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ON THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR LINEAR PARABOLIC EQUTAIONS OF HIGHER ORDER

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1. Introduction. In his paper [2] R. K. JUBERG has treated the Dirichlet problem for the homogeneous linear differential equation $D_x^4 u + D_t u = 0$ on the rectangle $(0, 1) \times (0, T\rangle$ with data $u(x, 0) = 0$ for $x \in (0, 1)$, $u(0, t) = a(t)$, $u(1, t) = b(t)$, $D_x u(0, t) = c(t)$, $D_x u(1, t) = d(t)$ for $t \in (0, T\rangle$ $(D_z = \partial/\partial z)$. Reducing this problem to solving a system of Voltera type integral equations with bounded and differentiable kernels, the author proved the existence and uniqueness of solution.

The present paper deals with two modifications of Dirichlet problem for the non-homogeneous parabolic equation $D_x^4 u + D_t u = \varphi(x, t)$ on $(0, 1) \times (0, T\rangle$. We study some properties of the fundamental solution of the operator $D_x^4 u + D_t u$ allowing to determine limits and derivatives of certain parametric integrals. Further, the Green function is constructed and with its help we seek an explicit representation of the solutions of the considered problems. The method introduced below affords an information on the behaviour of solutions at the points $(0, 0)$ and $(1, 0)$. The procedure may be directly applied to the equation $D_x^{2n} u + (-1)^n D_t u = \varphi(x, t)$.

2. The formulation of problems and some notions. Let R^n mean the n-dimensional Euclidean space and let $\bar{A}$ be the closure of $A \subset R^n$. By Ω_0, Ω and Ω_1 we shall denote the Cartesian products $(0, 1) \times (0, T)$, $(0, 1) \times (0, T\rangle$ and $(-\infty, \infty) \times \langle 0, \infty)$ respectively. Let A be an open set of R^2. The set of all functions $v(x, t) \in C_0(\bar{A})$ with continuous derivatives $D_x^m v(x, t)$ and $D_t v(x, t)$ on A, where m is a positive integer, will be denoted by $N_m(A)$.

We consider the following two boundary value problems

$$L(u; x, t) = D_x^4 u + D_t u = \varphi(x, t), \quad (x, t) \in \Omega \tag{1}$$

$$u(x, 0) = g(x), \quad 0 < x < 1 \tag{2}$$

$$\begin{aligned} D_x^{j-1} u(0, t) &= a_j(t), \quad D_x^{j-1} u(1, t) = b_j(t), \quad 0 < t \leqq T \\ D_x^{j+1} u(0, t) &= c_j(t), \quad D_x^{j+1} u(1, t) = d_j(t), \quad 0 < t \leqq T \end{aligned} \tag{3j}$$

for $j = 1, 2$, where $\varphi(x, t)$, $g(x)$, $a_j(t)$, $b_j(t)$, $c_j(t)$ and $d_j(t)$ are real functions of certain classes defined below.

The real functiõn $u(x, t)$ is said to be a solution of problem (1), (2), ($3j$), $j = 1, 2$ if $u \in C_0[\bar{\Omega} - \{(0, 0), (1, 0)\}]$ and $D_x^\nu u \in C_0(\Omega)$ for $\nu = 1, 2, 3$, $D_t u \in C_0(\Omega)$ and $u(x, t)$ satisfies conditions (1), (2), ($3j$).

Let $A \subset R^1$ and $f(x)$ be a real function on A. Let B mean a bounded and clõsed subset of A. If to each such defined B there is a constant $K(B)$ depending only on B such that

$$|f(x) - f(y)| \leqq K(B)\,|x - y|^{\varepsilon + n/4}, \quad n = 0, 2, \quad \varepsilon > 0$$

for every $x, y \in B$ and $0 < \varepsilon + n/4 < 1$, then the function $f(x)$ is called locally $(\varepsilon + n/4)$-Hölder continuous function on A. The set of all such functions is denoted by $S_n(x, A)$. If the function f depends on the parameter λ ($f = f(x; \lambda)$) and if the Hölder constant $K(B)$ does not depend on λ, the function $f(x; \lambda)$ is said to be locally $(\varepsilon + n/4)$-Hölder continuous with respect to x on A uniformly with respect to λ. We denote the set of all such functions by $S_n(x, A; \lambda)$.

Under *the fundamental solution of equation* $L(u; x, t) = 0$ *in* $\bar{\Omega}$ we understand a continuous function $\Gamma(x, t; \xi, \tau)$ for $(x, t; \xi, \tau) \in \bar{\Omega} \times \bar{\Omega}$, $(x, t) \neq (\xi, \tau)$ with the derivatives $D_t\Gamma$, $D_x\Gamma$, $D_x^2\Gamma$, $D_x^3\Gamma$, $D_x^4\Gamma$ such that the integral

$$u(x, t) = \int_0^T d\tau \int_0^1 \Gamma(x, t; \xi, \tau) f(\xi, \tau)\, d\xi$$

is a solution of the equation $L(u; x, t) = f(x, t)$ on Ω for any $f \in C_0(\bar{\Omega}) \cap S_0[x, (0, 1); t]$.

A continuous function $G_j(x, t; \xi, \tau)$ for $(x, t;, \xi, \tau) \in \bar{\Omega} \times \bar{\Omega}$, $t > \tau$, having the first derivative respect to t and the derivatives with respect to x up to the 4-th order, is called *the Green function of the problem* (1), (2), (3_j), $j = 1, 2$, if

$$G_j(x, t; \xi, \tau) = \Gamma(x, t; \xi, \tau) + v_j(x, t; \xi, \tau),$$

where Γ is the fundamental solution of $L(u; x, t) = 0$ in $\bar{\Omega}$ and v_j satisfies the following conditions:

a. $L(v_j; x, t) = 0$ for $t > \tau$.

b. $v_j|_{t=\tau} = 0$ for $(x; \xi, \tau) \in \langle 0, 1\rangle \times \bar{\Omega}$ if at leats one of the points x, ξ lies in the open interval $(0, 1)$.

c. $D_x^{j-1}G_j|_{x=0} = D_x^{j-1}G_j|_{x=1} = D_x^{j+1}G_j|_{x=0} = D_x^{j+1}G_j|_{x=1} = 0.$

3. The fundamental solution and its properties. Consider functions Γ_ν, $\nu = 0, 1, \ldots$ defined for $(x, t; \xi, \tau) \in \Omega_1 \times \Omega_1$, $(x, t) \neq (\xi, \tau)$ by the formula

(4) $$\Gamma_\nu(x, t; \xi, \tau) = \begin{cases} k_\nu(x, t;, \xi, \tau) & \text{if } 0 \leqq \tau < t \\ 0 & \text{if } \tau \geqq t, \end{cases}$$

where

$$(5) \qquad k_\nu(x, t; \xi, \tau) = \frac{(-i)^\nu}{2\pi} \int_{-\infty}^{\infty} \varrho^\nu \exp\{-i\varrho(x - \xi) - \varrho^4(t - \tau)\} \, d\varrho$$

and i means the imaginary unit.

In this section we investigate some properties of limits, integrals and derivatives of Γ_ν. The results are obtained by means of O. A. LADYZHENSKAYA's estimate given in [1]

$$(6) \qquad |D_x^\nu \Gamma_0(x, t; \xi, \tau)| \leqq c_1(\nu)\,(t - \tau)^{-(1+\nu)/4} \exp\{-c_2[(x - \xi)^4/(t - \tau)]^{1/3}\}$$

for $(x, t; \xi, \tau) \in \Omega_1 \times \Omega_1$, $\tau < t$, where the constant c_1 depends on ν and $c_2 > 0$ is an absolute constant. This estimate may be transformed to

$$(7) \qquad |D_x^\nu \Gamma_0(x, t;, \xi, \tau)| \leqq$$

$$\leqq c_1(\nu) \frac{|x - \xi|^{4\mu-\nu-1}}{(t - \tau)^\mu} \left[\frac{(x - \xi)^4}{t - \tau}\right]^{(1+\nu-4\mu)/4} \exp\left\{-c_2 \left[\frac{(x - \xi)^4}{t - \tau}\right]^{1/3}\right\} \leqq$$

$$\leqq K(\nu)\,(t - \tau)^{-\mu}\,|x - \xi|^{4\mu-\nu-1}$$

for $(x, t, \xi, \tau) \in \Omega_1 \times \Omega_1$, $\tau < t$, $\xi \neq x$ and $\mu \leqq (1 + \nu)/4$, where $\nu = 0\ 1, \ldots$ and $K(\nu)$ is a constant depending only on ν. In [2] the identities

$$(8) \qquad \int_{-\infty}^{0} \Gamma_0(x, 1; 0, 0)\, dx = \int_0^\infty \Gamma_0(x, 1; 0, 0)\, dx \, \frac{1}{2} \int_{-\infty}^{\infty} \Gamma_0(x, 1; 0, 0)\, dx = \frac{1}{2}$$

are established.

Lemma 1. *Let* $\nu = 0, 1, \ldots$ *and* $A = \{(x, t, \xi, \tau) \in \Omega_1 \times \Omega_1 : (\xi, \tau) \neq (x, t)\}$. *Then*

a. *The function* $\Gamma_\nu(x, t; \xi, \tau)$ *is continuous on* A *and the identities* $\Gamma_\nu = D_x^\nu \Gamma_0$, $(-1)^\nu D_x^{4\nu} \Gamma_0 = D_t^\nu \Gamma_0 = (-1)^\nu D_\tau^\nu \Gamma_0 = (-1)^\nu D_\xi^{4\nu} \Gamma_0$ *and* $D_x^\nu \Gamma_0 = (-1)^\nu D_\xi^\nu \Gamma_0$ *hold.*

b) $\Gamma_\nu(x, t; \xi, \tau)$ *is almost uniformly bounded on* A *in the sense that to each* $\delta > 0$ *there is* $N(\delta) > 0$ *such that* $|\Gamma_\nu(x, t;, \xi, \tau)| < N(\delta)$ *in* A *for* $(x - \xi)^2 + (t - \tau)^2 \geqq \delta^2$.

c. *If* $\nu = 0, 1, 2, 3$ *the integral* $\iint_\Omega |\Gamma_\nu(x, t; \xi, \tau)| \, d\xi \, d\tau$ *is uniformly convergent with respect to* $P(x, t) \in \Omega_1$ *that is, to each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$\int_{\Omega \cap S(P,\delta)} |D_x^\nu \Gamma_0(x, t; \xi, \tau)| \, d\xi \, d\tau < \varepsilon$$

for all $P \in \Omega_1$. $S(P, \delta)$ *denotes the circle* $(\xi - x)^2 + (\tau - t)^2 \leqq \delta^2$ *in* Ω_1.

Proof. a. For $t - \tau \geqq \delta$, $\delta > 0$ we have

$$\left|\varrho^{\nu} \exp\left\{-i\varrho(x-\xi) - \varrho^4(t-\tau)\right\}\right| \leqq \varrho^{\nu} \exp\left\{-\varrho^4 \delta\right\}$$

which ensures the locally uniform convergence of k_ν on $\Omega_1 \times \Omega_1$ for $\tau < t$. Hence and by $\lim_{\tau \to t-} \Gamma_\nu(x, t; \xi, \tau) = 0$ for $\xi \neq x$ (see (6)) the continuity of Γ_ν on A and the demanded identities follow.

b. For $(x-\xi)^2 + (t-\tau)^2 \geqq \delta^2$ again by the estimate (6)

$$\left|\Gamma_\nu(x, t; \xi, \tau)\right| \leqq c_1(\nu) \sup_{y \in (0, \delta\rangle} \varphi(y),$$

where $\varphi(y) = y^{-(1+\nu)/4} \exp\left\{-c_2(\delta^2 - y^2)^{2/3} y^{-1/3}\right\}$. The last inequality proves the almost uniform boundedness of Γ_ν.

c. Consider the rectangle O: $\Omega_1 \supset O = \left\{(\xi, \tau) : |x - \xi| \leqq \delta, |t - \tau| \leqq \delta\right\} \supset$ $\supset S(P, \delta)$. Then by (7) for $\nu/4 < \mu < (1+\nu)/4$ and $\nu = 0, 1, 2, 3$

$$\iint_{S(P,\delta)} \left|\Gamma_\nu(x, t; \xi, \tau)\right| \mathrm{d}\xi \, \mathrm{d}\tau \leqq 2\, K(\nu) \left[(4\mu - \nu)(1-\mu)\right]^{-1} \delta^{1+3\mu-\nu}.$$

The statement c is proved.

Lemma 2. *Let $f \in C_0(\bar{\Omega})$. Then the integral*

$$I_\nu(x, t; \tau) \equiv \int_0^1 \Gamma_\nu(x, t; \xi, \tau) f(\xi, \tau)\, \mathrm{d}\xi\,, \quad \nu = 0, 1, 2, \ldots$$

has the following properties:

a. *I_ν is continuous on $\bar{\Omega} \times \langle 0, T\rangle$, $\tau \neq t$ and*

(9) $$D_x^\nu I_0(x, t; \tau) = I_\nu(x, t; \tau)$$

for $(x, t; \tau) \in (0, 1) \times \langle 0, T\rangle \times \langle 0, T\rangle$ $\tau \neq t$.

b. *$D_t I_0(x, t; \tau)$ is continuous on $\langle 0, 1\rangle \times (0, T) \times \langle 0, T\rangle$, $\tau \neq t$ and*

(10) $$D_t I_0(x, t; \tau) = \int_0^1 D_t \Gamma_0(x, t; \xi, \tau) f(\xi, \tau)\, \mathrm{d}\xi = -I_4(x, t; \tau)$$

for $(x, t; \tau) \in \Omega \times \langle 0, T\rangle$, $\tau \neq t$.

c. *The uniform limit*

$$\lim_{t \to \tau+} I_0(x, t; \tau) = f(x, \tau) \; (\lim_{\tau \to t-} I_0(x, t; \tau) = f(x, t))$$

exists in any rectangle $R_1 = \langle a, b\rangle \times \langle 0, T)$ $(R_2 = \langle a, b\rangle \times (0, T\rangle)$, where $0 < a < b < 1$.

Proof. Putting $\mu = (1 + \nu)/4$ in (7) we get an integrable majorant for Γ_ν, $\nu = 0, 1, \ldots$ independent of x and locally independent of t and τ for $t \neq \tau$. Then from Lemma 1a and theorems for the differentiation of parametric integrals we have the assertions a, b.

To prove the statement c of this lemma write

$$I_0(x, t; \tau) = f(x, \tau) \int_0^1 \Gamma_0(x, t; \xi, \tau) \, d\xi +$$

$$+ \int_0^1 \Gamma_0(x, t; \xi, \tau) [f(\xi, \tau) - f(x, \tau)] \, d\xi \equiv U_1 + U_2 \,, \quad \tau < t.$$

If we transform the first integral by the substitution $(x - \xi)/(t - \tau)^{1/4} = -z$ then $(\Gamma_0(z, 1; 0, 0) = \Gamma_0(-z, 1; 0, 0))$

$$\int_0^1 \Gamma_0(x, t; \xi, \tau) \, d\xi = \int_0^1 \Gamma_0[x/(t - \tau)^{1/4}, 1;$$

$$\xi/(t - \tau)^{1/4}, 0] \, (t - \tau)^{-1/4} \, d\xi = \int_{-\omega_1}^{\omega_2} \Gamma_0(z, 1; 0, 0) \, dz \,,$$

where $\omega_1 = x/(t - \tau)^{1/4}$, $\omega_2 = (1 - x)/(t - \tau)^{1/4}$. In view of (8), $\lim_{t \to \tau+} U_1 = f(x, \tau)$ ($\lim_{\tau \to t-} U_1 = f(x, t)$) uniformly with respect to $(x, t) \in R_1$ ($(x, t) \in R_2$). Divide the second integral U_2 in two parts integrating on the interval $\langle x - \delta, x + \delta \rangle$, where $\delta > 0$, $x \in \langle a, b \rangle$ such that $x - \delta$, $x + \delta \in \langle 0, 1 \rangle$ and on the set $\langle 0, 1 \rangle - \langle x - \delta, x + \delta \rangle$. From the continuity of $f(x, t)$, for $\varepsilon > 0$ and sufficiently small δ

$$\left| \int_{x-\delta}^{x+\delta} \Gamma_0(x, t; \xi, \tau) [f(\xi, \tau) - f(x, \tau)] \, d\xi \right| \leqq \varepsilon \int_0^1 |\Gamma_0(x, t; \xi, \tau)| \, d\xi \leqq$$

$$\leqq \varepsilon \int_{-\infty}^{\infty} |\Gamma_0(z, 1; 0, 0)| \, dz \leqq \varepsilon K, \quad K > 0 \,.$$

The remaining part of U_2 is a continuous function of x, t, τ and $\lim_{t \to \tau+} U_2 = 0$ ($\lim_{\tau \to t-} U_2 = 0$) uniformly on R_1 (R_2). This completes the proof.

Remark 1. At any point $(x, t) \in \Omega$ the function I_0 may be continuously extended for $\tau = t$ by $I_0(x, t; t) = f(x, t)$.

Lemma 3. *Let*

$$T_\nu(x, t) \equiv \int_0^t \int_0^1 \Gamma_\nu(x, t; \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau \,.$$

Then

a. *For* $f \in C_0(\bar{\Omega})$ *and* $\nu = 0, 1, 2, 3$, T_ν *is continuous on* $\bar{\Omega}$ *and*

$$D_x^\nu T_0(x, t) = T_\nu(x, t), \quad (x, t) \in (0, 1) \times \langle 0, T \rangle . \tag{11}$$

b. *For* $f \in C_0(\bar{\Omega}) \cap S_0[x, (0, 1); t]$, T_4 *is continuous on* $\bar{\Omega}$ *and*

$$D_x^4 T_0(x, t) = T_4(x, t), \quad (x, t) \in (0, 1) \times \langle 0, T \rangle . \tag{12}$$

c. *For* $\in C_0(\bar{\Omega}) \cap S_0[x, (0, 1); t]$, *the derivative* $D_t T_0(x, t)$ *exists, is continuous and*

$$D_t T_0(x, t) = f(x, t) - \int_0^1 \Gamma_4(x, t; \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau = f(x, t) - T_4(x, t) \tag{13}$$

for $(x, t) \in \Omega$. $\Gamma_0(x, t; \xi, \tau)$ *is the fundamental solution of* $L(u; x, t) = 0$.

Proof. Since $T_\nu(x, t) = \int_0^t I_\nu(x, t; \tau) \, d\tau$ and the estimate

$$|I_\nu(x, t; \tau)| \leqq K(\nu) (t - \tau)^{-\mu}, \quad K(\nu) > 0$$

holds for $\nu/4 < \mu < (1 + \nu)/4$, $\nu = 0, 1, 2, 3$ and $(x, t; \tau) \in \langle 0, 1 \rangle \times \langle 0, T \rangle \times \times \langle 0, T \rangle$, $\tau \neq t$, the first assertion follows from Lemma 2a.

The part b will be proved if we find an integrable majorant of $I_4(x, t; \tau)$ with respect to τ. Let $y \in (0, 1)$ be an arbitrary point then by (9)

$$I_4(x, t; \tau) = D_x I_3(x, t; \tau) = f(y, \tau) [\Gamma_3(x, t; 0, \tau) - \Gamma_3(x, t; 1, \tau)]/_{y=x} + \\ + \int_0^1 \Gamma_4(x, t; \xi, \tau) [f(\xi, \tau) - f(y, \tau)] \, d\xi|_{y=x}$$

for $(x, t; \tau) \in (0, 1) \times \langle 0, T \rangle \times \langle 0, T \rangle$, $\tau \neq t$. In virtue of Lemma 1b the difference $\Gamma_3(x, t; 0, \tau) - \Gamma_3(x, t; 1, \tau)$ is a bounded function of (t, τ) at every point $x \in (0, 1)$ and has zero limit as $\tau \to t$-. The last integral may be estimated as follows:

$$\left| \int_0^1 \Gamma_4(x, t; \xi, \tau) [f(\xi, \tau) - f(x, \tau)] \, d\xi \right| \leqq \frac{K_1}{(t - \tau)^\mu} \int_0^1 \frac{d\xi}{|x - \xi|^{1+4-4\mu-\varepsilon}} \leqq \\ \leqq K/(t - \tau)^\mu, \quad 1 - (\tfrac{1}{4}\varepsilon) < \mu < 1$$

where K_1, K are constants independent of x, t, τ.

Prove c. Since $D_t I_0(x, t; \tau) = -I_4(x, t; \tau)$ (see (10)), it is

$$|D_t I_0(x, t; \tau)| \leqq K(t - \tau)^{-\mu}, \quad 1 - \tfrac{1}{4}\varepsilon < \mu < 1 , \tag{14}$$

whence we have the continuity of $D_t T_0$ on Ω. Let $t \in (0, T)$ and $h > 0$ such that $t + h < T$ and investigate the difference

$$(15)\quad \frac{1}{h}\{T_0(x, t + h) - T_0(x, t)\} = \frac{1}{h}\left\{\int_0^{t+h} I_0(x, t + h; \tau)\,\mathrm{d}\tau - \int_0^t I_0(x, t; \tau)\,\mathrm{d}\tau\right\} =$$
$$= \frac{1}{h}\int_t^{t+h} I_0(x, t + h; \tau)\,\mathrm{d}\tau + \int_0^t D_t I_0(x, t^*; \tau)\,\mathrm{d}\tau\,,$$

where $t < t^* < t + h$. In view of the continuity of $I_0(x, t; \tau)$ on $\bar{\Omega} \times \langle 0, T\rangle$ (see Lemma 2a and Remark 1) and the estimate (14), letting $h \to 0$ in (15) we obtain

$$D_t T_0(x, t) = I_0(x, t; t) - T_4(x, t) = f(x, t) - T_4(x, t)\,.$$

Lemma 3 is proved.

Lemma 4. *Put*

$$J_\nu(x, t; \xi) \equiv \int_0^t f(\tau)\Gamma_\nu(x, t; \xi, \tau)\,\mathrm{d}\tau\,.$$

Then

a. *For any $f \in C_0(\langle 0, T\rangle)$ and $\nu = 0, 1, \ldots$ the integral J_ν is continuous on $\bar{\Omega} \times \langle 0, 1\rangle$, $\xi \neq x$ and the equation*

$$(16)\qquad D_x^\nu J_0(x, t; \xi) = J_\nu(x, t; \xi)$$

holds for $(x, t; \xi) \in (0, 1) \times \langle 0, T\rangle \times \langle 0, 1\rangle$, $\xi \neq x$.

b. *For $f \in C_0(\langle 0, T\rangle)$, $D_t J_\nu$ is continuous and*

$$(17)\qquad D_t J_\nu(x, t; \xi) = \int_0^t f(\tau)\, D_t \Gamma_\nu(x, t; \xi, \tau)\,\mathrm{d}\tau = -J_{\nu+4}(x, t; \xi)$$

for $(x, t; \xi) \in \Omega \times \langle 0, 1\rangle$, $\xi \neq x$.

Proof. The proof follows from the estimate (7) for $0 < \mu < 1$.

Lemma 5. a. *For any fixed point $y \in (0, 1\rangle$ and $(x, t) \in (-\infty, \infty) \times (0, T\rangle$*

$$(18)\quad p(x, y) - \int_0^y \Gamma_0(x, t; \xi, 0)\,\mathrm{d}\xi = \int_0^t [D_\xi^3 \Gamma_0(x, t; y, \tau) - D_\xi^3\, \Gamma_0(x, t; 0, \tau)]\,\mathrm{d}\tau\,,$$

where

$$p(x, y) = \begin{cases} 1 & \text{if}\quad 0 < x < y \\ \frac{1}{2} & \text{if}\quad x = 0 \quad\text{or}\quad x = y \\ 0 & \text{if}\quad x < 0 \quad\text{or}\quad x > y\,. \end{cases}$$

b. *Let J_v mean the integral from Lemma 4. If $f \in C_0(\langle 0, T\rangle)$ then for every $t \in (0, T\rangle$*

$$\lim_{x \to z+} J_1(x, t; z) = 0\,, \quad z \in \langle 0, 1) \tag{19}$$

$$\lim_{x \to z-} J_1(x, t; z) = 0\,, \quad z \in (0, 1\rangle\,.$$

c. *Let $n = 2$ or $n = 3$. Then for $f \in S_{2n-4}[t, (0, T\rangle]$ at every point $t \in (0, T\rangle$*

$$\lim_{x \to z+} J_{2n-1}(x, t; z) = \tfrac{1}{2}(1 - |n - 2|) f(t)\,, \quad z \in \langle 0, 1) \tag{20}$$

$$\lim_{x \to z-} J_{2n-1}(x, t; z) = -\tfrac{1}{2}(1 - |n - 2|) f(t)\,, \quad z \in (0, 1\rangle\,.$$

Proof. a. For $x \neq y$ and $x \neq 0$ the convergence of the integrals in (18) follows by (7) and for $x = y$ or $x = 0$ by the identity $D_\xi^3\, \Gamma_0(y, t; y, \tau) = D_\xi^3\, \Gamma_0(0, t; 0, \tau) \equiv 0$. Let $y \in (0, 1\rangle$ be a fixed point. Integrating the equation

$$\int_0^y D_\tau\, \Gamma_0(x, t; \xi, \tau)\, \mathrm{d}\xi = D_\xi^3\, \Gamma_0(x, t; y, \tau) - D_\xi^3\, \Gamma_0(x, t; 0, \tau)\,, \quad t \neq \tau$$

with respect to τ from 0 te $t - \varepsilon$, $\varepsilon > 0$ we obtain

$$\int_0^{t-\varepsilon} [D_\xi^3\, \Gamma_0(x, t; y, \tau) - D_\xi^3\, \Gamma_0(x, t; 0, \tau)]\, \mathrm{d}\tau = \tag{21}$$

$$= \int_0^{t-\varepsilon} \left[\int_0^y D_\tau\, \Gamma_0(x, t; \xi, \tau)\, \mathrm{d}\xi\right] \mathrm{d}\tau = \int_0^y \Gamma_0(x, t; \xi, t - \varepsilon)\, \mathrm{d}\xi - \int_0^y \Gamma_0(x, t; \xi, 0)\, \mathrm{d}\xi$$

for $(x, t) \in \bar{\Omega}$. By the substitution $(x - \xi)/\varepsilon^{1/4} = -z$,

$$\int_0^y \Gamma_0(x, t; \xi, t - \varepsilon)\, \mathrm{d}\xi = \int_{-\omega_1}^{\omega_2} \Gamma_0(z, 1; 0, 0)\, \mathrm{d}z\,,$$

where $\omega_1 = x/\varepsilon^{1/4}$, $\omega_2 = (y - x)/\varepsilon^{1/4}$. Hence and by (8), letting $\varepsilon \to 0+$ in (21) we get the relation (18).

b. Since $\lim_{x \to z} \Gamma_1(x, t; z, \tau) = 0$ uniformly with respect to $\tau \in \langle 0, t - \varepsilon\rangle$, $\varepsilon > 0$, formulas (19) follow from estimate (6) for $v = 1$.

c. The function $f(t)$, $t \in (0, T\rangle$ may be continuously extended to the closed interval $\langle 0, T\rangle$. For $0 \leq z < x < 1$ and $0 \leq \tau < t \leq T$ we get

$$|H_{2n-1}| \overset{\mathrm{df}}{=} |[f(t) - f(\tau)]\, \Gamma_{2n-1}(x, t; z, \tau)| \leq$$
$$\leq K(t - \tau)^{\varepsilon - 1}\,, \quad 0 < \varepsilon < 1\,, \quad K > 0$$

($n = 2, 3$), whence $\lim_{x \to z+} \int_0^t H_{2n-1}\, \mathrm{d}\tau = 0$ at every point $t \in (0, T\rangle$.

In virtue of (18) for $y = z \neq 0$ $(D_\xi^3 \Gamma_0(z, t; z, \tau) \equiv 0)$

$$\lim_{x \to z+} \int_0^t D_x^3 \Gamma_0(x, t; z, \tau) \,\mathrm{d}\tau = \int_0^z \Gamma_0(z, t; \xi, 0) \,\mathrm{d}\xi - \int_0^t D_\xi^3 \Gamma_0(z, t; 0, \tau) \,\mathrm{d}\tau = \tfrac{1}{2}\,.$$

If $z = 0$ the formula (18) yields

$$\lim_{x \to 0+} \int_0^t D_x^3 \Gamma_0(x, t; 0, \tau) \,\mathrm{d}\tau =$$

$$= 1 - \int_0^y \Gamma_0(0, t; \xi, 0) \,\mathrm{d}\xi - \int_0^t D_\xi^3 \Gamma_0(0, t; y, \tau) \,\mathrm{d}\tau = 1 - \tfrac{1}{2} = \tfrac{1}{2}\,.$$

Since

$$J_{2n-1}(x, t; z) = -\int_0^t H_{2n-1} \,\mathrm{d}\tau + f(t)\, L_n\,,$$

where $L_2 = \int_0^t D_x^3 \Gamma_0(x, t; z, \tau) \,\mathrm{d}\tau$ and $L_3 = -D_x \Gamma_0(x, t; z, 0)$, the formula (20) is true for $x \to z+$. The reasoning for $0 < x < z \leqq 1$ is analogous.

4. The Green function. Theorem 1. *The function*

$$(22_j)\quad G_j(x, t; \xi, \tau) = \sum_{k=-\infty}^{\infty} [\Gamma_0(x, t; \xi, + 2k, \tau) + (-1)^j \Gamma_0(x, t; -\xi + 2k, \tau)]\,,$$
$$j = 1, 2$$

and its derivatives $D_x^\nu G_j$, $\nu = 1, 2, \ldots$ *are continuous for* $(x, t; \xi, \tau) \in \bar{\Omega} \times \bar{\Omega}$, $(\xi, \tau) \neq (x, t)$ *and* G_j *constitutes the Green function of problem* (1), (2), (3_j).

Proof. Investigate the convergence of the series

$$(23_j)\qquad u_0^{(j)}(x, t; \xi, \tau) = (-1)^j \Gamma_0(x, t; -\xi, \tau) +$$
$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} [\Gamma_0(x, t; \xi + 2k, \tau) + (-1)^j \Gamma_0(x, t; -\xi + 2k, \tau)]$$

$(G_j(x, t; \xi, \tau) = \Gamma_0(x, t; \xi, \tau) + u_0^{(j)}(x, t; \xi, \tau))$. For $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$ and $k = \pm 1, \pm 2, \ldots$, $|x \mp \xi - 2k| \geqq 2|k| - |\pm\xi - x| \geqq 2|k| - 2$. By (6) we get the estimate

$$|D_x^\nu \Gamma_0(x, t; \pm\xi + 2k, \tau)| \leqq$$
$$c_1(\nu)\,(t - \tau)^{-(1+\nu)/4} \exp\{-2^{4/3} c_2(t - \tau)^{-1/3} (|k| - 1)^{4/3}\}$$

for $\nu = 0, 1, \ldots$ and $\tau < t$. Thus number series

$$4c_1(\nu)\, \alpha^{-(1+\nu)/4} \sum_{l=1}^{\infty} \exp\{-2^{4/3} c_2 T^{-1/3} (l - 1)^{4/3}\}$$

is a convergent majorant of

$$(24)\qquad \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} [D_x^\nu \Gamma_0(x, t; \xi + 2k, \tau) + (-1)^j D_x^\nu \Gamma_0(x, t, -\xi + 2k, \tau)]\,,\quad j = 1, 2$$

if $0 < \alpha < t - \tau \leqq T$. Hence the continuity of $D_x u_0^{(j)}$ follows on $[\langle 0, 1\rangle \times \times (0, T\rangle] \times [\langle 0, 1\rangle \times \langle 0, T)]$, $\tau < t$. We easily see that in $\bar{\Omega} \times \bar{\Omega}$ the function $D_x^\nu \Gamma_0(x, t; -\xi, \tau)$ is discontinuous only for $x = \xi = 0$, $t = \tau$. The terms $D_x \Gamma_0(x, t; \pm\xi + 2k, \tau)$ of the series (24) for $k = 1, -1$ are continuous on $\bar{\Omega} \times \bar{\Omega}$ except the function $D_x^\nu \Gamma_0(x, t; -\xi + 2, \tau)$ which is discontinuous for $x = \xi = 1$, $t = \tau$. The following majorant of (24) for $k \neq 1, -1$

$$s(t, \tau) = 4c_1(\nu)(t - \tau)^{-(1+\nu)/4} \sum_{l=2}^{\infty} \exp\{-2^{4/3} c_2(t - \tau)^{-1/3}(l - 1)\} =$$
$$= 4c_1(\nu)(t - \tau)^{-(1+\nu)/4} q/(1 - q),$$

where $q = \exp\{-2^{4/3} c_2(t - \tau)^{-1/3}\}$, has the zero limit as $t \to \tau+$ or $\tau \to t-$. Consequently all derivatives $D_x^\nu u_0^{(j)}$ are continuous on $\bar{\Omega} \times \bar{\Omega}$, $(x, t) \neq (\xi, \tau)$. By Lemma 1a the continuity of $D_x^\nu G_j$ on $\bar{\Omega} \times \bar{\Omega}$, $(\xi, \tau) \neq (x, t)$ is evident for $\nu = 0, 1, \ldots$

From the preceding argument and Lemma 1a we see that the equation $L(u_0^{(j)}; x, t) = 0$ is not satisfied only at the points $(0, t; 0, t)$, $(1, t; 1, t) \in \bar{\Omega} \times \bar{\Omega}$. Let x or ξ be from the open interval $(0, 1)$ then by the estimate (6) one obtains $\lim_{t \to \tau+} u_0^{(i)}(x, t; \xi, \tau) = 0$. The properties c. of the Green function follow for $G_j(x, t; \xi, \tau)$ from the identity

$$(25) \qquad D_x^\nu G_j(x, t; \xi, \tau)|_{x=z} = \frac{(-i)^\nu}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \varrho^\nu\{[\cos \varrho(z - \xi - 2k) +$$
$$+ (-1)^j \cos \varrho(z + \xi - 2k)] - i[\sin \varrho(z - \xi - 2k) + (-1)^j \sin \varrho(z + \xi - 2k)]\} \,.$$
$$. \exp[-\varrho^4(t - \tau)] \, d\varrho$$

which may be obtained by the direct differentiation of (22_j) and by Eulerian identity. This concludes the proof.

Remark 2. In the compact set $\bar{\Omega} \times \bar{\Omega}$ the Green function G_j contains three singular terms $\Gamma_0(x, t; \xi, \tau)$, $\Gamma_0(x, t; -\xi, \tau)$ and $\Gamma_0(x, t; -\xi + 2, \tau)$ and the function $u_0^{(j)}$ is continuous on $\Omega \times \bar{\Omega}$ and $\bar{\Omega} \times \Omega$.

Theorem 2. *Let G_j be the Green function (22_j). Then for $u(x, t) \in N_4(\bar{\Omega})$ the identity*

$$(26) \qquad u(x, t) = \sum_{k=0}^{3} (-1)^k \int_0^t D_\xi^k u(1, \tau) D_\xi^{3-k} G_j(x, t; 1, \tau) \, d\tau -$$
$$- \sum_{k=0}^{3} (-1)^k \int_0^t D_\xi^k u(0, \tau) D_\xi^{3-k} G_j(x, t; 0, \tau) \, d\tau +$$
$$+ \int_0^1 u(\xi, 0) G_j(x, t; \xi, 0) \, d\xi + \int_0^t \int_0^1 G_j(x, t; \xi, \tau) L(u; \xi, \tau) \, d\xi \, d\tau$$

is true on Ω.

Proof. In virtue of the decomposition $G_j = \Gamma_0 + u_0^{(j)}$ and Remark 2 and Lemmas 2a, 3a, 4a we may assert that all integrals in (26) are continuous functions on Ω.

Let $u(x, t)$ and $v(x, t)$ be arbitrary functions of $N_4(\bar{\Omega})$ and let $M(v; \xi, \tau) = D_\xi^4 v - D_\tau v$. Integrating the identity

$$u\, M(v; \xi, \tau) - v\, L(u; \xi, \tau) = D_\xi[\sum_{k=0}^{3} (-1)^k D_\xi^k u D_\xi^{3-k} v] - D_\tau(uv)$$

over the closed domain $\bar{\Omega}$ we get by the Green formula

$$\iint_\Omega [u\, M(v; \xi, \tau) - v\, L(u; \xi, \tau)]\, \mathrm{d}\xi\, \mathrm{d}\tau = \int_{\partial\Omega} R(\xi, \tau)\, \mathrm{d}\xi + S(\xi, \tau)\, \mathrm{d}\tau\,,$$

where $R(\xi, \tau) = uv$ and $S(\xi, \tau) = \sum_{k=0}^{3} (-1)^k D_\xi^k u D_\xi^{3-k} v$ and $\partial\Omega$ is the boundary of Ω. Consider the positive oriented rectangle $\Omega' = (0 \leqq \xi \leqq 1) \times (0 \leqq \tau \leqq t - \varepsilon)$, $\varepsilon > 0$ with the vertices $A_1(0, 0)$, $A_2(1, 0)$, $M_1(0, t - \varepsilon)$, $M_2(1, t - \varepsilon)$ such that its one side passes through the point $P(x, t) \in \Omega$. In this rectangle we may put $v(\xi, \tau) = G_j(x, t; \xi, \tau)$ and the Green formula gives

$$\begin{aligned} (27) \qquad -\iint_{\Omega'} G_j\, L(u; \xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau &= \sum_{k=0}^{3} (-1)^k \int_{A_2M_2} D_\xi^k u D_\xi^{3-k} G_j\, \mathrm{d}\tau - \\ &- \sum_{k=0}^{3} (-1)^k \int_{A_1M_1} D_\xi^k u D_\xi^{3-k} G_j\, \mathrm{d}\tau + \int_{A_1A_2} u G_j\, \mathrm{d}\xi - \int_{M_1M_2} u \Gamma_0\, \mathrm{d}\xi - \\ &- \int_0^1 u(\xi, t - \varepsilon)\, u_0^{(j)}(x, t; \xi, t - \varepsilon)\, \mathrm{d}\xi\,. \end{aligned}$$

The integral $I(\varepsilon) = \{\int_{M_1M_2} u \Gamma_0\, \mathrm{d}\xi$ may be transformed to

$$I(\varepsilon) = \int_{-\omega_1}^{\omega_2} u(x + z \sqrt[4]{\varepsilon}, t - \varepsilon)\, \Gamma_0(z, 1; 0, 0)\, \mathrm{d}z\,, \qquad (x, t) \in \Omega_0\,,$$

where $\omega_1 = x/\varepsilon^{1/4}$, $\omega_2 = (1 - x)/\varepsilon^{1/4}$ by substituting $-z = (x - \kappa)\, \varepsilon^{1/4}$. With respect to the mean value theorem we obtain

$$\begin{aligned} I(\varepsilon) - u(x, t) &= \sqrt[4]{\varepsilon} \int_{-\omega_1}^{\omega_2} z D_\xi\, u(x + \theta z \sqrt[4]{\varepsilon},\ t - \theta\varepsilon)\, \Gamma_0(z, 1; 0, 0)\, \mathrm{d}z - \\ &- \varepsilon \int_{-\omega_2}^{\omega_1} D_\tau\, u(x + \theta z \sqrt[4]{\varepsilon}, t - \theta\varepsilon)\, \Gamma_0(z, 1; 0, 0)\, \mathrm{d}z - \\ &- u(x, t) \left(\int_{-\infty}^{\infty} - \int_{-\omega_1}^{\omega_2} \right) \Gamma_0(z, 1; 0, 0)\, \mathrm{d}z\,, \quad 0 < \theta < 1\,. \end{aligned}$$

If we denote $N = \max_{\bar{\Omega}} (|u|, |D_\xi u|, |D_\tau u|)$ then

$$|\tilde{I}(\varepsilon) - u(x, t)| < N \sqrt[4]{\varepsilon} \int_{-\infty}^{\infty} |z\, \Gamma_0(z, 1; 0, 0)|\, dz +$$

$$+ \varepsilon N \int_{-\infty}^{\infty} |\Gamma_0(z, 1; 0, 0)|\, dz + N \left| \left(\int_{-\infty}^{-\omega_1} + \int_{\omega_2}^{\infty} \right) \Gamma_0(z, 1; 0, 0)\, dz \right.$$

and by the formula (27) for $\varepsilon \to 0+$ we reach the assertion of this theorem.

Remark 3. The analogous formula to (26) may be shown for an arbitrary function $H(x, t; \xi, \tau) = \Gamma_0 + h$ (instead of G_j), where h has the following properties:

a. $L(h; x, t) \equiv 0$ for $t > \tau$.

b. $\lim_{t \to \tau+} h(x, t; \xi, \tau) = 0$ for $(x; \xi, \tau) \in \langle 0, 1 \rangle \times \bar{\Omega}$ if at least one of the points x and ξ lies in the open interval $(0, 1)$.

5. The solution of boundary problems. The following theorem gives a formula for the explicit representation of the solution of problem (1), (2), (3_j).

Theorem 3. *Let the right hand side of* (1) $\varphi(x, t)$ *be a function of the class* $C_0(\bar{\Omega}) \cap S_0[x, (0, 1); t]$ *and the boundary functions belong to the following classes:* $g(x) \in C_0(\langle 0, 1 \rangle)$; $a_j(t)$, $b_j(t) \in S_2[t, (0, T\rangle]$ *and* $c_j(t), d_j(t) \in S_0[t, (0, T\rangle]$ *for* $j = 1, 2$. *Then the function*

(28_j)

$$u_j(x, t) = (-1)^j \int_0^t \left[a_j(\tau)\, D_\xi^{4-j} G_j(x, t; 0, \tau) + c_j(\tau)\, D_\xi^{2-j}\, G_j(x, t; 0, \tau) \right] d\tau +$$

$$+ (-1)^{j+1} \int_0^t \left[b_j(\tau)\, D_\xi^{4-j}\, G_j(x, t; 1, \tau) + d_j(\tau)\, D_\xi^{2-j}\, G_j(x, t; 1, \tau) \right] d\tau +$$

$$+ \int_0^1 g(\xi)\, G_j(x, t; \xi, 0)\, d\xi + \int_0^t \int_1^0 G_j(x, t; \xi, \tau)\, \varphi(\xi, \tau)\, d\xi\, d\tau$$

is a solution of problem (1), (2), (3_j) *for* $j = 1, 2$.

Proof. First of all we see that there exists a continuous extension of a_j, b_j, c_j and d_j, $j = 1, 2$ on the closed interval $\langle 0, T \rangle$.

The functions $D_\xi^\nu \Gamma_0(x, t; -\xi, \tau)$ and $D_\xi^\nu \Gamma_0(x, t; \pm\xi + 2k, \tau)$ for $k = \pm 1, \pm 2, \ldots$ and $\nu = 0, 1, \ldots$ are continuously differentiable on $\Omega \times \bar{\Omega}$ up to an arbitrary order and satisfy the homogeneous equation $L(u; x, t) = 0$ on Ω for every fixed $(\xi, \tau) \in \bar{\Omega}$. With respect to formulas (10), (13), (17) and to the decomposition $D_x^\nu G_j = D_x^\nu \Gamma_0 + D_x^\nu u_0^{(j)}$ it is obvious that $u_j(x, t)$ satisfies condition (1).

Letting $t \to 0+$ in (28_j) we get condition (2) by Lemmas 2c, 3a and 4a.

For the proof of conditions (3_j) $j = 1, 2$ we shall need the following identity:

$$(29) \qquad D_x^{\nu} D_{\xi}^{\nu_1} G_j(z, t; \xi, \tau) =$$

$$= \frac{(-i)^{\nu+\nu_1}}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \varrho^{\nu+\nu_1} \{[(-1)^{\nu_1} \cos \varrho(z - \xi - 2k) + (-1)^j \cos \varrho(z + \xi - 2k)] -$$

$$- i[(-1)^{\nu_1} \sin \varrho(z - \xi - 2k) + (-1)^j \sin \varrho(z + \xi - 2k)]\} \exp[-\varrho^4(t - \tau)] \, d\varrho$$

which may be obtained analogously to formula (25) by (22_j). If the parameters (ν, ν_1, j, z) attain the values $(0, 0, 1, 0)$, $(0, 0, 1, 1)$, $(2, 0, 1, 0)$, $(2, 0, 1, 1)$ and $(1, 0, 2, 0)$, $(1, 0, 2, 1)$, $(3, 0, 2, 0)$, $(3, 0, 2, 1)$ then the derivative (29) is equal to zero for $\tau < t$. Hence and by Lemmas 2a, 3a for the same values of parameters (ν, ν_1, j, z) as above the derivatives of the both last integrals in (28_j) with respect to x up to the order $\nu = 0, 1, 2, 3$ converge to zero as $x \to 0+$ or $x \to 1-$ for every $t \in (0, T\rangle$. From the continuity of J_ν on $\bar{\Omega} \times \langle 0, 1\rangle$, $\xi \neq x$ and from (16) we have

$$(30) \qquad D_x^{\nu} \int_0^t f_j(\tau) \, D_{\xi}^{\nu_1} G_j(x, t; \xi, \tau) \, d\tau \to 0 \quad \text{if} \quad x \to 0+ \quad \text{or} \quad x \to 1-$$

(f_j represents any one of the functions a_j, b_j, c_j, d_j of (28_j)) for all values $(\nu, \nu_1, j, z) = (0, 1, 1, z)$, $(0, 3, 1, z)$, $(2, 1, 1, z)$, $(2, 3, 1, z)$ and $(1, 0, 2, z)$, $(1, 2, 2, z)$, $(3, 0, 2, z)$, $(3, 2, 2, z)$, where $z = 1 - \xi$ and $\xi = 0, 1$ $(\xi \neq z)$. If $\xi = z$ then Lemma 4a enables us only to interchange the differentiation and integration in (30). For the calculation of the limits of the integrals in (28_j) if $x \to 0+$ and $x \to 1-$ we have to use Lemma 5b,c. (There are two singular integrals in (30). If $z = 0$ then both integrals for $k = 0$ – see (22_j) – are singular and if $z = 1$ the first integral for $k = 0$ and the second integral for $k = 1$ are singular.) The remainder of integral (30) is a continuous function on $\bar{\Omega} \times \bar{\Omega}$ and we find out as well as in the previous case that its limit is zero. Thus the function $u_j(x, t)$ given in (28_j) satisfies condition (3_j).

The function $u_j(x, t)$ is sufficiently smooth. Indeed, the continuity of derivatives $D_x^{\nu} u_j$ for $\nu = 0, 1, 2, 3, 4$ and $D_t u_j$, $j = 1, 2$ in Ω follows successively by Lemma 2a,b and 3a,b,c and 4a,b and by the continuity of $u_0^{(j)}$ on $\Omega \times \bar{\Omega}$. Letting $t \to 0+$ in (28_j) for $x \in (0, 1)$ and then $x \to 0+$ and $x \to 1-$ for $t \in (0, T\rangle$ we get the continuity of $u_j(x, t)$ on $\bar{\Omega} - \{(0, 0), (1, 0)\}$ by Lemma 2c and 5a,b,c. Theorem 3 is proved.

Remark 4. In Theorem 3 we have proved the continuity of $u_j(x, t)$ in $\bar{\Omega} - \{(0, 0), (1, 0)\}$. For the continuity of $u_j(x, t)$ in the whole closed domain $\bar{\Omega}$ we have to put further conditions on the boundary functions. For instance, it is sufficient to assume that $g(x)$ and $a_1(t)$, $b_1(t)$ have a compact support in $\langle 0, 1\rangle$ and $\langle 0, T\rangle$ respectively. Really, substituting $(x - \xi)/t^{1/4}$ by $-z$ we obtain for a sufficiently small $\varepsilon > 0$

$$I_0(x, t; 0) = \int_0^1 g(\xi) \, \Gamma_0(x, t; \xi, 0) \, d\xi =$$

$$= \int_{\varepsilon}^{1-\varepsilon} g(\xi)\, \Gamma_0(x, t; \xi, 0)\, d\xi = \int_{-\omega_1}^{\omega_2} g(x + t^{1/4}z)\, \Gamma_0(z, 1; 0, 0)\, dz\,,$$

where $\omega_1 = (x - \varepsilon)/t^{1/4}$, $\omega_2 = (1 - \varepsilon - x)/t^{1/4}$, $x \in \langle 0, 1\rangle$, $t \in (0, T\rangle$ and $I_0(x, t; 0) \to 0$ as $(x, t) \to (0, 0)$ or $(x, t) \to (1, 0)$ for $(x, t) \in \Omega$. In virtue of (6) for $(x, t) \in \Omega$, $\xi \in \langle 0, 1\rangle$ and $f(t) \in C_0(\langle 0, T\rangle)$ we get

$$\left| \int_0^t f(\tau)\, D_\xi^{\nu_1}\, G_j(x, t; \xi, \tau)\, d\tau \right| \leqq \alpha_j(t)\,, \quad \nu_1 = 0, 1, 2$$

$$\left| \int_0^t \int_0^1 \varphi(\xi, \tau)\, G_j(x, t; \xi, \tau)\, d\xi\, d\tau \right| \leqq \beta_j(t)\,,$$

where $\lim\limits_{t \to 0+} \alpha_j(t) = \lim\limits_{t \to 0+} \beta_j(t) = 0$, $j = 1, 2$. Let $\xi = 0, 1$ then $\int_0^t f(\tau)\, D_\xi^3\, G_j(x, t; \xi, \tau)\, d\tau$ is a continuous function at the point $(1 - \xi, 0)$ (see Lemma 4a). Furthermore by (18) for $y = 1$ and $x \in (0, 1)$

$$|J_3(x, t; 0)| \leqq \left| a_1(t) \int_0^t D_\xi^3\, \Gamma_0(x, t; 0, \tau)\, d\tau \right| + \int_0^t |a_1(\tau) - a_1(t)|\, |D_\xi^3\, \Gamma_0(x, t; 0, \tau)|\, d\tau \leqq$$

$$\leqq |a_1(t)| \left\{ \left| \int_0^t D_\xi^3\, \Gamma_0(x, t; 1, \tau)\, d\tau \right| + \frac{K}{t^\mu} + 1 \right\} + \gamma(t)$$

where $K > 0$, $0 < \mu < 1$ and $\lim\limits_{t \to 0+} \gamma(t) = 0$. Thus $J_3(x, t; 0) \to 0$ if $(x, t) \to (0, 0)$. Analogously $J_3(x, t; 1) - 0$ as $(x, t) \to (1, 0)$.

Remark 5. The above procedures are directly applicable to the boundary value problems

$$L_n(u; x, t) = D_x^{2n} u + (-1)^n D_t u = \varphi(x, t)\,, \quad (x, t) \in \Omega$$

$$u(x, 0) = g(x)\,, \quad x \in (0, 1)$$

$$D_x^{2\nu + j - 1}\, u(0, t) = a_{\nu j}(t)\,, \quad D_x^{2\nu + j - 1}\, u(1, t) = b_{\nu j}(t)\,,$$

$$t \in (0, T\rangle\,, \quad \nu = 0, 1, \ldots, n - 1\,, \quad j = 1, 2\,.$$

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