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# Subdeterminants and subgraphs

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# SUBDETERMINANTS AND SUBGRAPHS

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## 1. INTRODUCTION

Given a square matrix $A$, assign to it the directed valuated graph $G(A)$ in the natural way. If some elements of $A$ are zero then many terms of det $A$ vanish, of course. In this paper, non-zero terms of principal and "almost principal" minors of the matrix $A$ and of matrices obtained by modifying its diagonal are described by means of certain classes of subgraphs of $G(A)$. This theory makes it possible to generate and to enumerate certain subgraphs of a given directed or non-directed graph and yields inequalities concerning minors of matrices of a certain kind. Some generalizations of results of the papers [1], [2] and [3] are given. Another application consists in expressing the solution of the system of linear equations and the coefficients of the characteristic polynomial of a matrix $A$ and of its modifications by means of subgraphs of $G(A)$. These formulae are well-known and frequently used in the field of electrical networks analysis (v. [3]).

## 2. PRELIMINARIES

Common concepts and terms from matrix and graph theory are used tacitly.

Let $n$ be an integer. Denote $N = \{1, 2, \ldots, n\}$. Let $F$ be a set. Denote by $|F|$ the cardinality of $F$.

Let $A = (a_{ik})$ be an $n \times n$ matrix and $\emptyset \neq K \subseteq N$, $\emptyset \neq L \subseteq N$. Denote by $A_{KL}$ the submatrix obtained from $A$ by deleting the rows and columns with indices from $N - K$ and $N - L$, respectively. Denote by $G(A)$ the directed valuated graph consisting of vertices $1, 2, \ldots, n$ and edges $(i, k)$ for each $a_{ik} \neq 0$. Each edge $(i, k)$ is assigned the value $a_{ik}$. By $\operatorname{diag}(d_1, d_2, \ldots, d_n)$ denote the $n \times n$ matrix $M = (m_{ik})$ such that $m_{ii} = d_i$ for each $i \in N$ and all the off-diagonal elements $m_{ik}$ are zero. By $D(A)$ denote the matrix $\operatorname{diag}\left(\sum_{k \neq 1} a_{1k}, \sum_{k \neq 2} a_{2k}, \ldots, \sum_{k \neq n} a_{nk}\right)$. By $I$ is denoted the identity matrix of the appropriate order.

Regard now det $B$ as a polynomial in the $n^2$ elements $b_{ik}$ of $B$. Let $r, s \in N$. It is easy to see that

$$(-1)^{r+s} \det B_{N-\{r\},N-\{s\}} = \frac{\partial}{\partial b_{rs}} \det B\,.$$

Suppose now $r, s \in V \subseteq N$, $r \neq s$. Put

$$m(V, r, s) = r + s + \left|\{j \in N - V \mid r < j < s\}\right| \quad \text{for} \quad r < s\,,$$
$$m(V, r, s) = m\,(V, s, r) \qquad\qquad\qquad \text{for} \quad s < r\,.$$

It is easy to see that

$$\det(B - I) = (-1)^n + \sum_{\emptyset \neq W \subseteq N} (-1)^{n-|W|} \det B_{WW}\,, \tag{2.1}$$

and more generally

$$\det(B - I)_{VV} = (-1)^{|V|} + \sum_{\emptyset \neq W \subseteq V} (-1)^{|V|-|W|} \det B_{WW}\,. \tag{2.2}$$

Differentiation of (2.2) with respect to $b_{rs}$ yields

(2.3)
$$(-1)^{m(V,r,s)} \det(B - I)_{V-\{r\},V-\{s\}} = \sum_{r,s \in W \subseteq V} (-1)^{|V|+|W|+m(W,r,s)} \det B_{W-\{r\},W-\{s\}}\,.$$

Let $G$ be a directed graph of vertices $1, 2, \ldots, n$ and $V \subseteq N$. By $G_V$ denote the subgraph of $G$ the vertex set of which is $V$ and the edges of which are all the edges of $G$ connecting two vertices from $V$. By $C(G)$ (resp. $D(G)$) denote the class of all the spanning subgraphs (resp. of all the subgraphs) of $G$ such that each component of them is a cycle (in other words, a directed circuit). Let, $j, k \in N$, $j \neq k$. By $P_{jk}(G)$ (resp. $Q_{jk}(G)$) denote the class of all the spanning subgraphs (resp. of all the subgraphs) of $G$ such that one component of each of them is a path from $j$ to $k$, the other components being cycles. Evidently, from each vertex and into each vertex of a subgraph from $C(G)$ or $D(G)$ leads exactly one edge. The same is true for all vertices of subgraphs from $P_{jk}(G)$ and $Q_{jk}(G)$ with the exception of $j$ and $k$. By adding the edge $(k, j)$ (provided that it is contained in $G$) to a subgraph from $P_{jk}(G)$ (resp. $Q_{jk}(G)$) a subgraph from $C(G)$ (resp. $D(G)$) is obtained. The empty subgraph (both the vertex and the edge sets are empty) belongs to $D(G)$. A root of a graph is a vertex from which no edge leads. By $L(G)$ denote the class of all the spanning subgraphs of $G$ such that each component of them is either a tree with exactly one root or a graph obtained from a tree with exactly one root $r$ by adding the edge $(r, r)$. Intuitively, each edge of a component of a subgraph from $L(G)$ is directed "towards" the root or the loop. Let $K \subseteq N$. By $L_K(G)$ (resp. $L_{(K)}(G)$) denote the subclass of $L(G)$ consisting of all the subgraphs such that the root set of each of them is $K$ (resp. contains $K$). In this notation, $L(G) = L_{(\emptyset)}(G)$. Let $r, s \in N$. By $L_K^{rs}(G)$ (resp. $L_{(K)}^{rs}(G)$) denote the subclass of $L_K(G)$ (resp. $L_{(K)}(G)$) consisting of all the subgraphs such that each of them contains the vertices $r$, $s$ in the same component.

Let $H$ be a subgraph of the graph $G(A)$ introduced above. Denote by $\pi(H)$, $\varrho(H)$ and $\sigma(H)$ the product of edge values, the number of components and the number of roots, respectively, of $H$. Let $K(G(A))$ be a class of subgraphs of $G(A)$. Denote

$$\Phi(K(G(A))) = \sum_{H \in K(G(A))} (-1)^{n+\varrho(H)+\sigma(H)} \pi(H),$$

$n$ being the number of vertices of $G(A)$.

## I. EXPANSIONS OF MINORS

Throughout this chapter, assume that $A = (a_{ik})$ is an $n \times n$ matrix over an integral domain, and that $\emptyset \neq V \subseteq N$. Wherever the symbols $R$, $S$, $r$ and $s$ appear, it is assumed that $r, s \in V \subseteq N$, $r \neq s$ and the following notation is used: $R = V - \{r\}$, $S = V - \{s\}$, $m(R, S) = m(V, r, s)$ (the function on the right side was introduced above).

### 3. A

**(3.1)** $$\det A = \Phi(C(G(A))).$$

Proof. Assign to each non-zero term $a_{1i_1}a_{2i_2} \dots a_{ni_n}$ of $\det A$ a subgraph $H$ consisting of edges $(1, i_1), (2, i_2), \dots, (n, i_n)$ of the graph $G(A)$. This is one-to-one correspondence between non-zero terms of $\det A$ and subgraphs from $C(G(A))$. Moreover,

$$\operatorname{sgn}\{i_1, i_2, \dots, i_n\} = (-1)^{n-\varrho(H)}.$$

Consequently,

$$\det A = \sum_{H \in C(G(A))} (-1)^{n-\varrho(H)} \pi(H) = \Phi(C(G(A))).$$

**(3.2)** $$\det A_{VV} = \Phi(C(G_V(A))).$$

Proof. Observe that $G_V(A) = G(A_{VV})$ and apply (3.1).

**(3.3)** $$(-1)^{m(R,S)} \det A_{RS} = -\Phi(P_{sr}(G_V(A))).$$

Proof. Differentiate (3.2) with respect to $a_{rs}$.

## 4. A—I

(4.1) $$\det(A - I) = \Phi(D(G(A)))\,.$$

Proof. Substitute (3.2) into (2.1). The term $(-1)^n$ corresponds to the empty subgraph.

(4.2) $$\det(A - I)_{VV} = \Phi(D(G_V(A)))\,.$$

Proof. Observe that $(A - I)_{VV} = A_{VV} - I$ and apply (4.1).

(4.3) $$(-1)^{m(R,S)} \det(A - I)_{RS} = -\Phi(Q_{sr}(G_V(A)))\,.$$

Proof. Differentiate (4.2) with respect to $a_{rs}$.

## 5. A—D(A)

(5.1) $$\det(A - D(A)) = \Phi(L_\emptyset(G(A)))\,.$$

Proof. If all the diagonal elements of $A$ are zero then $L_\emptyset(G(A)) = \emptyset$ and so $\Phi(L_\emptyset(G(A))) = 0$. Further, the matrix $A - D(A)$ is singular since all its row sums are zero. Thus (5.1) is true in this case.

Suppose now that there exists an $w \in N$ such that $a_{ww} \neq 0$. If $n = 1$ or if $A$ is the zero matrix then (5.1) is true. Suppose than $n > 1$ and that (5.1) is true for each square matrix of order less than $n$ and for each $n \times n$ matrix the number of non-zero elements of which is less than that of $A$.

Suppose first that $a_{wz} = 0$ for each $z \in N - \{w\}$. This implies

$$\det(A - D(A)) = a_{ww} \det(A - D(A))_{N-\{w\},N-\{w\}} = a_{ww} \det(B - D(B))$$

where

$$B = A_{N-\{w\},N-\{w\}} - \operatorname{diag}(a_{1w}, a_{2w}, \ldots, a_{nw})_{N-\{w\},N-\{w\}}\,.$$

By the induction hypothesis,

$$\det(B - D(B)) = \Phi(L_\emptyset(G(B)))\,.$$

It is easy to see that

$$\Phi(L_\emptyset(G(B))) = \Phi(L_\emptyset(G_{N-\{w\}}(A))) - \Phi(L_{\{w\}}(G(A)))\,.$$

Consequently,

$$\det(A - D(A)) = a_{ww}\,\Phi(L_\emptyset(G_{N-\{w\}}(A))) - a_{ww}\,\Phi(L_{\{w\}}(G(A))) = \Phi(L_\emptyset(G(A)))\,.$$

It remains to consider the case that $a_{wz} \neq 0$ for some $z \in N - \{w\}$. Denote by $A'$ (resp. $A''$) the matrix obtained from $A$ by replacing the element $a_{ww}$ (resp. the elements $a_{wz}$ for each $z \in N - \{w\}$) by zero. By the induction hypothesis,

$$\det(A' - D(A')) = \Phi(L_\emptyset(G(A'))),$$
$$\det(A'' - D(A'')) = \Phi(L_\emptyset(G(A''))).$$

It is easy to see that

$$L_\emptyset(G(A)) = L_\emptyset(G(A')) \cup L_\emptyset(G(A'')),$$

the sets on the right side being disjoint. Hence

$$\det(A - D(A)) = \det(A' - D(A')) + \det(A'' - D(A'')) = \Phi(L_\emptyset(G(A)))$$

which completes the proof.

**(5.2)** $$\det(A - D(A))_{VV} = (-1)^{n-|V|}\,\Phi(L_{N-V}(G(A)))\,.$$

Proof. Denote by $C = (c_{ik})$ the $n \times n$ matrix such that $C_{VN} = A_{VN}$ and $c_{ik} = \delta_{ik}$ for each $i \in N - V$, $k \in N$. According to (5.1),

$$\det(A - D(A))_{VV} = \det(C - D(C)) = \Phi(L_\emptyset(G(C)))\,.$$

It is easy to see that

$$L_\emptyset(G(C)) = L_{N-V}(G(A))\,.$$

**(5.3)** $$(-1)^{m(R,S)}\det(A - D(A))_{RS} = (-1)^{n-|R|}\,\Phi(L^{rs}_{N-R}(G(A)))\,.$$

Proof. Denote by $C = (c_{ik})$ the $n \times n$ matrix such that $C_{N-\{r\},N} = A_{N-\{r\},N}$, $c_{rr} = c_{rs} = 1$ and $c_{rw} = 0$ for each $w \in N - \{r, s\}$. According to (5.2),

$$(-1)^{m(R,S)}\det(A - D(A))_{RS} = \det(C - D(C))_{VV} = (-1)^{n-|V|}\,\Phi(L_{N-V}(G(C)))\,.$$

Denote by $L^{r/s}_{N-V}(G(C))$, $L^{r\to s}_{N-V}(G(C))$ and $L^{s\to r}_{N-V}(G(C))$ the subclass of $L_{N-V}(G(C))$ consisting of all the subgraphs such that none of them contains a path between $r$ and $s$, each of them contains a path from $r$ to $s$ and each of them contains a path from $s$ to $r$, respectively. It is easy to see that

$$L_{N-V}(G(C)) = L^{r/s}_{N-V}(G(C)) + L^{r\to s}_{N-V}(G(C)) + L^{s\to r}_{N-V}(G(C))\,,$$

the sets on the rigth side being disjoint, and

$$\Phi(L^{r/s}_{N-V}(G(C))) = -\Phi(L^{r/s}_{N-R}(G(A)))\,,$$
$$\Phi(L^{r\to s}_{N-V}(G(C))) = \Phi(L^{r/s}_{N-R}(G(A)))\,,$$
$$\Phi(L^{s\to r}_{N-V}(G(C))) = -\Phi(L^{rs}_{N-R}(G(A)))\,.$$

Consequently,

$$(-1)^{n-|V|}\,\Phi(L_{N-V}(G(C))) = (-1)^{n-|R|}\,\Phi(L^{rs}_{N-R}(G(A)))\,.$$

**(6.1)** $$\det(A - D(A) - I) = \Phi(L(G(A))).$$

Proof. Set $B = A - D(A)$ and substitute (5.2) into (2.1). The term $(-1)^n$ corresponds to the subgraph of $n$ roots, i.e. to the subgraph consisting of $n$ isolated vertices.

**(6.2)** $$\det(A - D(A) - I)_{VV} = (-1)^{n-|V|}\, \Phi(L_{(N-V)}(G(A)))\,.$$

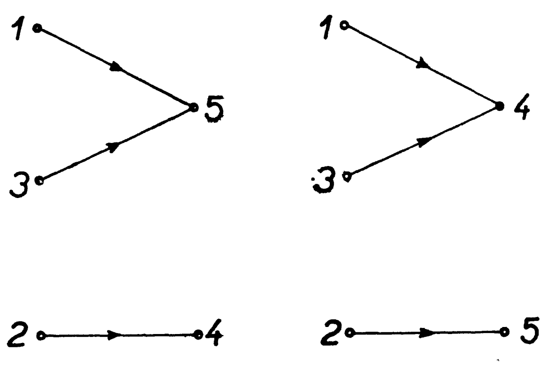
Proof. Set $B = A - D(A)$ and substitute (5.2) into (2.2).

**(6.3)** $$(-1)^{m(R,S)} \det(A - D(A) - I)_{RS} = (-1)^{n-|R|}\, \Phi(L^{rs}_{(N-R)}(G(A)))\,.$$

Proof. Set $B = A - D(A)$ and substitute (5.3) into (2.3).

* * *

Principal minors $M_{VV}$ and "almost principal" ones $M_{RS}$ of certain modifications $M$ of a matrix $A$ were dealt with in this chapter. The question about the other minors suggests itself. Indeed, it is not difficult to derive analogous expansions of them using analogous methods. However, these expansions lose the combinatorial character, i.e. the signs of their terms depend not merely on the appearance of corresponding subgraphs but also on the order of its vertices. For example, in the case $n = 5$, $R = \{1, 2, 3\}$, $S = \{3, 4, 5\}$, the terms $a_{15}a_{24}a_{35}$ and $a_{14}a_{34}a_{25}$ of $\det(A - D(A))_{RS}$, which correspond to the subgraphs




of $G(A)$, have opposite signs.

# II. APPLICATIONS

In this chapter, some applications of the results of the Chapter I are shown.

## 7. SYSTEMS OF LINEAR EQUATIONS

Let $A$ be an $n \times n$ matrix and $b$ be an $n$-dimensional column vector. Consider the system of linear algebraic equations $Ax = b$.

It holds

$$x_k \det A = \sum_{i=1}^{n} (-1)^{i+k} b_i \det A_{N-\{i\},N-\{k\}}$$

for each $k \in N$. According to (3.3) and (3.2),

$$x_k \det A = - \sum_{i \neq k} b_i \, \Phi(P_{ki}(G(A))) + b_k \, \Phi(C(G_{N-\{k\}}(A))) .$$

Put

$$B = \begin{pmatrix} A & b \\ o & 0 \end{pmatrix}$$

where $o$ is the $n$-dimensional zero row vector. It is easy to rewrite the last expansion into the form

$$x_k \det A = \Phi(P_{k,n+1}(G(B))) ,$$

the expansion of $\det A$ being given by (3.1).

Analogously,

$$x_k \det (A - I) = \Phi(Q_{k,n+1}(G(B))) ,$$

$$x_k \det (A - D(A)) = \Phi(L^{k,n+1}_{\{n+1\}}(G(B)))$$

and

$$x_k \det (A - D(A) - I) = \Phi(L^{k,n+1}_{\{n+1\}}(G(B)))$$

for each $k \in N$.

Such a formulae are frequently used in electrical engineering. They make it possible to read the solution of certain systems of linear equations which arise in network analysis immediately from the network diagram.

## 8. COEFFICIENTS OF CHARACTERISTIC POLYNOMIALS

Let $A$ be a $n \times n$ matrix. The following expression of the coefficients of its characteristic polynomial

$$\det (A - xI) = \sum_{t=0}^{n} a_t x^{n-t}$$

is well known:

$$a_t = (-1)^{n-t} \sum_{T \subseteq N\ T = t} \det A_{TT} \quad \text{fro each} \quad t \in N\,,$$

$$a_0 = (-1)^n\,.$$

According to (3.2), for each $t \in N$,

$$a_t = (-1)^{n-t} \sum_{T \subseteq N\ |T| = t} \Phi(C(G_T(A))) = \Phi(D_t(G(A))) \tag{8.1}$$

where by $D_t\,(G(A)))$ the subclass of $D(G(A))$ consisting of all the subgraphs of exactly $t$ vertices is denoted.

Further, according to (5.2), it holds for the coefficients of the characteristic polynomial

$$\det (A - D(A) - xI) = \sum_{t=0}^{n} c_t x^{n-t}$$

of the matrix $A - D(A)$, for each $t \in N$,

$$c_t = (-1)^{n-t} \sum_{T \subseteq N\ |T| = t} (-1)^{n-t}\, \Phi(L_{N-T}(G(A))) = \Phi({}_{n-t}L(G(A))) \tag{8.2}$$

where by ${}_zL(G(A)))$ the subclass of $L(G(A))$ consisting of all the subgraphs of exactly $z$ roots is denoted.

Consider now the characteristic polynomial

$$\det (A - I - xI) = \sum_{t=0}^{n} b_t x^{n-t}$$

of the matrix $A - I$. According to (2.2), for each $t \in N$,

$$\begin{aligned} b_t &= (-1)^{n-t} \sum_{T \subseteq N\ |T| = t} \det (A - I)_{TT} = (-1)^{n-t} \sum_{T \subseteq N |T| = t} [(-1)^t + \\ &\quad + \sum_{\emptyset \neq W \subseteq T} (-1)^{t-|W|} \det A_{WW}] = \\ &= (-1)^n \binom{n}{t} + \sum_{w=1}^{t} (-1)^{n-w} \binom{n-w}{n-t} \sum_{W \subseteq N\ |W| = w} \det A_{ww} = \sum_{w=0}^{t} \binom{n-w}{n-t} a_w\,. \end{aligned}$$

Thus, according to (8.1),

$$b_t = \sum_{w=0}^{t} \binom{n-w}{n-t} \Phi(D_w(G(A))) \tag{8.3}$$

for each $t \in N$.

Analogously, it works out for the coefficients of the characteristic polynomial

$$\det (A - D(A) - I - xI) = \sum_{t=0}^{n} d_t x^{n-t}$$

of the matrix $A - D(A) - I$, according to (8.2),

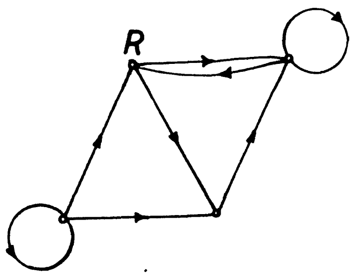
$$d_t^{*} = \sum_{w=0}^{t} \binom{n-w}{n-t} c_w = \sum_{w=0}^{t} \binom{n-w}{n-t} \Phi({}_{n-t}L(G(A)))$$

for each $t \in N$.

## 9. GENERATION OF SUBGRAPHS

Let $G$ be a finite directed graph. Observe that all its subgraphs of any class defined in the Preliminaries can be constructed in the following way. Order the vertices of $G$ and assign to each edge $(i, k)$ of $G$ a variable $x_{ik}$. Further, construct the matrix $X$ such that $G = G(X)$. Then the subgraphs of such a class correspond with the terms of the appropriate minor (which is regarded as a polynomial in $x_{ik}$) of the appropriate modification of the matrix $X$.

For example, let $G$ be the following graph




and construct all the subgraphs of the class $L_{\{R\}}(G)$. Number the vertices from left to right, then

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & x_{42} & 0 & x_{44} \end{pmatrix}.$$

Further,

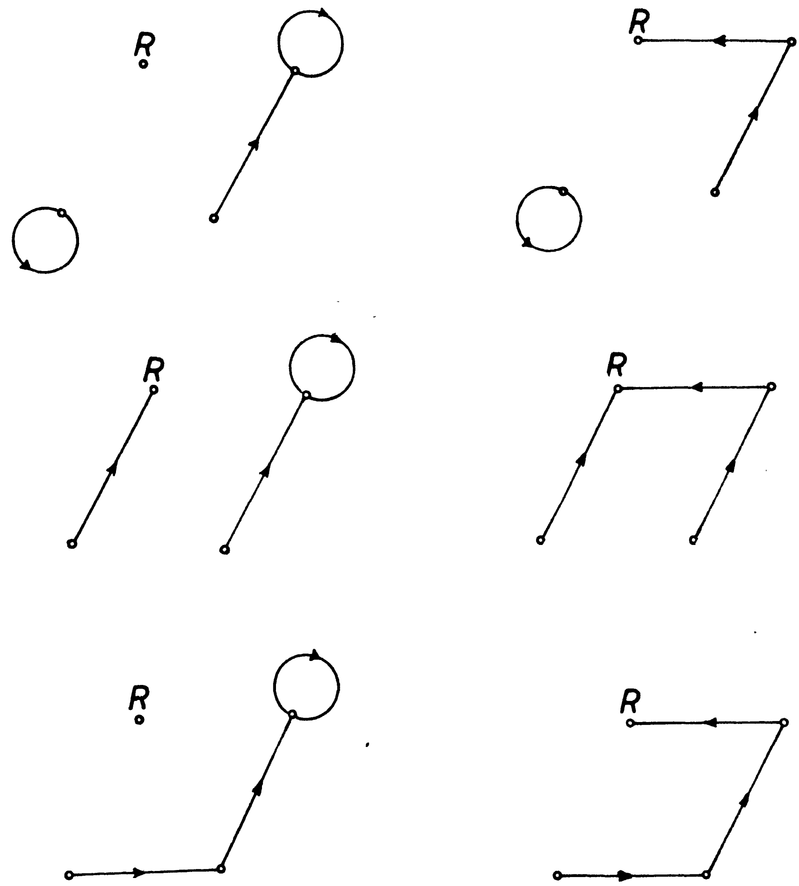
$$\det (X - D(X))_{\{1,3,4\},\{1,3,4\}} =$$

$$= \det \begin{pmatrix} x_{11} - x_{12} - x_{13} & x_{13} & 0 \\ 0 & -x_{34} & x_{34} \\ 0 & 0 & x_{44} - x_{42} \end{pmatrix} =$$

$$= -x_{11}x_{34}x_{44} + x_{11}x_{34}x_{42} + x_{12}x_{34}x_{44} - x_{12}x_{34}x_{42} + x_{13}x_{34}x_{44} -$$

$$- x_{13}x_{34}x_{42} \cdot$$

The corresponding subgraphs are




## 10. PROOF TECHNIQUES

The expansions of minors given in the Chapter I can be used to prove some relations concerning minors. For example, prove the following well-known formula.

*Let $n > 3$, $r, s \in N$, $r \neq s$ and $A = (a_{ik})$ be an $n \times n$ matrix. Then*

$$(-1)^{r+s+1} \det A_{N-\{r\},N-\{s\}} = a_{sr} \det A_{N-\{r,s\}N-\{r,s\}} -$$

$$- \sum_{i \in N-\{r,s\}} a_{ir} a_{si} \det A_{N-\{i,r,s\},N-\{i,s\}} -$$

$$- \sum_{\substack{i,k \in N-\{r,s\} \\ i \neq k}} (-1)^{m(N-\{i,r,s\},N-\{k,r,s\})} a_{ir} a_{sk} \det A_{N-\{i,r,s\},N-\{k,r,s\}} .$$

Proof. Denote by $P^l_{sr}(G(A))$ (resp. $P^{(l)}_{sr}(G(A))$) the subclass of the class $P_{sr}(G(A))$ consisting of all the subgraphs such that the path component of each is of length $l$ (resp. at least $l$). According to (3.3),

$$(-1)^{r+s+1} \det A_{N-\{r\},N-\{s\}} = \Phi(P_{sr}(G(A))) =$$
$$= \Phi(P^1_{sr}(G(A))) + \Phi(P^2_{sr}(G(A))) + \Phi(P^{(3)}_{sr}(G(A))) .$$

Further,

$$\Phi(P^1_{sr}(G(A))) = a_{sr}\, \Phi(C(G_{N-\{r,s\}}(A))) = a_{sr} \det A_{N-\{r,s\},N-\{r,s\}} ,$$

$$\Phi(P^2_{sr}(G(A))) = - \sum_{i\in N-\{r,s\}} a_{si}a_{ir}\, \Phi(C(G_{N-\{i,r,s\}}(A))) =$$

$$= - \sum_{i\in N-\{r,s\}} a_{si}a_{ir} \det A_{N-\{i,r,s\},N-\{i,r,s\}}$$

and

$$\Phi(P^{(3)}_{sr}(G(A))) = \sum_{i,k\in N-\{r,s\}\ i\neq k} a_{si}a_{kr}\Phi(P_{ik}(G_{N-\{r,s\}}(A))) =$$

$$= - \sum_{i,k\in N-\{r,s\}\ i\neq k} a_{si}a_{kr}(-1)^{m(N-\{i,r,s\},\ N-\{k,r,s\})} \det A_{N-\{i,r,s\},N-\{k,r,s\}} .$$

* * *

In the rest, formulae of the section 5 are applied. Observe that each subgraph $H \in L(G)$ contains exactly $n - \varrho(H)$ edges $(i, k)$ such that $i \neq k$, $n$ being the number of vertices of $G$. This makes it possible to eliminate the factor $(-1)^{n-\varrho(G(A))}$ in the terms of expansions of minos of the matrix $A - D(A)$ by changing the signs of all the off-diagonal elements of $A$.

For formal reasons, extend now the symbols $R$, $S$ defined in the introduction to the Chapter I, to the case $\emptyset \neq R = S \subseteq N$. Then put $m(R, S) = 0$, $r = s = 1$. This makes it possible to write the formulae (5.1)–(5.3) in the universal form (5.3).

Given an $n \times n$ matrix $A = (a_{ik})$, denote by $R(A) = (r_{ik})$ the $n \times n$ matrix such that $r_{ii} = a_{ii}$ and $r_{ik} = -a_{ik}$ for each $i, k \in N$, $i \neq k$. Then it holds for the elements $s_{ik}$ of the matrix $S(A) = R(A) - D(R(A))$ that $s_{ii} = \sum_{j=1}^{n} a_{ij}$ and $s_{ik} = -a_{ik}$ for each $i \in N$, $k \in N$, $i \neq k$. Further, $S(S(A)) = A$ and the formula (5.3) can be written in the form

$$(*)\quad (-1)^{m(R,S)} \det (S(A))_{RS} = (-1)^{n-|R|}\, \Phi(L^{rs}_{N-R}(G(A))) = \sum_{H\in L^{rs}{}_{N-R}(G(A))} \pi(H) .$$

## 11. ENUMERATION OF SUBGRAPHS

Let $G$ be a directed graph of $n$ vertices. Having chosen a fixed ordering of its vertices, assign to each edge of it the value 1 and construct the matrix $A(G)$ such that $G = G(A(G))$ and the matrix $S(G) = S(A(G))$. (The matrix $A(G)$ is usually called the incidence matrix of $G$.) Then the elements $s_{ik}$ of the matrix $S(G)$ satisfy $s_{ik} = -1$ if $(i, k) \in G$ and $s_{ik} = 0$ otherwise for each $i, k \in N$, $i \neq k$, $s_{ii}$ being equal to the number of edges of $G$ leading from the vertex $i$ for each $i \in N$. The formula (*) yields

$$|L^{rs}_{N-R}(G)| = (-1)^{m(R,S)} \det (S(G))_{RS} .$$

Let $G$ be a non-directed graph of $n$ vertices without loops now. Having chosen a fixed ordering of its vertices, assign to it the matrix $S(G) = (s_{ik})$ such that $s_{ik} = s_{ki} = -1$ if $(i, k) \in G$ and $s_{ik} = s_{ki} = 0$ otherwise for each $i, k \in N$, $i \neq k$, $s_{ii}$ being equal to the number of edges of $G$ which are incident with the vertex $i$ (the degree of the vertex $i$) for each $i \in N$. It is easy to see that $S(G) = S(G')$ where $G'$ is the directed graph obtained from $G$ by replacing all the non-directed edges $(i, k)$ of $G$ by the pair of directed edges $(i, k)$, $(k, i)$. It is easy to see that there is a one-to-one correspondence between spanning forests of $G$ and subgraphs from $L(G')$. (A forest is a subgraph each component of which is a tree.) Consequently, $(-1)^{m(R,S)} \det (S(G))_{RS}$ is equal to the number of all the spanning forests of $G$ such that each consists of exactly $n - |R|$ components, each vertex from $N - R$ being contained in exactly one component, the vertices $r$, $s$ being contained in the same component. Especially, for any $i \in N$ the minor $\det (S(G))_{N-\{i\},N-\{i\}}$ is equal to the number of spanning trees of $G$(cf. [1], [2]).

## 12. INEQUALITIES CONCERNING MINORS

**(12.1)** *Let $M = (m_{ik})$ be a real $n \times n$ matrix such that $m_{ik} \leqq 0$ and $\sum_{j=1}^{n} m_{ij} \geqq 0$ for each $i, k \in N$, $i \neq k$. Then*

$$(-1)^{m(R,S)} \det M_{RS} \geqq 0 .$$

*Equality is attained if and only if $L^{rs}_{N-R}(G(S(M))) = \emptyset$.*

Proof. Obviously, $M = S(A)$ where $A$ is a non-negative matrix, so each term in (*) is non-negative. Further, $A = S(S(A)) = S(M)$ and the sum in (*) is non-zero if and only if $L^{rs}_{N-R}(G(A)) \neq \emptyset$.

**(12.2)** *Let $M = (m_{ik})$ be a real $n \times n$ matrix such that $m_{ik} < 0$ and $\sum_{j=1}^{n} m_{ij} > 0$ for each $i, k \in N$, $i \neq k$. Then*

$$(-1)^{m(R,S)} \det M_{RS} > 0 .$$

Proof. This is an easy corollary of (12.1).

**(12.3)** *Let $M$ be a matrix satisfying the assumptions of* (12.2). *Let $D = (d_{ik})$ be a non-negative diagonal $n \times n$ matrix. Then*

$$(-1)^{m(R,S)} \det (M + D)_{RS} \geqq (-1)^{m(R,S)} \det M_{RS} .$$

*Equality is attained if and only if any subgraph from $L^{rs}_{N-R}(G(S(M)))$ contains no edge $(i, i)$ such that $d_{ii} > 0$ and $|L^{rs}_{N-R}(G(S(M)))| = |L^{rs}_{N-R}(G(S(M + D)))|$.*

Proof. $M = S(A)$ where $A$ is non-negative and $M + D = S(A + D)$. The theorem follows immediately from $(*)$.

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