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Časopis pro pěstování matematiky, roč. 99 (1974), Praha

RECONSTRUCTION OF A TREE FROM CERTAIN MAXIMAL PROPER SUBTREES

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B. MANVEL [4] has proved that a (finite) tree T can be reconstructed up to an isomorphism from its set of non-isomorphic maximal proper subtrees, excluding the cases when either (a) T has exactly four vertices, or (b) T has exactly six vertices and exactly three of them are terminal. In the present paper we shall demonstrate that we can obtain the same result by replacing the set of non-isomorphic maximal proper subtrees by its certain subset. For Manvel's reconstruction of a tree and also for other ways of reconstructing a tree from subgraphs (see [3], [2], and [1]) the notion of a center of a tree is important. In the present reconstruction an important role will be ascribed to certain vertices which we shall call a focus and a pseudofocus.

First we introduce the necessary notions and symbols (for basic notions of theory of graphs, see [5]). By a tree we shall mean a finite tree. Let T be a tree. By a subtree of T we shall mean a connected subgraph of T. By a maximal proper subtree of T we mean a tree which we obtain from T by deleting one of the terminal vertices of T. By $V(T)$, $I(T)$, $C(T)$, $d(T)$ and $r(T)$ we denote its vertex set, its set of terminal vertices, its set of centers, its diameter and its radius, respectively. If u, $v \in V(T)$, then by $d_T(u, v)$ we denote the distance between u and v in T and if $w \in V(T)$, we denote

$$r_T(w) = \max\{d_T(w, c) \mid c \in C(T)\}\,. \tag{1}$$

Moreover, we denote

$$I_j(T) = \{v \in I(T) \mid r_T(v) \leqq j\}\,, \quad \text{for} \quad 1 \leqq j \leqq r(T)\,; \tag{2}$$

$$\bar{I}_j(T) = \{v \in I(T) \mid r_T(v) = j\}\,, \quad \text{for} \quad 1 \leqq j \leqq r(T)\,;$$

$$\gamma(T) = \min\{j \mid \text{either } j = r(T) \text{ or } |I_j(T)| \geqq 3\}\,. \tag{3}$$

We say that a maximal proper subtree T' of the tree T is a γ-subtree of T if there is $u \in I_{\gamma(T)}(T)$ such that we obtain T' from T by deleting the verex u.

If T_0 is a tree, $|V(T_0)| \geqq 2$ and $v \in I(T_0)$, we say that (T_0, v) is a branch or a branch rooted in v. Branches (T_1, v_1) and (T_2, v_2) will be isomorphic if the trees T_1 and T_2 are isomorphic and if there exists an isomorphism between T_1 and T_2 in which v_1 and v_2 correspond to each other. We say that a branch B is a branch of T rooted in v, if $B = (T_0, v)$ where T_0 is a subtree of T such that the only vertex w in T_0 which is joined to v by an edge fulfils the following condition: $u \in V(T_0) - \{v\}$ if and only if $u \in V(T)$ and w lies on the path connecting u and v in T.

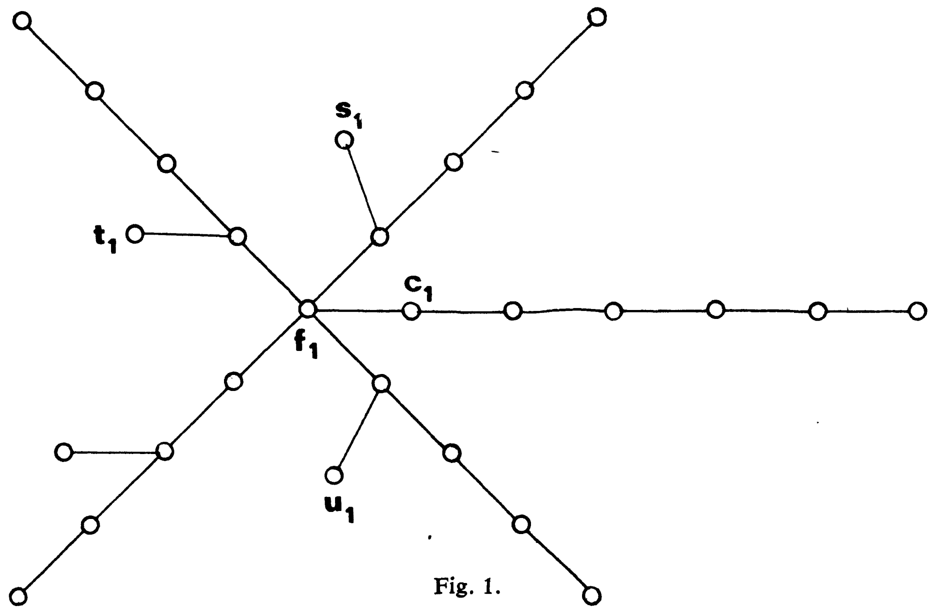

Fig. 1.

Let T be a tree with at least one edge, and if $|I(T)| \geqq 3$, let T' be such a γ-subtree of T that $d(T') = d(T)$. We say that $u \in V(T)$ or $u' \in V(T')$ is a focus of T or a pseudofocus of T', respectively, if the following conditions hold: (i) there are at least two branches of T or T' rooted in u or u', respectively, such that each of them contains at least one vertex from $I_{\gamma(T)}(T)$ or $I_{\gamma(T)}(T')$, respectively; (ii) if $v \in V(T)$ and $r_T(v) < r_T(u)$ or if $v' \in V(T')$ and $r_{T'}(v') < r_{T'}(u')$, then all the vertices of $I_{\gamma(T)}(T)$ or $I_{\gamma(T)}(T')$, respectively, lie in exactly one of the branches of T that are rooted in v or in exactly one of the branches of T' rooted in v', respectively. By $F(T)$ or $P(T')$ we denote the set of foci of T or the set of pseudofoci of T', respectively. It is easy to see that $1 \leqq |F(T)| \leqq 2$, $1 \leqq |P(T')| \leqq 2$, and if $|F(T)| = 2$, then $F(T) = C(T)$ and if $|P(T')| = 2$, then $P(T') = C(T')$. If $|F(T)| = 1$, we denote $f(T) = r_T(w)$, where w is the focus; if $|F(T)| = 2$, we put $f(T) = 1$. If $|P(T')| = 1$, we denote $p(T') = r_{T'}(w')$, where w' is the pseudofocus; if $|P(T')| = 2$, we put $p(T') = 1$.

Example. Let T_i denote the tree in Fig. i, $1 \leqq i \leqq 3$. Then $\gamma(T_1) = \gamma(T_2) = \gamma(T_3) = 3$, $I_3(T_1) = \{s_1, t_1, u_1\}$, $I_3(T_2) = \{s_2, t_2, u_2, v_2, w_2\}$, $I_3(T_3) =$

$= \{s_3, t_3, u_3, v_3, w_3\}$, $C(T_1) = \{c_1\}$, $F(T_1) = \{f_1\}$, $C(T_2) = \{c_2', c_2''\} = F(T_2)$, $C(T_3) = \{c_3', c_3''\}$, $F(T_3) = \{c_3'\}$, and $f(T_1) = f(T_2) = f(T_3) = 1$.

For the remainder of the present paper we shall assume that a tree T is given such that $|V(T)| \geqq 2$, $|V(T)| \neq 4$, and that if $|V(T)| = 6$, then $|I(T)| \neq 3$; by R we denote its set of non-isomorphic γ-subtrees. If R contains trees of different diameters or if it contains a tree with at most two terminal vertices, then $\gamma(T) = r(T)$; thus $I_{\gamma(T)}(T) = I(T)$ and from Theorem 1 in Manvel [4] it follows directly that T can be reconstructed up to an isomorphism from R. This proposition is complemented by the following theorem:

Theorem. *Let the set R not contain any tree T_0 with $|I(T_0)| \leqq 2$, and let all the trees in R have same diameter. Then T can be reconstructed up to an isomorphism from the set R.*

Proof. It is easy to see that $d(T') = d(T)$, $r(T') = r(T)$ and $|C(T')| = |C(T)|$, for every $T' \in R$. First we determine $\gamma(T)$.

(A) Let there be m, $1 \leqq m \leqq r(T)$, and $T_1, T_2 \in R$ such that $|\tilde{I}_m(T_1)| \neq |\tilde{I}_m(T_2)|$ and that for any n, $m < n \leqq r(T)$, and for any $T', T'' \in R$ it holds that $|\tilde{I}_n(T')| = |\tilde{I}_n(T'')|$. Denote $h = \max\{|I_m(T_0)| \mid T_0 \in R\}$. Obviously, $h \geqq 1$. If $h = 1$, then for every γ-vertex of T it holds that $r_T(v) = m + 1$; thus also $\gamma(T) = m + 1$. If $h \geqq 2$, then there is a γ-vertex u such that $r_T(u) = m$ and there is no γ-vertex w such that $r_T(w) > m$; thus $\gamma(T) = m$.

(B) For any $T_1, T_2 \in R$ and for any m, $1 \leqq m \leqq r(T)$, let $|\tilde{I}_m(T_1)| = |\tilde{I}_m(T_2)|$. Then for any γ-vertices v_1 and v_2 of T, $r_T(v_1) = r_T(v_2)$. Thus $|\tilde{I}_{\gamma(T)}(T)| \geqq 3$, and if $\gamma(T) > 1$, then $I_{\gamma(T)-1}(T) = \emptyset$. For any $T_0 \in R$, $|\tilde{I}_{\gamma(T)}(T_0)| \geqq 2$, and if $\gamma(T) > 1$, then $|I_{\gamma(T)-1}(T_0)| \leqq 1$. Thus $\gamma(T) = \min\{j \mid |I_j(T_0)| \geqq 2\}$, for any $T_0 \in R$.

As we know $\gamma(T)$, we also know $I_{\gamma(T)}(T')$ for any $T' \in R$ and thus we easily determine $P(T')$ and $p(T')$, for any $T' \in R$. Obviously, $|F(T)| = 1$ if and only if $|P(T_0)| = 1$, for any $T_0 \in R$; $|F(T)| = 2$ if and only if there is $T' \in R$ such that $|P(T')| = 2$. Thus we have $|F(T)| = \max\{|P(T_0)| \mid T_0 \in R\}$. If $|F(T)| = 2$, then $f(T) = 1$. Let $|F(T)| = 1$. It is easy to see that there exists $T_0 \in R$ such that $p(T_0) \neq f(T)$ if and only if there exists a branch of T rooted in the focus which contains exactly $|I_{\gamma(T)}(T)| - 1$ γ-vertices of T. Thus if for $T_0 \in R$, $p(T_0) \neq f(T)$, then $p(T_0) > f(T)$. Obviously, there is $T' \in R$ such that $p(T') = f(T)$. Thus we have $f(T) = \min\{p(T_0) \mid T_0 \in R\}$. Now we shall demonstrate the remaining part of the reconstruction of the tree T.

(I) Let $|R| = 1$, let T' be the only element of R, $|P(T')| = 1$ and let there be at least two different branches B_1 and B_2 of T' rooted in the pseudofocus such that B_1 and B_2 each have exactly one edge. Then T can be reconstructed by adding an edge to the pseudofocus of T'.

(II) Let case (I) not hold. Consider any $T_0 \in R$. If $|F(T)| = 1$, then by v_0 we denote the vertex of T_0 such that $r_{T_0}(v_0) = f(T)$ and that v_0 lies on the path connecting the pseudofocus with the center if $|C(T_0)| = 1$, or with the nearer center if $|C(T_0)| = 2$; any branch of T_0 rooted in v_0 will be called a fundamental branch of T_0. If $|F(T)| = 2$, we shall describe as fundamental branches of T_0 the two branches of T_0 each of which is rooted in one center of T_0 while also containing the corresponding second center

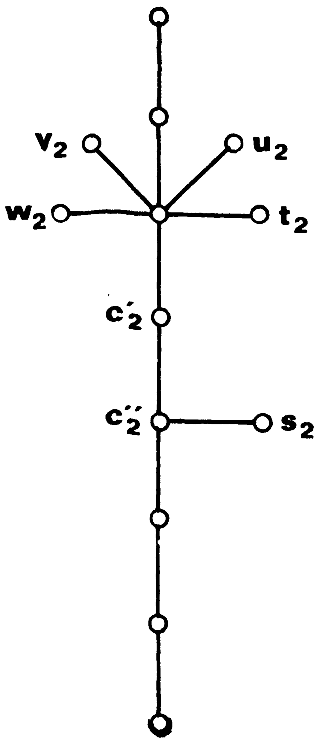

Fig. 2.

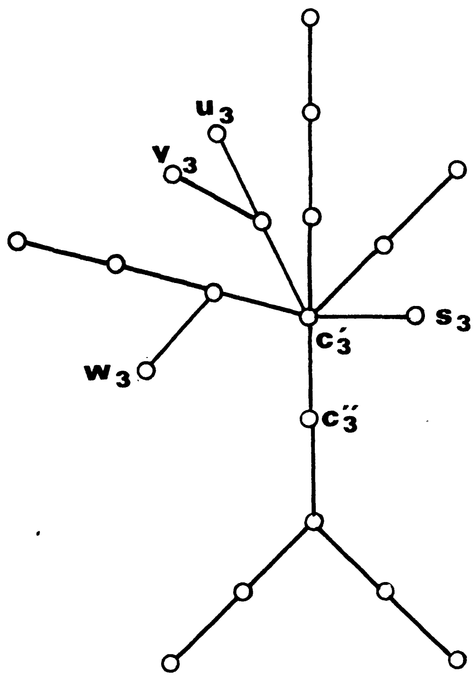

Fig. 3.

If $|F(T)| = 1$, then by a focus branch of T we call every branch of T rooted in the focus; if $|F(T)| = 2$, then we describe as focus branches of T the two branches of T each of which is rooted in one focus of T while also containing the corresponding second focus. It is easy to see that for any focus branch B of T there is $T' \in R$ such that B is isomorphic to a fundamental branch of T'.

By X we denote such a set of non-isomorphic branches that it hqlds (i) if $B \in X$, then there is $T' \in R$ which has a fundamental branch isomorphic to B; (ii) if $T_0 \in R$ and if B_0 is a fundamental branch of T_0, then there is $B \in X$ such that B_0 and B are isomorphic. If $B \in X$, $T' \in R$, then by $g(B, T')$ we shall denote the number of fundamental branches of T' which are isomorphic to B. Moreover, we denote $g(B) = \max\{g(B, T') \mid T' \in R\}$. Let $B_1, B_2 \in X$; we shall write $B_1 \to B_2$ if there is $T_0 \in R$

such that it has a fundamental branch B_0 isomorphic to B_1 and containing a vertex $v \in I_{\gamma(T)}(T_0)$ such that if we delete v from B_0 we obtain a branch isomorphic to B_2.

We say that $B \in X$ is extraordinal if simultaneously (a) there is no $B' \in X$ such that $B' \to B$, (b) there is $B'' \in X$ such that $B \to B''$, and (c) $g(B, T_0) = g(B)$, for any $T_0 \in R$. If $B \in X$ fulfils (a), then it is isomorphic to a focus branch B_0 of T; if moreover B fulfils (b), then B_0 has at least two edges and it contains a γ-vertex. It is easy to see that if B is extraordinal then all focus branches of T which contain any γ-vertex are isomorphic to B; thus X contains at most one extraordinal branch.

By G we denote the directed graph with the vertex set X which is defined by the binary relation $\to$. Obviously, G is acyclic. Every vertex B of G is evaluated by the positive integer $g(B)$. Now, we define a new evaluation $h(B)$, for every $B \in X$, as follows: (i) if B is extraordinal, then $h(B) = g(B) + 1$; (b) if B is not extraordinal and if there is no $B' \in X$ such that both $B' \to B$ and $h(B') \neq 0$, then $h(B) = g(B)$; (c) if B is not extraordinal and if there is $B' \in X$ such that $B' \to B$ and $h(B') \neq 0$, then $h(B) = g(B) - 1$. As G is acyclic, $h(B)$ is uniquely determined for every $B \in X$.

Let $B \in X$. Then B is isomorphic to no focus branch of T if and only if $g(B) = 1$ and there is $B' \in X$ such that $B' \to B$ and B' is isomorphic to a branch of T. B is isomorphic to exactly $n \geqq 1$ focus branches of T if and only if either (a) B is extraordinal, and $g(B) = n - 1$, or (b) B is not extraordinal, $g(B) = n$, and there is no $B' \in X$ such that $B' \to B$ and B' is isomorphic to focus branch of T, or (c) B is not extraordinal, $g(B) = n + 1$ and there is $B' \in X$ such that $B' \to B$ and B' is isomorphic to any focus branch of T. By induction we have the result that every $B \in X$ is isomorphic to exactly $h(B)$ focus branches of T. As every focus branch of T is isomorphic to some $B \in X$ and since we know the number of foci of T, then T can be reconstructed. The case when $|F(T)| = 1$ is obvious. If $|F(T)| = 2$, then T has exactly two focus branches; they have one common edge joining the foci.

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