpha_1}\,\partial\xi_2^{\alpha_2}|^2]^{1/2}}{|-\xi_1^n\xi_2^n \pm ia\xi_1^n \pm ib\xi_2^n + c|} < \infty$$

the fundamental solution of the operator (2) is in the space $\mathscr{B}_{\infty P}$.

As in [4], we call the linear manifold

$$\Lambda(P) = \{\eta : \eta \textit{ is real and } P(\xi + t\eta) = P(\xi) \textit{ for any } \xi \textit{ and } t\}$$

the lineality space of the polynomial $P$, and we say that a polynomial $P$ is complete if its lineality space consists of the origin only.

It is evident that the polynomial (3) is complete. Since $P^{(\alpha)}(\xi)/P(\xi) \to 0$ if $\xi$ is real and $\to \infty$ for every $\alpha$ with $|\alpha| \neq 0$, the operator (2) is of local type (see [4], p. 222).

Since every fundamental solution of the operator $P(D)$, being complete and of local type, is proper in the sense of L. Hörmander's definition, the fundamental solution of the differential operator (2) is proper.

It means that $Q(D)\,(E * f) \in L_2^{\text{loc}}$ holds for $f \in L_2$ with compact support and for every differential polynomial $Q$ weaker than $P$.

Consider the special case when $n = 1$, $ab = c = 0$, $a > 0$, $b > 0$. Then

$$F[E]\,(\xi) = -1/(\xi_1 - ib)\,(\xi_2 - ia)\,.$$

As $F[g(x_1)f(x_2)] = F[g]\,(\xi_1)\,.\,F[f]\,(\xi_2)$ and $F[\Theta(x_2)\,e^{-ax_2}] = -i/(\xi_2 - ia)$ if $a > 0$, $F[\Theta(x_1)\,e^{-bx_1}] = -i/(\xi_1 - ib)$ if $b > 0$, we have the temperate fundamental solution of the differential operator

$$P(D) = -D_1D_2 + iaD_1 + ibD_2 + c$$

in the form

$$E(x_1, x_2) = \Theta(x_1, x_2)\,e^{-(bx_1 + ax_2)}\,,$$

where $\Theta(x_1, x_2) = 1$ if $x_1 > 0$, $x_2 > 0$, $\Theta(x_1, x_2) = 0$ if $x_1 < 0$, $x_2 < 0$.

The support of this fundamental solution is the convex angle $A = \{(x_1, x_2) : x_1 > 0,\ x_2 > 0\}$, with vertex at the origin.

Let $n$ be even. Then we have for the differential operator (1) with constant coefficients and $abc > 0$ the following

**Theorem 2.** *Let* $n = 2m$, $m = 1, 2, 3, \ldots$. *Then there exists only one temperate* *fundamental solution E of the differential operator* (1) *if m is odd and* $a < 0$, $b < 0$ *or if m is even and* $a > 0$, $b > 0$. *This fundamental solution is in the space* $\mathscr{B}_{\infty,p}$ *and it is proper.*

*If we assume that* $ab - c = 0$, *then we can write*

$$F[E](\xi) = 1/(\xi_1^{2m} + \beta^2)(\xi_2^{2m} + \alpha^2),$$

*where* $a = \alpha^2$, $b = \beta^2$ *if* $a > 0$, $b > 0$ *and* $-a = \alpha^2$, $-b = \beta^2$ *if* $a < 0$, $b < 0$.

Proof. The corresponding polynomial is

$$P(\xi) = \xi_1^{2m}\xi_2^{2m} \pm a\xi_1^{2m} \pm b\xi_2^{2m} + c,$$

where the sign $+$ is for the even $m$ and the sign $-$ is for the odd $m$.

It is $P(\xi) \neq 0$ if $\xi$ is real. We prove analogously to Theorem 1 that $P(\xi)$ is complete and of local type.

In particular, for $n = 2$ and $ab - c = 0$ we have the differential operator

$$P(D) = D_1^2 D_2^2 - aD_1^2 - bD_2^2 + ab, \quad a < 0, \quad b < 0$$

and

$$F[E] = 1/(\xi_1^2 + \beta^2)(\xi_2^2 + \alpha^2),$$

so that the temperate fundamental solution is

$$E = \frac{1}{4\alpha\beta} e^{-\alpha|x_2| - \beta|x_1|}, \quad \alpha > 0, \quad \beta > 0.$$

The support of this temperate solution is the whole plane $(x_1, x_2)$.

## 2. THE OPERATOR $-D_1D_2 + iaD_1 + ibD_2 + c$ WITH VARIABLE COEFFICIENTS

Consider now the case $n = 2$ and the differential operator

$$P(D, x) = -D_1D_2 + ia(x_1, x_2)D_1 + ib(x_1, x_2)D_2 + c(x_1, x_2), \tag{4}$$

where $a(x_1, x_2)$, $b(x_1, x_2)$, $c(x_1, x_2) \in C^\infty(R_2)$.

We say that a distribution $u(x_1, x_2) \in \mathscr{D}'(R_2)$ is a generalized solution of the differential equation

$$P(D, x)u = f \tag{5}$$

in the domain $G \subset R_2$, if the relation

$$\iint_G (-D_1D_2u + iaD_1u + ibD_2u + cu)\varphi \, dx_1 \, dx_2 = \iint_G f\varphi \, dx_1 \, dx_2$$

holds for all $\varphi \in C_0^\infty(G)$, supp $\varphi \in G$, $f \in \mathscr{D}'(G)$.

Let $G = \{(x_1, x_2) : \alpha_1 < x_1 < \beta_1,\ \alpha_2 < x_2 < \beta_2,\ \alpha_1, \alpha_2, \beta_1, \beta_2$ are real and positive constants$\}$

and assume that

$$\frac{\partial a}{\partial x_1} + ab - c = 0\,. \tag{6}$$

Then the differential equation (5) may be replaced by the system

$$\frac{\partial u}{\partial x_2} + au = z\,, \tag{7}$$

$$\frac{\partial z}{\partial x_1} + bz = f\,. \tag{8}$$

For the generalized solution of the differential equation (8) we have

**Lemma 1.** *Let $f \in \mathscr{E}'(G)$ and let $K \in \mathscr{D}'(G)$ be an arbitrary distribution which is independent of $x_1$* (see [2], p. 55). *Then the generalized solution of the differential equation* (8) *in G is given by*

$$z(x_1, x_2) = K \exp\left[-B(x_1, x_2)\right] +$$
$$+ \exp\left[-B(x_1, x_2)\right] \int f(x_1, x_2) \exp\left[B(x_1, x_2)\right] \mathrm{d}x_1\,,$$

*where $B(x_1, x_2) = \int b(x_1, x_2)\, \mathrm{d}x_1$.*

Similarly, for the differential equation (7) we obtain

**Lemma 2.** *Let $H \in \mathscr{D}'(G)$ be an arbitrary distribution independent of $x_2$ and let $z(x_1, x_2) \in \mathscr{E}'(G)$. Then*

$$u(x_1, x_2) = H \exp\left[-A(x_1, x_2)\right] +$$
$$+ \exp\left[-A(x_1, x_2)\right] \int z(x_1, x_2) \exp\left[A(x_1, x_2)\right] \mathrm{d}x_2\,,$$

*where $A(x_1, x_2) = \int a(x_1, x_2)\, \mathrm{d}x_2$, is a generalized solution of the differential equation* (7).

Hence under the condition (6) we have

**Theorem 3.** *Let $G = \{(x_1, x_2) : \alpha_1 < x_1 < \beta_1;\ \alpha_2 < x_2 < \beta_2,\ \alpha_1, \alpha_2, \beta_1, \beta_2$ are real positive constants$\}$; let $f \in \mathscr{E}'(G)$, let $K \in \mathscr{D}'(G)$ be an arbitrary distribution which is independent of $x_1$, $H \in \mathscr{D}'(G)$ an arbitrary distribution which is independent of $x_2$. Let $\Theta(x_1, x_2) = 1$ if $(x_1, x_2) \in G$, $\Theta(x_1, x_2) = 0$ if $(x_1, x_2) \in \mathbf{C}G$.*

*Then the generalized solution $u(x_1, x_2)$ of the differential equation* (5) *in G is given by*

$$u(x_1, x_2) = H \exp\left[-A(x_1, x_2)\right] +$$
$$+ \exp\left[-A(x_1, x_2)\right] \int \Theta K \exp\left[-B(x_1, x_2) + A(x_1, x_2)\right] \mathrm{d}x_2 +$$

$$+ \exp\left[-A(x_1, x_2)\right] \int \left\{ \exp\left[-B(x_1, x_2) + A(x_1, x_2)\right] \cdot \right.$$

$$\left. \cdot \int f(x_1, x_2) \exp B(x_1, x_2)\, \mathrm{d}x_1 \right\} \mathrm{d}x_2 \,.$$

This is a simple consequence of Lemma 1 and Lemma 2. If we put $f = \delta$, where $\delta$ is the Dirac distribution, we obtain the fundamental solution $E$ of the differential operator $P(D, x)$.

In the case $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = \infty$, $a = \text{const.}$, $b = \text{const.}$, $H = \frac{1}{2}\, \Theta(x_1, x_2)\, \mathrm{e}^{-bx_1}$, $K = \frac{1}{2}\, b\mathrm{e}^{-ax_2}$ we obtain the temperate fundamental solution which has been derived in the first section.

Let us assume the relation $(\partial b/\partial x_2) + ab - c = 0$ to be valid instead of (6). Then we have the fundamental solution of the differential operator $P(D, x)$ in the form

$$E(x_1, x_2) = K \exp\left[-B(x_1, x_2)\right] +$$

$$+ \exp\left[-B(x_1, x_2)\right] \int \Theta H \exp\left[-A(x_1, x_2) + B(x_1, x_2)\right] \mathrm{d}x_1 +$$

$$+ \exp\left[-B(x_1, x_2)\right] \int \left\{ \exp -A(x_1, x_2) + B(x_1, x_2) \int \delta \exp A(x_1, x_2)\, \mathrm{d}x_2 \right\} \mathrm{d}x_1 \,.$$

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