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BOUNDARY VALUE PROBLEMS FOR LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS.

HOMOGENEOUS EQUATION IN THE HALF PLANE

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Certain types of functional spaces corresponding to each type of partial differential equations are known which are, in a certain sense, suitable for their investigation. This is a common approach to this theory. It is also possible, however, to look for such a space in which suitable boundary problems could be solved for the largest possible class of equations. The summary of the corresponding results is given in the book [1] where boundary value problems in the half space for the operator $p_n(\partial/\partial x)\,\partial^n/\partial t^n + \ldots + p_0(\partial/\partial x)$ ($p_0, \ldots, p_n$ being polynomials) with $p_n(x) = 1$ are studied. The last condition is removed in the presented paper.

We investigate a nonstandard space of distributions where boundary problems for an arbitrary differential operator with constant coefficients in a given half plane can be formulated and solved. Although the results are more complicated than those in the book [1], it is possible to give quite elementary proofs, which are more concise, technically simpler and perhaps even more complete. Our conception seems to be useful even if the variable x is a vector. These considerations seem to be connected very closely with the papers [2], [3] where the elliptical case is studied.

The case when $p_n \not\equiv 1$ and the polynomial p_n even has real roots, leads after Fourier transformation to an equation the solution of which has singularities. We can see that this case occurs in some types of problems even if $p_n \equiv 1$. It appears, e.g., in the case of nonhomogeneous equations which will be dealt with in the next part of the paper.

1. Function and distribution spaces. Let $\boldsymbol{L}$ be the space of all functions $w(x)$ $(-\infty < x < \infty)$ such that $\|w\| = \int |w(x)|\,dx < \infty$, with usual topology. Let P be a polynomial such that $P(x) \neq 0$ for real x. We denote by $\boldsymbol{PL}$ the space of all functions v ($v = Pw$, $w \in \boldsymbol{L}$) with the norm $\|v\|_P = \|v/P\|$. Put $\mathscr{L} = \bigcup_P \boldsymbol{PL}$ with the topology of inductive limit.

Let Q be a polynomial such that $Q(x) \not\equiv 0$. We denote by $Q^{-1}\mathscr{L}$ the space of all distributions u defined in the domain $-\infty < x < \infty$ such that $Qu \in \mathscr{L}$. The map $u \to Qu$ has a finite-dimensional kernel and the topology in the space $Q^{-1}\mathscr{L}$ is, by definition, the weakest separated topology for which the exactly given map is continuous. Put $\mathscr{K} = \bigcup_Q Q^{-1}\mathscr{L}$ with the topology of inductive limit.

2. Explicit form of a distribution from the space $\mathscr{K}$. Distribution u is defined if the values $\langle u, \varphi \rangle$ are given for all functions $\varphi \in C_0^\infty$ with a sufficiently small support. Choose a real number a and let $Q(x) = (x - a)^s\, Q_1(x)$ $(s \geqq 0,\ Q_1(a) \neq 0)$. Let $\alpha \in C_0^\infty$ be a function such that $\alpha(x) \equiv 1$ in a neighbourhood of the point $x = a$. Then for all functions $\varphi \in C_0^\infty$, the support of which is in a small neighbourhood of the point $x = a$, the equation

(1) $$\langle u, \varphi \rangle = \langle Qu, \psi \rangle + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, u_{a,\alpha}^j$$

holds where we denote

(2) $$\psi = Q^{-1}\left(\varphi - \alpha \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}(x - a)^j\right), \quad u_{a,\alpha}^j = \langle u, \alpha(x - a)^j \rangle .$$

Clearly $\psi \in C_0^\infty$ and according to the suppositions $Qu = v \in \mathscr{L}$. We also see that the function $U = v/Q$ is determined by the distribution u almost everywhere and at the same time we can write the equation (1) in the form

(1′) $$\langle u, \varphi \rangle = \int UQ\psi \, dx + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, u_{a,\alpha}^j .$$

Function U will be called an (*ordinary*) *component* of the distribution u. It is clear that a function $U(x)$ $(-\infty < x < \infty)$ is a component of a distribution $u \in \mathscr{K}$ if and only if there exists a polynomial Q $(Q(x) \not\equiv 0)$ such that $v = QU \in \mathscr{L}$.

Unfortunately, the numbers $a_{a,\alpha}^j$ depend on the choice of function α and therefore they are of no invariant significance. It is, however, important that the numbers $u_{a,\alpha}^j$ $(j = s, s + 1, \ldots)$ are already defined by the component U:

(3) $$u_{a,\alpha}^j = \langle u, \alpha(x - a)^j \rangle = \int U\alpha(x - a)^j \, dx \quad (j = s, s + 1, \ldots)$$

and generally that the knowledge of the component U and of the numbers $u_{a,\alpha}^j$ $(j = c, c + 1, \ldots;\ c \geqq 0)$ is equivalent to the knowledge of values $\langle u, \varphi \rangle$ for those functions φ for which $\varphi(a) = \ldots = \varphi^{(c-1)}(a) = 0$.

A a rule, for the sake of brevity, the indices $_{a,\alpha}$ will be omitted.

3. Fourier transform. We shall limit ourselves to some brief remarks concerning the notation. By $\hat{u}$ we denote the Fourier transform of the distribution u. This transform is defined for $u = w \in \boldsymbol{L}$ to be

$$\hat{w}(\xi) = \int u(x)\, e^{-i\xi x}\, \mathrm{d}x .$$

In other cases the distributions $\hat{u}$ will be defined on the domain $-\infty < \xi < \infty$ as well. The spaces $\boldsymbol{L}$, $\boldsymbol{PL}$, $\mathscr{L}$, $Q^{-1}\mathscr{L}$, $\mathscr{K}$ lead to the spaces $\boldsymbol{L}^\wedge$, $(\boldsymbol{PL})^\wedge = \boldsymbol{PL}^\wedge, \ldots$ $\ldots, \mathscr{K}^\wedge$. The topology is defined in the usual way for the transform $^\wedge$ to be a homeomorphism. E.g., to the space $\mathscr{K}^\wedge$ there belong such distributions $\hat{u}$ that $Q(-iD)\,\hat{u} = P(-iD)\,\hat{w}$, where P, Q are the corresponding polynomials, $w \in \boldsymbol{L}$, $D = D_\xi$ is the derivative in the distributional sense.

4. Distribution depending on parameter. If, e.g., the distribution $u = u(t) \in \mathscr{K}$ depends on a parameter t, we shall denote its derivatives according to the parameter in the sense of topology of the space $\mathscr{K}$ by $(\mathrm{d}^j/\mathrm{d}t^j)\,(t)$. Analogous notation will be used for other spaces as well. For example,

$$\left(\frac{\mathrm{d}^j}{\mathrm{d}t^j}\, u(t)\right)^\wedge = \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, \hat{u}(t) \quad \text{where we denote} \quad \hat{u}(t) = (u(t))^\wedge .$$

Further, we shall need the rules

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\, p\, u(t) = p\, \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, u(t), \quad \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, \varphi\, u(t) = \varphi\, \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, u(t),$$

$$\left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, u(t), \varphi \right\rangle = \langle u(t), \varphi \rangle^{(j)},$$

where p is a polynomial, $\varphi \in C_0^\infty$, $^{(j)}$ is the derivative in the elementary sense.

5. Conclusions from inductive limit topology. Let the distribution $u = u(t) \in \mathscr{K}$ $(a \leqq t \leqq b)$ depend continuously on the parameter t. The interval $a \leqq t \leqq b$ is compact, therefore there exist polynomials P, Q such that $u(t) \in Q^{-1}\mathscr{L}$, $Q\, u(t) \in \boldsymbol{PL}$ $(a \leqq t \leqq b)$. At the same time the function $Q\, u(t)$ depends continuously on the parameter t in the space $\boldsymbol{PL}$.

The preceding conclusion holds to a certain extent also for the noncompact intervals. Let us say that the distribution $u(t)$ $(a \leqq t < \infty)$ is *tempered* if there exists N $(N \geqq 0)$ such that

$$\lim_{t \to \infty} u(t)\, t^{-N} = 0 \tag{4}$$

in the sense of the corresponding topology. We can see that if the distribution $u(t) \in \mathscr{K}$

$(a \leqq t < \infty)$ is tempered and depends continuously on the parameter t, then again $u(t) \in Q^{-1}\mathscr{L}$, $Q\, u(t) \in PL$ $(a \leqq t < \infty)$ for the appropriate polynomials P, Q. At the same time the function $Q\, u(t)$ is tempered in the space PL.

6. Equations in the spaces $\mathscr{K}$, $\mathscr{K}^\wedge$. We shall deal with the conditions

$$(5)^\wedge \qquad p_n(-iD) \frac{\mathrm{d}^n}{\mathrm{d}t^n} \hat{u}(t) + \ldots + p_0(-iD)\, \hat{u}(t) = 0\,,$$

$$\hat{u}(t) \in \mathscr{K}^\wedge \quad (0 \leqq t < \infty)\,, \quad \hat{u}(t) \text{ is tempered}.$$

Moreover, we suppose that $p_n(x), \ldots, p_0(x)$ $(n \geqq 0)$ are given polynomials, $p_n(x) \not\equiv 0$, $D = D_\xi$ is the derivative in the sense of distribution theory. By Fourier transform we get an equivalent problem

$$(5) \qquad p_n(x) \frac{\mathrm{d}^n}{\mathrm{d}t^n} u(t) + \ldots + p_0(x)\, u(t) = 0$$

$$u(t) \in \mathscr{K} \quad (0 \leqq t < \infty)\,, \quad u(t) \text{ is tempered}.$$

7. Ordinary differential equation with a parameter. Let us consider the conditions

$$(6) \quad p_n(x)\, U^{(n)} + \ldots + p_0(x)\, U = 0\,, \quad U = U(x, t) \quad (-\infty < x < \infty, 0 \leqq t < \infty)\,,$$

$$\text{there exists} \quad N \;(N \geqq 0) \quad \text{such that} \quad \lim_{t \to \infty} U(x, t)\, t^{-N} = 0\,.$$

The index ${}^{(j)}$ means the derivative according to the parameter t, both the derivative and the limit being understood in the elementary sense of the classical analysis. Thus we have an ordinary linear differential equation the coefficients of which are constants dependent on the parameter x. In the following lemmas we imagine that this parameter is fixed. The function U will interest us except sets of measure 0 with regard to the parameter x.

8. Main problem. The aim of this paper is to prove that if (5) holds for the distribution $u(t)$, then its component U fulfils (6) provided that we adapt it conveniently on a set of measure 0. Conversely, we wish to prove that if $U(x, t)$ is the component of a distribution $u(t) \in \mathscr{K}$ dependent on the parameter t and (6) holds at the same time, then U is the component of a distribution $u(t)$ for which (5) holds.

9. General lemma. *Let $\mathscr{M}$ be a compact topological space. Let $\mathscr{V}$, $\mathscr{W}$ be normed vector space where the space $\mathscr{V}$ has a finite dimension. Let T_Z $(Z \in \mathscr{M})$ be a linear transformation of the space $\mathscr{V}$ into the space $\mathscr{W}$ such that every vector $T_Z Y$ $(Z \in \mathscr{M}$, $Y \in \mathscr{V})$ depends continuously on the point Z. Then there exists a constant M for which $\|T_Z\| \leqq M$ $(Z \in \mathscr{M})$.*

This lemma is evident.

10. Lemma, *Let* $z_1, \ldots, z_n$ *be the roots of the polynomial* $p_n z^n + \ldots + p_0$ $(n \geqq 0, p_n \neq 0)$. *Then* $|z_j| \leqq C = 1 + |p_{n-1}/p_n| + \cdots + |p_0/p_n|$ $(j = 1, \ldots, n)$.

This lemma is well known.

11. Lemma. *There exists a constant* M_n *such that for every solution* $y(t)$ $(-\infty < t < \infty)$ *of the differential equation* $p_n y^{(n)} + \ldots + p_0 y = 0$ *the inequalities* $|y^{(j)}(t)| \leqq M_n C^{n-1+j}(1 + |t|^{n-1})\, e^{|Ct|}(|y(0)| + \ldots + |y^{(n-1)}(0)|)(-\infty < t < \infty, j = 0, 1, \ldots)$ *hold where* C *is the constant from* **10.**

Proof. Let $\mathscr{V}$ be the vector space of all n-tuples $Y = (y_0, \ldots, y_{n-1})$ with the norm $\|Y\|_{\mathscr{V}} = |y_0| + \ldots + |y_{n-1}|$. For every vector $Z = (z_1, \ldots, z_n)$ let $T_Z Y = y$ be the solution of the equation $y^{(n)} - \sigma_1 y^{(n-1)} + \ldots + (-1)^n \sigma_n y = 0$ $(\sigma_1 = z_1 + \ldots + z_n, \ldots, \sigma_n = z_1 \ldots z_n)$ determined by the conditions $y(0) = y_0, \ldots, y^{(n-1)}(0) = y_{n-1}$. It is easily verified that

$$(7) \quad T_Z(y_0, \ldots, y_{n-1})(t/c) \equiv T_{Z/c}(y_0, y_1/c, \ldots, y_{n-1}/c^{n-1})(t) \quad (-\infty < t < \infty).$$

Let $\mathscr{W}$ be the vector space of all functions $w(t)$ $(-\infty < t < \infty)$ such that $\|w\|_{\mathscr{W}} = \sup |w(t)|\,(1 + |t|^{-n+1})\, e^{-|t|} < \infty$. (The continuous dependence of the vector $T_Z Y$ on the vector Z follows quite simply from the fact that the space of all vectors of the form $T_Z Y$ is finite dimensional and therefore any two norms are equivalent in it. It suffices to take for one of these norms the usual norm "sup" on a nonvoid interval (for which the continuous dependence is well known), for the other the norm $\|\cdot\|_{\mathscr{W}}$.) Let $\mathscr{M}$ be the set of those Z for which $|z_1| \leqq 1, \ldots, |z_n| \leqq 1$. According to **9** there exists a constant M_n for which $\|T_Z\| \leqq M_n$ $(Z \in \mathscr{M})$.

If the polynomial $p_n z^n + \ldots + p_0 z$ has the roots $z_1, \ldots, z_n$, then evidently $y(t) \equiv T_Z(y(0), \ldots, y^{(n-1)}(0))(t)$. Let C be the constant from **10.** Then $Z/C \in \mathscr{M}$ and therefore according to (7) the inequality

$$\sup |y(t)|\,(1 + |Ct|^{-n+1})\, e^{-|Ct|} = \|y(t/C)\|_{\mathscr{W}} =$$
$$= \|T_{Z/C}(y(0), \ldots, y^{(n-1)}(0)/C^{n-1})(t)\|_{\mathscr{W}} \leqq M_n \|(y(0), \ldots, y^{(n-1)}(0)/C^{n-1})\|_{\mathscr{V}} \leqq$$
$$\leqq M_n(|y(0)| + \ldots + |y^{(n-1)}(0)|)$$

holds. This is, however, the required inequality for the case $j = 0$. The other cases $j = 1, 2, \ldots$ are already easy consequences of the preceding one.

12. Lemma. *Let the polynomial* $p_n z^n + \ldots + p_0$ $(n \geqq 0, p_n \neq 0)$ *have the roots* $z_1, \ldots, z_n$ *such that*

$$(8) \qquad \operatorname{Re} z_1 \leqq \ldots \leqq \operatorname{Re} z_m \leqq 0 < \operatorname{Re} z_{m+1} \leqq \ldots \leqq \operatorname{Re} z_n$$

for some m $(0 \leqq m \leqq n)$. *Then for every number* $y_0, \ldots, y_{m-1}$ *there exists exactly one solution* $y(t)$ $(-\infty < t < \infty)$ *of the differential equation* $p_n y^{(n)} + \ldots + p_0 y = 0$

for which $y(0) = y_0, \ldots, y^{(m-1)}(0) = y_{m-1}$ *and such that for every positive* ε *it is* $\lim_{t\to\infty} y(t)\, e^{-\varepsilon t} = 0$.

Proof. Let the curve $\mathscr{C}$ circle the roots $z_1, \ldots, z_n$ in the plane of the complex variable z. It is well-known that the function

$$y(t) \equiv \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{y_0(p_n z^{n-1} + \ldots + p_1) + y_1(p_n z^{n-2} + \ldots + p_2) + \ldots + p_n}{p_n z^n + \ldots + p_0}\, e^{tz}\, dz$$

is the only solution of the equation $p_n y^{(n)} + \ldots + p_0 y = 0$ for which $y(0) = y_0, \ldots$ $\ldots, y^{(n-1)}(0) = y_{n-1}$.

The condition with the limit holds if and only if $z_{m+1}, \ldots, z_n$ are regular points of the function after the symbol of integral. It occurs if and only if there are suitable constants $a_{m-1}, \ldots, a_0$, $b_0 = a_{m-1}$, $b_1, \ldots, b_m$ for which

$$\frac{y_0(p_n z^{n-1} + \ldots + p_1) + \ldots + p_n}{p_n z^n + \ldots + p_0} = \frac{a_{m-1} z^{m-1} + \ldots + a_0}{(z - z_1)\ldots(z - z_m)} =$$

$$= \frac{b_0(z^{m-1} - \sigma'_1 z^{m-2} + \ldots + (-1)^{m-1}\sigma'_{m-2}) + b_1(z^{m-2} - \sigma'_1 z^{m-3} + \ldots + (-1)^{m-2}\sigma'_{m-1} + \ldots + b_{m-1}}{(z - z_1)\ldots(z - z_m)}$$

holds, where we denote $\sigma'_1 = z_1 + \ldots + z_m, \ldots, \sigma'_m = z_1 \ldots z_m$. We see that

$$(9) \qquad y(t) \equiv \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{b_0(z^{m-1} + \ldots + \sigma'_{m-1}) + \ldots + b_{m-1}}{(z - z_1)\ldots(z - z_m)}\, e^{tz}\, dz \quad (-\infty < t < \infty),$$

at the same time it is clear that

$$(9') \qquad b_0 = y(0) = y_0, \ldots, b_{m-1} = y^{(m-1)}(0) = y_{m-1}.$$

Hence the existence and uniqueness follows.

Remark. From equation (9) it follows that the function $y(t)$ is a solution of the equation

$$(\text{'} - z_1)\ldots(\text{'} - z_m)\, y = y^{(m)} - \sigma'_m y^{(m-1)} + \ldots + (-1)^m \sigma'_m y = 0.$$

13. Lemma. *Under suppositions* **12** *there exists a constant* M_n^m *such that the inequalities* $|y^{(j)}(t)| \leqq M_n^m C^{m-1+j}(1 + |t|^{m-1})(|y(0)| + \ldots + |y^{(m-1)}(0)|)$ $(0 \leqq t < \infty, j = 0, 1, \ldots)$ *hold for the corresponding solutions.*

Proof. Let $\mathscr{V}$ be the vector space of m-tuples $Y = (y_0, \ldots, y_{m-1})$ with the norm $\|Y\|_{\mathscr{V}} = |y_0| + \ldots + |y_{m-1}|$. For every vector $Z = (z_1, \ldots, z_n)$ such that (8) holds, let $T_Z Y = y$ be the solution of the differential equation $y^{(n)} + \sigma_1 y^{(n-1)} + \ldots + \sigma_n y =$ $= 0$ determined by conditions $y(0) = y_0, \ldots, y^{(m-1)}(0) = y_{m-1}$, $\lim_{t\to\infty} y(t)\, e^{-\varepsilon t} = 0$

($\varepsilon > 0$). Let $\mathscr{W}$ be the vector space of the functions $w(t)$ $(0 \leqq t < \infty)$ such that $\|w\|_{\mathscr{W}} = \sup |w(t)| (1 + |t|^{-m+1}) < \infty$. Let $\mathscr{M}$ be the set of such vectors $Z = (z_1, \ldots, z_n)$ that (8) holds, $|z_1| + \ldots + |z_n| \leqq 1$. By (9), (9′) the vector $T_Z Y \in \mathscr{W}$ depends continuously on $Z \in \mathscr{M}$. The reasoning is then analogous to that of **11.**

14. Lemma. *Let $F(t)$ $(0 \leqq t < \infty)$ be a continuous function such that $\lim_{t\to\infty} F(t) \,.\, t^{-N'} = 0$ for a suitable positive N'. There exists at least one function $y(t)$ $(0 \leqq t < \infty)$ for which*

$$(10) \qquad p_n y^{(n)} + \ldots + p_0 y = F\,, \quad \lim_{t\to\infty} y(t)\, t^{-N} = 0 \quad \text{for a suitable positive } N\,.$$

Proof. Let us take the function $\tilde{y}(t)$ $(-\infty < t < \infty)$ for which

$$p_n \tilde{y}^{(n)} + \ldots + p_0 \tilde{y} = 0\,, \quad \tilde{y}(0) = \ldots = \tilde{y}^{(n-2)}(0) = 0\,, \quad \tilde{y}^{(n-1)}(0) = 1\,.$$

It is possible to write $\tilde{y} = \tilde{y}_+ + \tilde{y}_-$ where $p_n \tilde{y}_\pm^{(n)} + \ldots + p_0 \tilde{y}_\pm = 0$ holds again and at the same time $\tilde{y}_+(t)$ $(-\infty < t < \infty)$ is a function such that $\lim_{t\to\infty} \tilde{y}_+(t)\, t^{-n} = 0$ and $\tilde{y}_-(t)$ $(-\infty < t \leqq 0)$ is a bounded function.

The function

$$(11) \qquad y(t) = \int_0^t \tilde{y}_+(t - \tau)\, F(\tau)\, d\tau - \int_t^\infty \tilde{y}_-(t - \tau)\, F(\tau)\, d\tau \quad (0 \leqq t < \infty)$$

is sought. It follows from the obvious fact that there is

$$y(t) \equiv \int_0^t y(t - \tau)\, F(\tau)\, d\tau - \int_0^\infty y_-(t - \tau)\, F(\tau)\, d\tau$$

where the first summand is a solution of the non-homogeneous equation and the second is a solution of the corresponding homogeneous one.

15. Lemma. *Under the suppositions* **12, 14** *there exists exactly one function $y(t)$ $(0 \leqq t < \infty)$ such that (10) holds, $y(0) = y_0, \ldots, y^{(m-1)}(0) = y_{m-1}$.*

This is an easy consequence of **13, 14.**

16. Lemma. *Let (5) hold. There exists a function $y(x, t)$ such that for every N $(N = 0, 1, \ldots)$ there exist polynomials P, Q $(Q(x) \not\equiv 0)$ for which $(d^j/dt^j)\, Q\, u(t) = y^{(j)}(\cdot, t) = (d^j/dt^j)\, y(\cdot, t) \in PL$ $(0 \leqq t < \infty,\ j = 0, 1, \ldots, N)$.*

Proof. It is convenient to choose the number N sufficiently large. Then let $N > n$. Further choose the constant N'. By **5** there exist polynomials P, Q $(Q(x) \not\equiv 0)$ such that $(d^j/dt^j)\, Q\, u(t) \in PL$ $(0 \leqq t \leqq N',\ j = 1, \ldots, N)$, $Qu(t) \in PL$ $(0 \leqq t < \infty)$. Let us denote $Q\, u(t) = v(\cdot, t)$ and let $y(x, t)$ be the function defined by the conditions

$y(x, 0) \equiv v(x, 0), \ldots, y^{(n-1)} x, 0) \equiv (\mathrm{d}^{n-1}/\mathrm{d}t^{n-1})\, v(x, 0),$

$$p_n(x)\, y^{(n)} + \ldots + p_0(x)\, y \equiv 0\,. \tag{12}$$

We suppose that x is a parameter, $^{(j)}$ is the derivative according to the variable t. (The function y is determined only for almost all values of the parameter x but the sets of measure 0 are not essential.)

First we shall prove that $y^{(j)}(\cdot, t) = (\mathrm{d}^j/\mathrm{d}t^j)\, v(\cdot, t)$ $(0 \leqq t \leqq N', j = 0)$.

Let Ω be an arbitrary compact interval where

$$(1 + |p_{n-1}(x)/p_n(x)| + \ldots + |p_0(x)/p_n(x)|) = C(x) \leqq \text{constant}\,.$$

From the inequalities in **13** and from the Lebesque theorem on majorant convergence it follows that for an arbitrary polynomial p,

$$\int_\Omega p y^{(j)}\,\mathrm{d}x = \left(\int_\Omega p y\,\mathrm{d}x\right)^{(j)} \quad (-\infty < t < \infty, j = 0, 1, \ldots, n)\,.$$

Therefore

$$\left(\int_\Omega p_n y\,\mathrm{d}x\right)^{(n)} + \ldots + \int_\Omega p_0 y\,\mathrm{d}x \equiv 0 \quad (-\infty < t < \infty)\,,$$

$$\left(\int_\Omega p y\,\mathrm{d}x\right)^{(j)} = \int_\Omega p \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, v\,\mathrm{d}x = \left(\int_\Omega p v\,\mathrm{d}x\right)^{(j)} \quad (t = 0, j = 0, \ldots, n-1)\,.$$

From these equations together with the equation (5) multiplied by the polynomial Q it follows that the function $\Delta = y - v$ satisfies the relations

$$\left(\int_\Omega p_n \Delta\,\mathrm{d}x\right)^{(n)} + \ldots + \int_\Omega p_0 \Delta\,\mathrm{d}x = 0 \quad (0 \leqq t \leqq N')\,,$$

$$\left(\int_\Omega p \Delta\,\mathrm{d}x\right)^{(j)} = 0 \quad (t = 0, j = 0, \ldots, n-1)\,.$$

Integration yields

$$\int_\Omega p_n(x)\, \Delta(x, t)\,\mathrm{d}x + \int_0^t \int_\Omega p_{n-1}(x)\, \Delta(x, t_1)\,\mathrm{d}x\,\mathrm{d}t_1 + \ldots$$

$$\ldots + \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-2}} \int_\Omega p_0(x)\, \Delta(x, t_{n-1})\,\mathrm{d}x\,\mathrm{d}t_{n-1} \ldots \mathrm{d}t_1 \equiv 0 \quad (0 \leqq t \leqq N')\,.$$

This identity holds if we take instead of the interval Ω its arbitrary subset $\Theta \subset \Omega$. Let us fix t and let Θ be a subset such that the functions $\operatorname{Re} p_n\Delta$, $\operatorname{Im} p_n\Delta$ of the variable x do not change the sign on it. Then

$$\int_\Theta |p_n\Delta| \, dx \leqq \operatorname{Re}\int_\Theta |p_n\Delta| \, dx + \operatorname{Im}\int_\Theta |p_n\Delta| \, dx =$$

$$= \left|\operatorname{Re}\int_\Theta p_n\Delta \, dx\right| + \left|\operatorname{Im}\int_\Theta p_n\Delta \, dx\right| \leqq 2\left|\int_\Theta p_n\Delta \, dx\right| \leqq$$

$$\leqq 2\left(\int_0^t\int_\Theta |p_{n-1}\Delta| \, dx \, dt_1 + \ldots + \int_0^t\int_0^{t_1}\ldots\int_0^{t_{n-2}}\int_\Omega |p_0\Delta| \, dx \, dt_{n-1}\ldots dt_1\right).$$

Since the interval Ω can be expressed as a union of four subsets Θ of the above mentioned type, it must be

$$\int_\Omega |p_n\Delta| \, dx \leqq 2\left(\int_0^t\int_\Omega |p_{n-1}\Delta| \, dx \, dt_1 + \ldots + \int_0^t\ldots\int_\Omega |p_0\Delta| \, dx \ldots dt_1 \leqq\right.$$

$$\left.\leqq 2 \text{ const.} \left(\int_0^t\int_\Omega |p_n\Delta| \, dx \, dt_1 + \ldots + \int_0^t\ldots\int_\Omega |p_n\Delta| \, dx \ldots dt_1\right).$$

This inequality holds for every t $(0 \leqq t \leqq N')$ and hence it easily follows that $\int_\Omega |p_n\Delta| \, dx = 0$, therefore $p_n(x)\,\Delta(x, t) \equiv 0$ $(x \in \Omega)$ for every t $(0 \leqq t \leqq N')$ and for almost all x and even, with regard to the arbitrariness of the interval Ω and the constant N', $p_n\Delta \equiv 0$ almost everywhere.

Thus we proved that $y = v + \Delta \equiv v$, therefore

$$y^{(j)} = \frac{d^j}{dt^j} v \quad (0 \leqq t < \infty, j = 0).$$

The case when $j = 1, 2, \ldots$ can be investigated in an analogous way by means of relations (5), (12) derived according to t. At the same time it is necessary to consider that our functions y, v fulfil $y^{(j)}(x, 0) \equiv (d^j/dt^j)\, v(x, 0)$ $(j = 0, 1, \ldots)$.

17. Theorem. *Let* (5) *hold. Then for the corresponding component* (*suitably modified on a set of measure* 0) (6) *holds.*

Proof. In **16** choose $N > n$. From the equation $(d^j/dt^j)\, Q\, u(t) = y^{(j)}(\cdot, t)$ $(j = 0)$ it follows that we can choose $U(x, t) = y(x, t)/Q$. (The roots of the polynomial Q are inessential.) From (12) it follows that $p_n y^{(n)}/Q + \ldots + p_0 y/Q \equiv 0$, which is the first one of the equations (6).

According to **5**, (5) the polynomials P, Q $(Q(x) \not\equiv 0)$ exist such that $\lim_{t\to\infty} \|Q\, u(t)\, t^{-N}\|_P = 0$ for a suitable N. We can suppose P, Q, N to be the same as the corresponding items from **16**. Then $Q\, u(t) = y(\cdot, t)$, therefore $\lim_{t\to\infty} \|y(x, t)\, t^{-N}\|_P = 0$. By the Riesz theorem it is possible to choose from every sequence $t_1, t_2, \ldots, \to\infty$ a subsequence $t_{n_1}, t_{n_2}, \ldots, \to\infty$ such that for almost all x, $\lim_{t\to\infty} y(x, t_{n_j})\, t_{n_j}^{-N} \equiv 0$. The function y is, however, a solution of the differential equation (12), therefore it must be, moreover, $\lim_{t\to\infty} y(x, t)\, t^{-N} \equiv 0$ for almost all x. The sets of measure 0 are, however, inessential.

18. Analysis of equations (5). Let (5) hold. Let Q be a polynomial from **16**. If a, α, φ have an analogous meaning as in **2**, then the relation corresponding to (1):

$$\langle u(t), \varphi\rangle = \langle Q\, u(t), \psi\rangle + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, u^j(t) \tag{13}$$

holds where $\psi \in C_0^\infty$, $u^j(t) = \langle u(t), \alpha(x-a)^j\rangle$. The function $Q\,u(t) = y(\cdot, t)$ satisfies (12) where instead of derivatives ${}^{(j)}$ we can write $\mathrm{d}^j/\mathrm{d}t^j$ therefore according to the rules from **4** we have

$$0 = \left\langle p_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} u(t) + \ldots + p_0\, u(t), \varphi \right\rangle = \left\langle p_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} y(\cdot, t) + \ldots + p_0\, y(\cdot, t), \varphi \right\rangle +$$
$$+ \sum_{j=0}^{s-1} \frac{\varphi^j(a)}{j!} (p_n\, u^{j,(n)}(t) + \ldots + p_0\, u^j(t)) = \sum_{j=0}^{s-1} \frac{\varphi^j(a)}{j!} \sum_{k=0}^{n} \sum_{l} p_k^l\, u^{j+l,(k)}(t)\,.$$

where we denote

$$p_k(x) = \sum_l p_k^l (x-a)^l \quad (k = 0, \ldots, n)\,.$$

Since the function φ is in general arbitrary, the equation

$$\sum_{k=0}^{n} \sum_{l} p_k^l\, u^{j+l,(k)}(t) \equiv 0 \quad (j = s-1, \ldots, 0)\,. \tag{14}$$

must hold.

It is also evident that

(15) there exists N $(N \geqq 0)$ for which $\lim_{t\to\infty} u^j(t)\, t^{-N} = 0 \quad (j = 0, 1, \ldots)\,.$

19. Definition. Let Ω_∞ be the set of those numbers $x = a$ for which $p_n(a) = \ldots = p_0(a) = 0$. For $x \notin \Omega_\infty$ let $z_1(x), \ldots, z_r(x)$ $(r = r(x),\ 0 \leqq r(x) \leqq n)$ be all roots of the polynomial $p_n(x)\, z^n + \ldots + p_0(x)$ with the corresponding multiplicity. For the sake of definiteness, let $\operatorname{Re} z_1(x) \leqq \ldots \leqq \operatorname{Re} z_m(x) \leqq 0 < \operatorname{Re} z_{m+1}(x) \leqq \ldots \leqq \operatorname{Re} z_r(x)$ $(m = m(x),\ 0 \leqq m(x) \leqq r(x))$. Denote by Ω_c the set of those numbers x for which $m(x) = c$. Further, denote $\Omega^0 = \Omega_0 \cup \ldots \cup \Omega_n$, $\Omega^1 = \Omega_1 \cup \ldots \cup \Omega_n, \ldots, \Omega^n = \Omega_n$.

20. Solution of equations (6). Choose the functions $G_0(x), \ldots, G_{n-1}(x)$ $(-\infty < x < \infty)$. According to **15** the function U satisfying the relations (6) is uniquely determined by the conditions

$$(16) \qquad U(x, 0) \equiv G_0(x) \, (x \in \Omega^1), \ldots, U^{(n-1)}(x, 0) \equiv G_{n-1}(x) \quad (x \in \Omega^n)$$

From the inequalities in **13** it follows that if the functions $G_0, \ldots, G_{n-1}$ are components of some distributions, then for every t $(0 \leqq t < \infty)$ this function $U(x, t)$ is also a component of a distribution of the space $\mathscr{K}$. From the same inequalities it is clear that there exist polynomials P, Q $(Q(x) \not\equiv 0)$ for which

$$(17) \qquad Q \, U^{(j)}(\cdot, t) \in \boldsymbol{PL} \quad (0 \leqq t < \infty, j = 0, \ldots, n)$$

$$(18) \qquad \lim_{t \to \infty} U^{(j)}(x, t) \, t^{-n} = 0 \quad (j = 0, \ldots, n) \,.$$

Now the question is whether the distribution $u(t)$ whose component is $U(x, t)$ can be chosen so that (5) may hold. It sufficies if we solve this problem locally. Let us choose the point $x = a$ and suppose that (1), (1)′ hold or, in more detail, (13). At the same time let us choose the polynomial Q so that (17) holds for a suitable polynomial P.

21. Form of equations (14), (15). The functions $u^s, u^{s+1}, \ldots$ are defined by (3). It is important to realize that the corresponding relations (15) hold not only for $j = s, s + 1, \ldots$, but according to (18) it is, moreover,

$$(19) \qquad \lim_{t \to \infty} u^{s,(j)}(t) \, t^{-n} = \lim_{t \to \infty} u^{s+1,(j)}(t) \, t^{-n} = \ldots = 0 \quad (j = 0, \ldots, n) \,.$$

Now, we have to define the functions $u^{s-1}, \ldots, u^0$. They satisfy the system of equations (14):

$$(14) \qquad \begin{aligned} & p_n^0 u^{s-1,(n)} + \ldots + p_0^0 u^{s-1} = F_{s-1} \,, \\ & p_n^0 u^{s-2,(n)} + \ldots + p_0^0 u^{s-2} + \\ & \qquad + p_n^1 u^{s-1,(n)} + \ldots + p_0^1 u^{s-1,n} = F_{s-2} \,, \\ & \ldots \\ & p_n^0 u^{0,(n)} + \ldots + p_0^0 u^0 + \\ & \qquad + p_n^1 u^{1,(n)} + \ldots + p_0^1 u^1 + \ldots = F_0 \,. \end{aligned}$$

The functions $F_{s-1} = -p_n^1 u^{s,(n)} - \ldots - p_0^1 u^s - \ldots$, $F_0 = -p_n^s u^{s,(n)} - \ldots - p_0^s u^s - \ldots$ which occur there are well-known already. Moreover, let us realize that according to (19),

$$(20) \qquad \lim_{t \to \infty} F_j(t) \, t^{-n} = 0 \quad (j = s - 1, \ldots, 0) \,.$$

22. Solution of equations (14), (15) **if** $a \notin \Omega_\infty$. In this case it is not p_j^0 $(=p_j(a)) \equiv 0$ $(j = 0, \dots, n)$ and from the system (14) the functions $u^{s-1}, \dots, u^0$ can be successively determined in such a manner that (15) holds. In more detail: Let $a \in \Omega_m$. Then according to (20), (15) these functions are uniquely determined by the numbers $u^j(0), \dots, u^{j,(m-1)}(0)$ $(j = s - 1, \dots, 0)$ which can be taken arbitrarily.

23. Solution of equations (14), (15) **if** $a \in \Omega_\infty$. Then $p_n^0 = \dots = p_0^0 = 0$ and the first equation of the system (14) has the form $F_{s-1} = 0$, therefore it is a condition assigned to the functions already known. We prove that it is an identity so that no nontrivial compatibility relations occur.

First of all we shall write the equation (12) in this form:

$$(p_n^1 + p_n^2(x - a) + \dots)\, y^{(n)} + \dots + (p_0^1 + p_0^2(x - a) + \dots)\, y = 0\,.$$

We know that $y(\cdot, t)^{(J)} = (\mathrm{d}^J/\mathrm{d}t^J)\; Q\, u(t)$ where $Q = (x - a)^s\, Q_1$ $(Q_1(a) \neq 0)$. The identity which is to be proved follows therefore easily in this way:

$$0 = \left\langle (p_n^1 + p_n^2(x - a) + \dots)\, y^n + \dots + (p_0^1 + p_0^2(x - a) + \dots)\, y, \frac{\alpha}{Q_1} \right\rangle =$$

$$= \left\langle p_n^1 y^{(n)} + \dots + p_0^1 y, \frac{\alpha}{Q_1} \right\rangle + \left\langle p_n^2 y^{(n)} + \dots + p_0^2 y, \frac{\alpha}{Q_1}(x - a) \right\rangle + \dots =$$

$$= \langle p_n^1 u^{(n)} + \dots + p_0^1 u, \alpha(x - a)^s \rangle + \langle p_n^2 u^{(n)} + \dots + p_0^2 u, \alpha(x - a)^{s+1} \rangle + \dots =$$

$$= p_n^1 u^{s,(n)} + \dots + p_0^1 u^s + p_n^2 u^{s+1,(n)} + \dots + p_0^2 u^{s+1} + \dots = -F_{s-1}\,.$$

It can be proved quite analogously that if $p_n^1 = \dots = p_0^1 = 0$ as well, then the second equation of the system (14) has the form $F_{s-2} = 0$ and it is again the identity, etc. Thus we can see that there are no compatibility relations of the system (14) and if the equations with the indices $j = s - 1, \dots, j = s - c$ are identically fulfilled, then only the functions $u^{s-1}, \dots, u^{s-c}$ occur in this system. For the functions $u^{s-c-1}, \dots, u^0$ we have only the conditions (15).

24. Summary. An arbitrary distribution $u(t)$ for which (5) holds can be obtained in this manner: Functions $G_0(x), \dots, G_{n-1}(x)$ are chosen which are components of some distributions from the space $\mathscr{K}$. Then the component $U(x, t)$ of the distribution $u(t)$ is defined by the conditions (6), (16). A polynomial Q $(Q(x) \not\equiv 0)$ is chosen so that (17) holds. In the neighbourhood of the point $x = a$ the sought distribution is defined by the equations of the type (1), (1)′, (13), where the functions $u^j(t)$ $(j = s - 1, \dots, 0)$, hitherto unknown, occur. They can be computed from the equations (14), (15) where the functions $F_{s-1}, \dots, F_0$ are already uniquely defined by the known component U. In more detail: Let $a \notin \Omega_\infty$ and for the sake of definiteness, $a \in \Omega_m$. Then the functions $u^{s-1}, \dots, u^0$ are uniquely determined according to **15** if we choose arbitrarily the numbers $u^j(0), \dots, u^{j,(m-1)}(0)$ $(j = s - 1, \dots, 0)$.

On the contrary, let $a \in \Omega_\infty$ and let c be the largest number such that $p_j^0 = \ldots = p_j^{c-1} \equiv 0$ $(j = 0, \ldots, n)$. From the equations (14), (15) only the functions $u^{s-1}(t), \ldots, u^{s-c}(t)$ can be computed (initial conditions at the point $t = 0$ must be still chosen) but the functions $u^{s-c-1}(t), \ldots, u^0(t)$ are arbitrary in the main.

Corollary. *If the component $U(x, t)$ of a distribution satisfies the conditions* (6), *then the corresponding distribution $u(t)$ can be chosen in such a manner that* (5) *holds.*

Corollary. *If* (5) *holds, then for every N $(N = 0, 1, \ldots)$ there exist polynomials P, Q $(Q(x) \not\equiv 0)$ such that $Q(\mathrm{d}^j/\mathrm{d}t^j)\, u(t) \in PL$ $(j = 0, \ldots, N; 0 \leqq t < \infty)$. This derivative is to be understood as the derivative in the space PL.*

In **2** we have already noted that the functions $u^j(t)$ have no invariant significance by themselves but they are very closely connected with the restriction of the form $\langle u, \varphi \rangle$ to certain invariantly defined subspaces of the space C_0^∞. It is advantageous to understand our results exactly in this manner which is, however, a little clumsy and therefore we shall not use it explicitly.

25. Definition. A point is called an *ordinary point of the distribution u*, if it has such a neighbourhood that in this neighbourhood the distribution u is equal to its component U. We say that the distribution $u \in \mathscr{K}$ (with the component U) is equal to zero on a set Ω if $U(x) \equiv 0$ $(x \in \Omega)$ and if all points of the set Ω are ordinary points of the distribution u. If Ω is an open set, then this definition agrees with the usual one.

26. Theorem. *Let Ω_∞ be an empty set. Let us denote by Ω the union of boundaries of all sets $\Omega_0, \ldots, \Omega_n$. Let $p_n(x) \neq 0$ $(x \in \Omega)$, let $g_0, \ldots, g_{n-1}$ be distributions of the space $\mathscr{K}$ such that all points of the set Ω are their ordinary points. Then there exists exactly one distribution $u(t)$ having these properties: It satisfies the equation* (5), *every point $x \in \Omega$ is an ordinary point of the distribution $u(t)$ $(0 \leqq t < \infty)$, every distribution $(\mathrm{d}^k/\mathrm{d}t^k)\, u(0) - g_k$ $(k = 0, \ldots, n-1)$ is equal to zero on the set Ω^{k+1}.*

Proof. Let us choose the point $x = 0$ and suppose that in its neighbourhood the relations (1)′ hold:

$$\langle g_k, \varphi \rangle = \int G_k Q \psi \,\mathrm{d}x + \sum_{j=0}^{s-1} \frac{\varphi^{(j)}(a)}{j!}\, g_k^j \quad (k = 0, \ldots, n-1)\,.$$

The component U is defined by the conditions (6), (16). In the neighbourhood of the point $a \in \Omega$, according to the inequalities 13, the function $U(\cdot, t)$ $(0 \leqq t < \infty)$ is integrable and therefore we can take $s = s(a) = 0$, $u(t) \equiv U(\cdot, t)$. Let $a \notin \Omega$ and for the sake of definiteness let $a \in \Omega_m$. Then $a \in \Omega^1 \cup \ldots \cup \Omega^m$ and then the equations

$$u^{j,(k)}(0) = g_k^j \quad (k = 0, \ldots, m-1; j = 0, \ldots, s-1)$$

must hold.

It is clear that the functions $u^0(t), \ldots, u^{s-1}(t)$ and thus also the distributions $u(t)$ are uniquely determined by these conditions and equations (14). At the same time it is necessary to observe that the functions $F_{s-1}, \ldots, F_0$ can be computed by means of the relations (3).

27. Example. Let us deal with the equation $-iD(\mathrm{d}/\mathrm{d}t)\,\hat{u} + c\hat{u} = 0$, where $n = 1$, $p_1(x) = x$, $p_0(x) = c$. $x(\mathrm{d}/\mathrm{d}t)\,u + cu = 0$ holds for the function u, $xU' + cU = 0$ holds for the component U and the properties of these equations depend on the behaviour of the polynomial $xz + c$.

Let $c = 0$. Then $0 \in \Omega_\infty$ and for $x \neq 0$ we have $x \in \Omega_1 = \Omega^1$, $r(x) = m(x) = 1$. Let us choose the distribution g_0. From the equations (6), (16) it follows $U \equiv G_0$. For the polynomial Q we can take an arbitrary polynomial such that $Qg_0 \in \mathscr{L}$. From the equations (14) it follows that for $a \neq 0$ it is $u_a^j = g_{0,a,\alpha}^j$ and for $a = 0$ we have the solution $u_{\alpha,a}^{s-1} = c_{s-1}, \ldots, u_{a,\alpha}^1 = c_1$, $u_{a,\alpha}^0 = f(t)$. In general

$$u = g_0 + \sum_{j=1}^{s-1} c_{s-1} D^j(\delta/j!) + f(t)\,\delta\,, \quad \hat{u} = \hat{g}_0 + \sum_{j=1}^{s-1} c_{s-1}(i\xi)^j/j! + f(t)\,.$$

Let $c \neq 0$. Then $\Omega_\infty = 0$, for $x \leqq 0$ it is $x \in \Omega_0$, $m(x) = r(x) = 1$ and for $x > 0$ it is $x \in \Omega_1$, $m(x) = r(x) = 1$. Clearly $U(x, t) = f(x)\,e^{-ct/x}$ $(x > 0)$, $U(x, t) \equiv 0$ $(x < 0)$. First let us choose $f(x) \equiv 1$. Then the definition equation of the distribution $u(t)$ can be written globally in the form

$$\langle u, \varphi \rangle = \int_0^\infty e^{-ct/x}\,\varphi(x)\,\mathrm{d}x + \sum_{a,j} c_a^j(t)\,\varphi^{(j)}(a)$$

where the sum is finite and ranges over $a > 0$, $j = 0, 1, \ldots$. We have

$$\hat{u} = \langle u, e^{i\xi x} \rangle = \int_0^\infty e^{-ct/x - i\xi x}\,\mathrm{d}x + \sum_{a,j} c_a^j(t)\,(-i\xi)^j\,e^{i\xi a}\,;$$

the integral must be understood in the generalized sense and it could be possible to express it by means of Bessel functions. The functions $c_a^j(t)$ must be chosen in such a way that the sum $\sum_{a,j}$ is a solution of the equation $u_{xt} + cu = 0$. From the general theory it follows that the numbers $c_a^j(0)$ can be chosen arbitrarily. Secondly, let us choose $f(x) = 1/(1 - x)$. The definition formula for the distribution $u(t)$ can be written in the form

$$\langle u, \varphi \rangle = \int_0^\infty e^{-ct/c}(\varphi(x) - \alpha(x)\,\varphi(0))\frac{\mathrm{d}x}{1 - x} + \sum_{a,j} c_a^j(t)\,\varphi^{(j)}(a)$$

(where $\alpha \in C_0^\infty$, $\alpha(0) = 1$) and in the sum the term $c_1^0(t)\,\varphi^{(j)}(1)$ must be given since in our case the corresponding polynomial Q is a multiple of the polynomial $x - 1$. We see that

$$\hat{u} = \int_0^\infty e^{-ct/x}(e^{i\xi x} - \alpha(x))\,\frac{dx}{1 - x} + \sum_{a,j} c_a^j(t)\,(-i\xi)^j\,e^{i\xi a}\,.$$

The functions $c_a^j(t)$ must be still calculated and $c_1^0(t)$ plays a significant role.

It is clear that it could be possible to choose

$$c_1^0(t) = \int_0^\infty e^{-ct/x}\,\alpha(x)\,\frac{dx}{1 - x}, \quad c_1^j(t) \equiv 0 \quad (j = 1, 2, \ldots)\,.$$

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