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Georgij Alexandrovič Kamenskij; Anatolij Dmitrievich Myshkis; Alexander L. Skubachevskii

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GENERALIZED AND SMOOTH SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MANY SENIOR MEMBERS

G. A. KAMENSKII, A. D. MYSHKIS, A. L. SKUBACHEVSKII, Moscow

Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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INTRODUCTION

This article is concerned with the boundary value problem (BVP)

$$\sum_{i=1}^{n} [a_i(x)\, u'(h_i(x))]' = F[u](x), \quad (\alpha < x < \beta), \tag{1}$$

$$u(x) = \varphi(x) \ (\alpha_1 \leqq x \leqq \alpha), \quad u(x) = \psi(x) \ (\beta \leqq x \leqq \beta_1) \tag{2}$$

where $1 \leqq n < \infty$, $-\infty < \alpha_1 < \alpha < \beta < \beta_1 < +\infty$.

Let $W_q^p[a, b]$ be the Sobolev spaces of functions $u\colon [a, b] \to \mathbb{R}$, absolutely continuous together with their derivatives of orders $< p$ and with $u^{(p)} \in L_q[a, b]$; we assume $a_i \in W_1^1[\alpha, \beta]$, $h_i \in W_1^2[\alpha, \beta]$, $h_i \neq 0$ and $h_i([\alpha, \beta]) \subseteqq [\alpha_1, \beta_1]$. The operator F in general is a nonlinear operator to be defined later.

The BVP of type (1), (2) were first mentioned in [1]. The definition of the solution in that paper was proposed by analogy to the $n = 1$ case. In the investigations which followed it was found that the case $n > 1$ has principal peculiarities. Therefore another definition of the solution analogous to the "weak" solutions was given in [2] (see also [3]). Both these definitions are equivalent for sufficiently smooth solutions (e.g. for $u \in W_1^2$), but differ for piecewise smooth solutions which appear, as a rule, even in the case when all given functions are analytic.

In this paper we study the properties of solutions defined analogously to the definition in [3]. The solutions in the sense of [1] will be called pseudosolutions. We investigate conditions of smoothness of solutions and of continuous dependence of solutions on the boundary functions. It enables us in some cases (in particular for linear equations with integer deviations and with conditions of regularity) to formulate the definition of a generalized solution continuously depending on the boundary functions. This together with some other properties makes it possible to distinguish solutions from pseudosolutions. It is the reason why solutions and not pseudosolutions appear in variational and some other problems. (See [4], [5].)

1. MAIN DEFINITIONS

We denote by $Q[a, b]$ the subspace of the space $W^1_\infty[a, b]$ containing the functions f with finite limits $f'(x^+), f'(x^-)$ and $f'(a^+), f'(b^-)$ with the norm $\|f\|_Q = \max\{\max_{x\in[a,b]} |f(x)|, \operatorname{vrai}\max_{x\in[a,b]} |f'(x)|\}$. We also denote $Q_b = Q[\alpha_1, \alpha] \times Q[\beta, \beta_1]$, $C_b = C[\alpha_1, \alpha] + C[\beta, \beta_1]$, $(W^p_q)_b = W^p_q[\alpha_1, \alpha] \times W^p_q[\beta, \beta_1]$. It is easy to prove that $Q[a, b]$ is closed in $W^1_\infty[a, b]$ (cf. [6]).

There are many ways to choose the solution space. Let $\{\varphi, \psi\} \in Q_b$ and $F: Q[\alpha_1, \beta_1] \to L_1[\alpha, \beta]$. Under a solution (it is of course a generalized solution) of equation (1) we understand any function $u \in Q[\alpha_1, \beta_1]$ satisfying for some $C_1 \in \mathbb{R}$ the equation

$$\sum_{i=1}^{n} a_i(x)\, u'(h_i(x)) - \int_\alpha^x F[u](s)\, ds = C_1 \tag{3}$$

for almost all $x \in (\alpha, \beta)$. (In our case this is equivalent to – "for all, except, may be, a denumerable set of values".) It is evident that for u to be a solution of (3) it is necessary and sufficient that $R[u] = \sum_{i=1}^{n} a_i(\cdot)\, u'(h_i(\cdot)) \in W^1_1[\alpha, \beta]$ and

$$[R[u](x)]' = F[u](x) \quad \text{for almost all} \quad x \in (\alpha, \beta)\,. \tag{4}$$

In particular, a function $u \in W^2_1[\alpha, \beta_1]$ is a solution of equation (1) iff it satisfies (1) for almost all $x \in (\alpha, \beta)$.

By integrating both parts of (3), we have

$$\sum_{i=1}^{n} \left[\frac{a_i(x)}{h'_i(x)}\, u(h_i(x)) - \int_\alpha^x \left(\frac{a_i(s)}{h'_i(s)}\right)' u(h_i(s))\right] ds - \tag{5}$$
$$- \int_\alpha^x (x - s)\, F[u](s)\, ds = C_1 x + C_2 \quad (\alpha \leqq x \leqq \beta)$$

with some $C_1, C_2 \in \mathbb{R}$ which, under the condition that $u \in Q[\alpha_1, \beta_1]$, may serve as another definition of a solution of equation (1), equivalent to that given above.

The differentiation of both parts of (3) is possible if we understand $u''(h_i(\cdot))$ as distributions on (α, β). Equation (1) may be written in the equivalent form

$$\sum_{i=1}^{n} a_i(x)\, h'_i(x)\, u''(h_i(x)) = F[u](x) - \sum_{i=1}^{n} a'_i(x)\, u'(h_i(x)) \quad (\alpha < x < \beta)\,.$$

2. CONDITIONAL PROPOSITIONS

The theorems proved in this section are of conditional type. We formulate a number of assumptions concerning the behaviour of solutions of equation (1) and some other equations and deduce the properties of smooth and generalized solutions of BVP (1), (2). In the subsequent sections we obtain sufficient conditions for these assumptions to be fulfilled.

In what follows the equation

$$(6) \qquad \sum_{i=1}^{n} a_i(x)\,(\operatorname{sgn} h_i')\, z(h_i(x)) = 0 \quad (\alpha < x < \beta)$$

plays an essential role. The boundary conditions for z are given on $(\alpha_1, \alpha] \cup [\beta, \beta_1)$ and the solution $z\colon (\alpha_1, \beta_1) \to \mathbb{R}$ has to satisfy (6) for all $x \in (\alpha, \beta)$.

Assumption A-1. The solution z of equation (6) satisfying zero boundary conditions and the condition

$$(7) \qquad \forall \varepsilon > 0 \quad \text{the set} \quad \{x\colon x \in (\alpha, \beta);\ |z(x)| \geqq \varepsilon\} \quad \text{is finite},$$

is identically equal to zero.

Theorem 1. *Let* A-1 *hold,* $\{\varphi, \psi\} \in C_b^1$ *and let the solution* u *of BVP* (1), (2) *satisfy the conditions*

$$(8) \qquad u'(\alpha^+) = \varphi'(\alpha^-), \quad u'(\beta^-) = \psi'(\beta^+).$$

Then $u \in C^1[\alpha_1, \beta_1]$.

Proof. Let $\Delta u'(x) = u'(x^+) - u'(x^-)$. Since $u \in Q[\alpha_1, \beta_1]$, this function is defined for all $x \in (\alpha_1, \beta_1)$ and $z = \Delta u'$ has the property (7). From (3) we obtain

$$(9) \qquad \sum_{i=1}^{n} a_i(x)\,(\operatorname{sgn} h_i')\, \Delta u'(h_i(x)) = 0$$

for all $x \in (\alpha, \beta)$. Besides, from the conditions on $\{\varphi, \psi\}$ and from (8) we have $\Delta u'(x) = 0$ on $(\alpha_1, \alpha] \cup [\beta, \beta_1)$. Therefore A-1 implies $\Delta u'(x) \equiv 0$.

Corollary 1. *If in addition* $h_i(x) \geqq \alpha\ (\leqq \beta)$ *and the conditions on* φ *(on* ψ*) in Theorem* 1 *are omitted, then* $u \in C^1[\alpha, \beta_1]$ $(C^1[\alpha_1, \beta])$.

Proof. Indeed, it is possible to change the function φ (ψ) and fulfil the omitted conditions on $[\alpha_1, \alpha]$ ($[\beta, \beta_1]$) without affecting the solution on $[\alpha, \beta_1]$ ($[\alpha_1, \beta]$).

Assumption A-2. If $z \in C[\alpha_1, \beta_1]$ is a solution of the equation

$$(10) \qquad \sum_{i=1}^{n} a_i(x)\, z(h_i(x)) = f(x) \quad (\alpha < x < \beta)$$

with zero boundary conditions and $f \in W_1^1[\alpha, \beta]$, then $z \in W_1^1[\alpha, \beta]$.

Corollary 2. *Let* A-1 *and* A-2 *hold,* $\{\varphi, \psi\} \in (W_1^2)_b$ *and let the solution* u *of BVP* (1), (2) *satisfy* (8). *Then* $u \in W_1^2[\alpha_1, \beta_1]$.

Proof. Indeed, it follows from Theorem 1 and (3) that $z = u' - v$, where $v \in W_1^1[\alpha_1, \beta_1]$ is any continuation of φ' and ψ', satisfies A-2.

Assumption A-3. The solution $u_{\varphi,\psi}$ of BVP (1), (2) exists and is unique for any given $\{\varphi, \psi\} \in C_b^1$, and $\{\varphi, \psi\} \to u_{\varphi,\psi}$ is a continuous mapping of C_b^1 into $Q[\alpha_1, \beta_1]$.

Let us denote by S the set of $\{\varphi, \psi\} \in C_b^1$ such that $\varphi'(\alpha^-) = u'_{\varphi,\psi}(\alpha^+)$, $u'_{\varphi,\psi}(\beta^-) = \psi'(\beta^+)$. If A-3 holds then S is a closed subset of C_b^1. It follows from Theorem 1 that if A-1 holds then S coincides with the set of $\{\varphi, \psi\}$ for which $u_{\varphi,\psi} \in C^1[\alpha_1, \beta_1]$. In some cases it is possible, under the assumption A-3, to extend the conditions for the existence of solutions of BVP (1), (2) to a class of boundary functions wider than C_b^1 by introducing another definition of the generalized solution.

Assumption A-4. For any $\{\varphi, \psi\} \in C_b$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(\forall\{\varphi^i, \psi^i\} \in C_b^1\colon \|\{\varphi^i - \varphi, \psi^i - \psi\}\|_{C_b} < \delta,\ i = 1, 2) \Rightarrow$$
$$\Rightarrow \|u_{\varphi^1,\psi^1} - u_{\varphi^2,\psi^2}\|_{C[\alpha_1,\beta_1]} < \varepsilon\,.$$

Theorem 2. *Let* A-3 *and* A-4 *hold. Then the mapping* $C_b^1 \mapsto Q[\alpha_1, \beta_1]$ *defined in* A-3 *may extended to the continuous mapping from* C_b *to* $C[\alpha_1, \beta_1]$.

Proof. For any $\{\varphi, \psi\} \in C_b$ there is a sequence $\{\varphi^k, \psi^k\} \in C_b^1$ approximating $\{\varphi, \psi\}$ in C_b. That proves the theorem.

The mapping defined by Theorem 2 is denoted as above $\{\varphi, \psi\} \mapsto u_{\varphi,\psi}$ and the function u is called a *c-generalized solution* of BVP (1), (2) for $\{\varphi, \psi\} \in C_b$. Thus, if A-4 holds, then there exists a unique c-generalized solution and it depends continuously on $\{\varphi, \psi\}$. By $\{\varphi, \psi\} \in C_b^1$ the c-generalized solution coincides with the above defined solution of BVP (1), (2). By analogy to Theorem 2 we can prove Theorems 3 and 4.

Theorem 3. *Let* A-3 *and* A-4 *hold and let it be possible to extend the operator*

$$u \mapsto \int_\alpha^{\cdot} (\cdot - s)\, F[u](s)\, \mathrm{d}s\colon Q[\alpha_1, \beta_1] \to C[\alpha, \beta]$$

to a continuous operator $C[\alpha_1, \beta_1] \to C[\alpha, \beta]$. *Then every c-generalized solution of BVP* (1), (2) *satisfies equation* (5).

Assumption A-5. If any two solutions of equation (5) coincide on $(\alpha_1, \alpha] \cup [\beta, \beta_1)$, then they coincide on (α, β).

Theorem 4. *Let the conditions of Theorem* 3 *and* A-5 *hold. Then any solution from* $C[\alpha_1, \beta_1]$ *of equation* (5) *is a c-generalized solution of BVP* (1), (2).

Pseudosolutions. In accordance with the definition in [1] for given $\{\varphi, \psi\} \in (W_1^2)_b$, a function $u \in Q[\alpha_1, \beta_1]$ such that its restriction on $[\alpha, \beta]$ belongs to

$W_1^2[\alpha, \beta]$ and satisfying equation (1) for almost all $x \in (\alpha, \beta)$ and boundary conditions (2) will be called a pseudosolution of BVP (1), (2).

Further in this section we suppose that A-1, A-2, A-3 hold if we insert $(W_1^2)_b$ instead of C_b^1.

Assumption A-6. Suppose $\forall\{\varphi, \psi\} \in (W_1^2)_b$, there exists a unique pseudosolution of BVP (1), (2) $\tilde{u}_{\varphi,\psi}$, and the mapping $\{\varphi, \psi\} \mapsto \tilde{u}_{\varphi,\psi}$ is a continuous mapping from $(W_1^2)_b$ to $Q[\alpha_1, \beta_1]$.

It follows then from Corollary 2 that for $\{\varphi, \psi\} \in S \cap (W_1^2)_b$ we have $\tilde{u}_{\varphi,\psi} = u_{\varphi,\psi}$. On the other hand, in general we have $\tilde{u}_{\varphi,\psi} \neq u_{\varphi,\psi}$ for $\{\varphi, \psi\} \in (W_1^2)_b \setminus S$. For example, this will be the case if

$$[u'_{\varphi,\psi}(\alpha^+) - \varphi'(\alpha^-)] \sum_{h_i(x)=\alpha} a_i(x) \operatorname{sgn} h'_i +$$
$$+ [\psi'(\beta^+) - u'_{\varphi,\psi}(\beta^-)] \sum_{h_i(x)=\beta} a_i(x) \operatorname{sgn} h'_i \not\equiv 0 \quad (\alpha < x < \beta).$$

The proof can be obtained by way of contradiction.

The essential difference between solutions and pseudosolutions is the impossibility of extending the latter in the general case to the boundary functions from C_b preserving the continuity of the mapping $\{\varphi, \psi\} \mapsto \tilde{u}_{\varphi,\psi}$ as a mapping from C_b to $C[\alpha_1, \beta_1]$.

Let us formulate a simple result in this direction.

Theorem 5. *Let* A-3, A-4 *and*

$$(12) \qquad \exists x_1, x_2 \in (\alpha, \beta): \sum_{h_i(x_1)=\alpha} a_i(x_1) \operatorname{sgn} h'_i \neq 0,$$
$$\sum_{h_i(x_1)=\beta} a_i(x_1) \operatorname{sgn} h'_i = 0, \quad \sum_{h_i(x_2)=\alpha} a_i(x_2) \operatorname{sgn} h'_i = 0, \quad \sum_{h_i(x_2)=\beta} a_i(x_2) \operatorname{sgn} h'_i \neq 0$$

hold and let $S \cap (W_1^2)_b$ be dense in C_b. Then the mapping $\{\varphi, \psi\} \mapsto \tilde{u}_{\varphi,\psi}$ ($\{\varphi, \psi\} \in (W_1^2)_b$) as a mapping from C_b to $C[\alpha_1, \beta_1]$ is discontinuous at every point of $\{\varphi, \psi\} \in (W_1^2)_b \setminus S$.

Proof. Suppose that the assertion is false for a pair $\{\varphi, \psi\} \in (W_1^2)_b \setminus S$. Choose a sequence $\{\varphi^i, \psi^i\} \in S \cap (W_1^2)_b$, $\{\varphi^i, \psi^i\} \to \{\varphi, \psi\}$, $i \to \infty$ in C_b. Then our assumption yields $u_{\varphi^i,\psi^i} \to \tilde{u}_{\varphi,\psi}$, $i \to \infty$ in $C[\alpha_1, \beta_1]$, but A-4 implies $u_{\varphi^i,\psi^i} \to u_{\varphi,\psi}$, $i \to \infty$ in $C[\alpha_1, \beta_1]$ and we obtain that $\tilde{u}_{\varphi,\psi} = u_{\varphi,\psi}$ which contradicts (12) for $\{\varphi, \psi\} \bar{\in} S$.

This also implies that $(12) \Rightarrow (\{\varphi, \psi\} \in S \cap (W_1^2)_b \Leftrightarrow \tilde{u}_{\varphi,\psi} = u_{\varphi,\psi})$.

3. EQUATIONS WITH A MAJORANT

In this and the subsequent sections we obtain sufficient conditions for the assumptions A-1 – A-6 to be fulfilled. These conditions are to be used together with Theorems 1 – 5.

Theorem 6. *Let* $\forall\gamma \in (\alpha, \beta)$, $\exists\gamma_1 \in (\alpha, \beta)$, $i_1 \in \{1, \ldots, n\}$:

$$h_{i_1}(\gamma_1) = \gamma, \sum_{i \neq i_1, h_i(\gamma_1) \in (\alpha,\beta)} |a_i(\gamma_1)| < |a_{i_1}(\gamma_1)| .$$

Then A-1 *holds. On the stronger supposition that* $\exists i_1 \in \{1, \ldots, n\}$:

$$h_{i_1}([\alpha, \beta]) \supseteq [\alpha, \beta] , \quad \min_{[\alpha,\beta]} |a_{i_1}| > 0 , \quad \sum_{i \neq i_1} \max \left|\frac{a_i}{a_{i_1}}\right| < 1$$

the assumption A-2 *holds, too.*

Proof. If the weaker supposition is true and equation (6) has the solution $z(x) \not\equiv 0$ with properties mentioned in A-1, then denoting $\gamma = \max \{x: |z(x)| = \max |z|\}$ and putting $x = \gamma_1$ in (6), we obtain a contradiction.

Let now the stronger supposition of the theorem and A-2 hold. Note that equation (10) is satisfied. For $x = \alpha, \beta$ let us rewrite it in the form

$$z(h_{i_1}(x)) = \sum_{i \neq i_1} b_i(x)\, z(h_i(x)) + g(x) \quad (\alpha \leqq x \leqq \beta) .$$

In virtue of our assumptions we have $h_{i_1}([\alpha, \beta]) \supseteq [\alpha, \beta]$ for all b_i and $g \in W_1^1[\alpha, \beta]$, $b = \sum_{i \neq i_1} \max |b_i| < 1$, $z \in C[\alpha_1, \beta_1]$, $z(x) \equiv 0$ $(x \in [\alpha_1, \alpha] \cup [\beta, \beta_1])$ and we have to prove that $z \in W_1^1[\alpha, \beta]$.

In $W_1^1[\alpha_1, \beta_1]$ let us introduce the equivalent norm

$$\|u\|_1 = \max \left\{ \max_{[\alpha_1,\beta_1]} |u|, \quad p \int_{\alpha_1}^{\beta_1} |u'(x)|\, dx \right\} ,$$

where $p > 0$ will be chosen later. Define the operator A: $C[\alpha, \beta_1] \to C[\alpha_1, \beta_1]$ by the formula

$$(Au)(x) = \begin{cases} \sum_{i \neq i_1} b_i(h_{i_1}^{-1}(x))\, u(h_i(h_{i_1}^{-1}(x))) + g(h_{i_1}^{-1}(x)) & (\alpha \leqq x \leqq \beta) , \\ (Au)(\alpha) & (\alpha_1 \leqq x \leqq \alpha) , \\ (Au)(\beta) & (\beta \leqq x \leqq \beta_1) . \end{cases}$$

This operator maps $W_1^1[\alpha_1, \beta_1]$ in to itself and $Az = z$. For any $u_1, u_2 \in W_1^1[\alpha_1, \beta_1]$ we have

$$\|Au_1 - Au_2\|_1 = \max \left\{ \max_{[\alpha,\beta]} \left| \sum_{i \neq i_1} b_i(h_{i_1}^{-1}(x)) \left[u_1(h_i(h_{i_1}^{-1}(x))) - u_2(h_i(h_{i_1}^{-1}(x)))\right]\right| , \right.$$

$$p \int_\alpha^\beta \left| \sum_{i \neq i_1} [b_i(h_{i_1}^{-1}(x)) (u_1(h_i(h_{i_1}^{-1}(x))) - u_2(h_i(h_{i_1}^{-1}(x)))]' \, dx \right. \leqq$$

$$\leqq C \max \left\{ b \|u_1 - u_2\|_1, \; p \int_\alpha^\beta \sum_{i \neq i_1} \right| [b_i(h_{i_1}^{-1}(x))]' \, dx \|u_1 - u_2\|_1 +$$

$$+ p \int_{h_i(h_{i_1}^{-1}(\alpha))}^{h_i(h_{i_1}^{-1}(\beta))} |b_i(h_{i_1}^{-1}(s)) [u_1(s) - u_2(s)]'| \, ds \Big\} \leqq$$

$$\leqq \max \left\{ b\|u_1 - u_2\|_1, \left(p \int_\alpha^\beta \sum_{i \neq i_1} |[b_i(h_{i_1}^{-1}(x))]'| \, dx + b \right) \|u_1 - u_2\|_1 \right\}.$$

Therefore, if $p > 0$ is sufficiently small, then the operator A is a contraction operator not only in the space $C[\alpha_1, \beta_1]$ but also in the space $W_1^1[\alpha_1, \beta_1]$.

Thus the function $z \in C[\alpha_1, \beta_1]$ satisfying the equation $Az = z$ belongs to the space $W_1^1[\alpha_1, \beta_1]$. Hence the proof is complete.

Now consider the case when equation (1) has the form

$$[u'(x)]' + \sum_{i=2}^{n} [a_i(x) \, u'(h_i(x))]' = F[u](x) \quad (\alpha < x < \beta) \tag{13}$$

with the same conditions on the given functions as in Sec. 1.

Denoting $a_i/h_i' = b_i$ and transforming (13) to the form (5) we obtain

$$u(x) = \sum_{i=2}^{n} \left[-b_i(x) \, u(h_i(x)) + \int_\alpha^x b_i'(x) \, u(h_i(s)) \, ds \right] +$$
$$+ \int_\alpha^x (x - s) \, F[u](s) \, ds + C_1 x + C_2 .$$

Finding the values of C_1 and C_2 by substituting $x = \alpha$ and $x = \beta$, we get the equation

$$u(x) = \sum_{i=2}^{n} \left[-b_i(x) \, u(h_i(x)) + \int_\alpha^x b_i'(s) \, u(h_i(s)) \, ds \right] + \tag{14}$$
$$+ \int_\alpha^x (x - s) \, F[u](s) \, ds + \Big\{ u(\beta) + \sum_{i=2}^{n} \Big[b_i(\beta) \, u(h_i(\beta)) -$$
$$- \int_\alpha^\beta b_i'(s) \, u(h_i(s)) \, ds \Big] - \int_\alpha^\beta (\beta - s) \, F[u](s) \, ds \Big\} \frac{x - \alpha}{\beta - \alpha} -$$
$$- [u(\alpha) + \sum_{i=2}^{n} b_i(\alpha) \, u(h_i(\alpha))] \frac{x - \beta}{\beta - \alpha} \quad (\alpha \leqq x \leqq \beta) .$$

For a given pair $\{\varphi, \psi\} \in C_b^1$ we denote by $M_{\varphi,\psi}$ the set of functions from $Q[\alpha_1, \beta_1]$ satisfying conditions (2). Define an operator $A_{\varphi,\psi}: M_{\varphi,\psi} \to M_{\varphi,\psi}$ in such a way that $A_{\varphi,\psi}[u](x)$ for $\alpha < x < \beta$ is equal to the right hand part of (14), and for $\alpha_1 \leqq x \leqq \alpha$ and $\beta \leqq x \leqq \beta_1$ is equal to the functions $\varphi(x)$ and $\psi(x)$, respectively.

Suppose that for some $L, L_1 \geqq 0$ and for all $u_1, u_2 \in Q[\alpha_1, \beta_1]$,

$$\|\Delta F[u]\|_{L_1[\alpha_1,\beta_1]} \leqq L\|\Delta u\|_{Q[\alpha_1,\beta_1]} , \tag{15}$$

$$\left\| \int_\alpha^x (x - s) \, \Delta F[u](s) \, ds \right\|_{C[\alpha,\beta]} \leqq L_1 \|\Delta u\|_{C[\alpha_1,\beta_1]} , \tag{16}$$

where $\Delta u = u_1 - u_2$, $\Delta F[u] = F[u_1] - F[u_2]$.

Considering the expressions for $\Delta A_{\varphi,\psi}[u](x)$, $\Delta[A_{\varphi,\psi}[u](x)]'$ we can verify that if $\max_{i,x}|a_i(x)|$, $\max_{i,x}|b_i(x)|$, $\max_i \int_\alpha^\beta |b_i'(x)|\,dx$ $(i \geqq 2)$ and the constant L in the inequality (15) are sufficiently small, then the operator $A_{\varphi,\psi}$ is a contraction operator. This proves that the solution of BVP (13), (2) exists and is unique. By using equation (14) and the equation obtained by differentiating (14) we can prove that the mapping $\{\varphi, \psi\} \mapsto u_{\varphi,\psi}$ is a continuous mapping from C_b^1 to $Q[\alpha_1, \beta_1]$. In an analogous way it is possible to prove that if the constant L_1 in inequality (16) is sufficiently small, then all conditions of Theorem 4 are fulfilled. Thus we conclude that the following theorem is true.

Theorem 7. *If the operator F in* (13) *satisfies conditions* (15) *and* (16) *and all constants $L, L_1, \max_{x,i}[a_i]$, $\max_{x,i}[b_i]$ and $\max_i \int_\alpha^\beta |b_i'(x)|\,dx$ $(i \geqq 2)$ are sufficiently small, then all conditions of Theorem* 4 *are fulfilled.*

4. THE EQUATION WITH FINITE TRANSITIVITY PROPERTY

Theorem 8. *Let* $\forall \gamma \in (\alpha, \beta)$, $\exists M^\gamma = \{x_1^\gamma, \ldots, x_{k_\gamma}^\gamma\} \subset (\alpha, \beta)$: $\gamma \in M^\gamma$: $\bigcup_i h_i(M^\gamma) \subset$ $\subset M^\gamma \cup [\alpha_1, \alpha] \cup [\beta, \beta_1]$ *and*

$$\det\Big(\sum_{i:\,h_i(x_p^\gamma)=x_q^\gamma} a_i(x_p^\gamma)\operatorname{sgn} h_i'\Big) \neq 0,\; p, q = 1, \ldots, k_\gamma\,. \tag{17}$$

Then A-1 *holds.*

Proof. The assertion of the theorem we immediately obtain if we put $x = x_1^\gamma, \ldots, x_{k_\gamma}^\gamma$ in (6) and consider the resulting equalities as a system of equations for $z(x_1^\gamma), \ldots, z(x_{k_\gamma}^\gamma)$.

Remark. The property of existence for any $\gamma \in (\alpha, \beta)$ of the set M^γ having all the properties mentioned above, except, may be, (17), is naturally called the finite transitivity of equation (1).

By analogy to Theorem 8 it is possible to formulate a sufficient condition for the validity of assumption A-2 ,which is a nondegeneracy condition on a matrix-function constructed with the use of the functions a_i and h_i.

These conditions assume the simplest form when $h_i(x) = x + h_i$, where all h_i are commeasurable. In this case we may suppose without loss of generality that all h_i's are integers $(h_i \in \mathbb{Z})$ and write equation (1) in the form

$$\sum_{i=-m}^{m} [a_i(x)\,u'(x+i)]' = F[u](x) \quad (\alpha < x < \beta)\,, \tag{18}$$

where $m = [\beta - \alpha]$, $\alpha_1 = \alpha - m$, $\beta_1 = \beta + m$. Denote now $\delta = \beta - \alpha - m$. Condition (17) for equation (18) has the form

$$(19)\qquad \det\{a_{q-p}(x+p)\} \neq 0 \quad (\alpha < x < \beta),$$

where $p, q \in \mathbb{Z}$ and $x + p$, $x + q \in (\alpha, \beta)$, $a_i(x) \equiv 0$ for $|i| > m$. The left hand part of (19) is 1-periodic and absolutely continuous with the exception, in general, of jumps at $x = \alpha + i$, $x = \beta - i$ $(i = 1, \ldots, m)$. If $a_i(x) = \text{const}$, then condition (19) has the form of two inequalities

$$(20)\qquad \det\{a_{q-p}\}_{p,q=0}^{m} \neq 0, \quad \det\{a_{q-p}\}_{p,q=0}^{m-1} \neq 0.$$

Inequality (19) for $\alpha \leqq x \leqq \beta$ and, in the case of $a_i(x) = \text{const}$ inequality (20), is a sufficient condition for assumption A-2 to be fulfilled.

5. LINEAR EQUATIONS WITH INTEGER DEVIATIONS OF ARGUMENT

A simple class of problems, for which it is easy to formulate conditions for the validity of assumptions A-3 to A-6, arises by studying the equations of the form

$$(21)\qquad \sum_{i=-m}^{m} [a_i(x)\, u'(x+i)]' = \sum_{i=-m}^{m} [b_i(x)\, u'(x+i) + c_i(x)\, u(x+i)] + f(x)$$

with boundary conditions (2) and the condition

$$(22)\qquad \det\{a_{q-p}(x+p)\} \neq 0 \quad (\alpha \leqq x \leqq \beta),$$

which is sufficient for the validity of assumptions A-1 and A-2. In correspondence with Sec. 1 we assume that all $a_i \in W_1^1[\alpha, \beta]$, $b_i, c_i, f \in L_1[\alpha, \beta]$. For the moment we suppose that $\{\varphi, \psi\} \in (W_1^2)_b$ and the number $\beta - \alpha$ is not an integer. Define vector-functions u^1, u^2 with coordinates

$$(23)\qquad \begin{aligned} u_i^1(x) &= u(x + \alpha + i - 1) \quad (0 \leqq x \leqq \delta,\ i = 1, \ldots, m+1), \\ u_i^2(x) &= u(x + \alpha + i - 1) \quad (\delta \leqq x \leqq 1,\ i = 1, \ldots, m). \end{aligned}$$

By analogy with (3) and from (23) it follows that these functions belong to the space W_1^2. Therefore on each of the intervals $(\alpha, \alpha + \delta)$, $(\alpha + \delta, \alpha + 1)$, $(\alpha + 1, \alpha + 1 + \delta), \ldots, (\beta - \delta, \beta)$ it is possible to use the formula of differentiation of a product for the left hand part of equation (21). Using also (22) and (23), we obtain two systems of equations

$$(24)\qquad \begin{aligned} u_i^{1''} &= \sum_{j=1}^{m+1} [d_{ij}^1(x)\, u_j^{1'} + e_{ij}^1(x)\, u_j^1] + g_i^1(x) \quad (0 \leqq x \leqq \delta,\ i = 1, \ldots, m+1), \\ u_i^{2''} &= \sum_{j=1}^{m} [d_{ij}^2(x)\, u_j^{2'} + e_{ij}^2(x)\, u_j^2] + g_i^2(x) \quad (\delta \leqq x \leqq 1,\ i = 1, \ldots, m). \end{aligned}$$

Here all d_{ij}^r. e_{ij}^r, g_{ij}^r are integrable and the functions g_i^r are constructed with the use of the boundary functions φ and ψ. Additional conditions on (24) arise from the continuity conditions on u at the points $\alpha, \alpha+\delta, \alpha+1, \ldots, \beta-\delta, \beta$ and the relations for the jumps of u' at the points $\alpha+\delta, \ldots, \beta-\delta$:

$$(25)\qquad u_1^1(0) = \varphi(\alpha)\,, \quad u_i^1(\delta) = u_i^2(\delta)\,, \quad u_i^2(1) = u_{i+1}^1(0) \quad (i = 1, \ldots, m)\,,$$
$$u_{m+1}^1(\delta) = \psi(\beta)\,,$$

$$(26)\qquad \sum_{j=-i}^{m-i} a_j(\alpha + i)\,[u_{i+j+1}^{1\prime}(0) - u_{i+j}^{2\prime}(1)] = 0 \quad (i = 1, \ldots, m)\,,$$
$$\sum_{j=-i+1}^{m-i+1} a_j(\alpha + i - 1 + \delta)\,[u_{i+j}^{2\prime}(\delta) - u_{i+j}^{1\prime}(\delta)] = 0 \quad (i = 1, \ldots, m)\,,$$

where we put $u_0^{2\prime}(1) = \varphi'(\alpha)$, $u_{m+1}^2(\delta) = \psi'(\beta)$.

The equalities (25) and (26) can be written in the short form

$$(27)\qquad l_i^1[u^1] + l_i^2[u^2] = \sum_{j=1}^{m+1} [a_{ij}^1 u_j^1(0) + b_{ij}^1 u_j^1(\delta)] +$$
$$+ \sum_{j=1}^{m} [a_{ij}^2 u_j^2(\delta) + b_{ij}^2 u_j^2(1)] = e_i \quad (i = 1, \ldots, 2m + 2)\,,$$
$$\lambda_i^1[u^1] + \lambda_i^2[u^2] = \sum_{j=1}^{m+1} [\alpha_{ij}^1 u_j^{1\prime}(0) + \beta_{ij}^1 u_j^{1\prime}(\delta)] +$$
$$+ \sum_{j=1}^{m} [\alpha_{ij}^2 u_j^{2\prime}(\delta) + \beta_{ij}^2 u_j^{2\prime}(1)] = \varepsilon_i \quad (i = 1, \ldots, 2m)\,.$$

Suppose that the general solution of each of the systems (24) has the form

$$(28)\qquad u_i^1 = \sum_{j=1}^{2m+2} A_j u_i^{j1}(x) + S_i^1(x) \quad (i = 1, \ldots, m + 1)\,,$$
$$u_i^2 = \sum_{j=1}^{2m} B_j u_i^{j2}(x) + S_i^2(x) \quad (i = 1, \ldots, m)\,,$$

where A_i, B_i are arbitrary constants. The main regularity condition for the BVP considered is

$$(29)\qquad \det \begin{bmatrix} l_1^1[u^{11}] & \ldots & l_1^1[u^{2m+2,1}] & l_1^2[u^{12}] & \ldots & l_1^2[u^{2m,2}] \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ l_{2m+2}^1[u^{11}] & \ldots & l_{2m+2}^1[u^{2m+2,1}] & l_{2m+2}^2[u^{12}] & \ldots & l_{2m+2}^2[u^{2m,2}] \\ \lambda_1^1[u^{11}] & \ldots & \lambda_1^1[u^{2m+2,1}] & \lambda_1^2[u^{12}] & \ldots & \lambda_1^2[u^{2m,2}] \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_{2m}^1[u^{11}] & \ldots & \lambda_{2m}^1[u^{2m+1,1}] & \lambda_{2m}^2[u^{12}] & \ldots & \lambda_{2m}^2[u^{2m,2}] \end{bmatrix} \neq 0\,.$$

If (29) holds, then inserting (28) in (29) we can determine A_i and B_i in a unique way. It means that BVP (21), (2) has a unique solution.

Notice that the determinant in (29) depends only on a_i, b_i, c_i and not on the functions φ, ψ and f. Therefore condition (29) is equivalent to the requirement that the corresponding homogeneous BVP has no nontrivial solutions. Notice also that the numbers A_i, B_i depend continuously on $\{\varphi, \psi\}$ as on an element of C_b^1 and on the set of S_i^r as elements of C^1, i.e. on the set of $\int_0^\bullet g_i^r(s)\,ds$ as elements of C.

Let now $\{\varphi, \psi\} \in C_b^1$. Construct a sequence $\{\varphi_p, \psi_p\} \in (W_1^2)_b$, $\{\varphi_p, \psi_p\} \to \{\varphi, \psi\}$, $p \to \infty$ in C_b^1, and the corresponding solutions u_p. All sequences $\{\int_0^\bullet g_{ip}^r(s)\,ds\}$ converge in C and therefore all sequences $\{u_{ip}^r\}$ converge in C^1. By integrating both parts of equation (21) and passing to the limit as $p \to \infty$ we obtain that the function $u = \lim u_p$ is a solution of BVP (21), (2). The continuity of the mapping $\{\varphi, \psi\} \mapsto$ $\mapsto u_{\varphi,\psi}$ as a mapping from C_b^1 to $Q[\alpha - m, \beta + m]$ is proved analogously to the proof of existence of a solution. Uniqueness follows from condition (29). Thus A-3 holds.

For verifying A-4 we suppose in addition that in (21) all $b_i \in W_1^1[\alpha, \beta]$. If A-4 does not hold, then we can construct, as made above, a sequence of boundary functions $\{\varphi_p, \psi_p\} \in (W_1^2)_b$ having the property $\|\{\varphi_p, \psi_p\}\|_{C_b} \to 0$. At the same time the corresponding solutions u_p of (21) with $f(x) \equiv 0$ satisfy $\max_{[\alpha,\beta]} |u_p| = 1$. In accordance with (23) denote

$$u_{ip}^r(0) = M_{ip}^r\,, \quad u_{ip}^{r\prime}(0) = N_{ip}^r$$

and rewrite equations (24), after double integration of both parts of the equations and subsequent integration by parts, in the form

$$(30) \qquad u_{ip}^1(x) = M_{ip}^1 + N_{ip}^1 x + \sum_{j=1}^{m+1} \int_0^x m_{ij}^1(x, s)\, u_{jp}^1(s)\,ds + h_{ip}^1(x)$$

$$(0 \leqq x \leqq \delta;\ i = 1, \ldots, m + 1)\,,$$

$$u_{ip}^2(x) = M_{ip}^2 + N_{ip}^2 x + \sum_{j=1}^{m} \int_0^x m_{ij}^2(x, s)\, u_{jp}^2(s)\,ds + h_{ip}^2(x)$$

$$(\delta \leqq x \leqq 1;\ i = 1, \ldots, m)\,.$$

Here our conditions guarantee that all $h_{ip}^r \to 0$, $p \to \infty$ in C and all $\{u_{ip}^r\}$ and $\{M_{ip}^r\}$ are bounded. It follows that all $\{N_{ip}^r\}$ are also bounded and therefore, passing to a subsequence, we obtain that $M_{ip}^r \to M_{i\omega}^r$, $N_{ip}^r \to N_{i\omega}^r$. Then (30) yields that $u_{ip}^r \to u_{i\omega}^r$ in C, i.e. $u_p \to u_\omega$ in $C[\alpha_1, \beta_1]$. It means that $u_\omega \in Q[\alpha - m, \beta + m]$, $u_\omega(x) \equiv 0$ on $[\alpha - m, \alpha] \cup [\beta, \beta + m]$ and $\max_{[\alpha,\beta]} |u_\omega(x)| = 1$. On the other hand, the equation for u_p analogous to (5) is

$$(31) \qquad \sum_{i=-m}^{m} a_i(x)\, u_p(x + i) = \sum_{i=-m}^{m} \int_\alpha^x \{a_i'(s) + b_i(s) +$$

$$+ (x - s)\left[C_i(s) - b_i'(s)\right]\}\, u_p(s + i)\,ds + A_p x + B_p \quad (\alpha \leqq x \leqq \beta)$$

where, as above, $A_p \to A_\omega$, $B_p \to B_\omega$.

Therefore, passing to the limit as $p \to \infty$ we obtain that u_ω is a solution of equation (21) and $u_\omega(x) \equiv 0$ since the boundary functions are equal to zero. This contradiction proves that A-4 is fulfilled.

It is evident that the remaining requirements of Theorem 4 with $\forall b_i \in W_1^1[\alpha, \beta]$ are fulfilled.

If we consider the pseudosolutions, it is necessary to change (26) to

$$u_i^{1'}(\delta) = u_i^{2'}(\delta), \quad u_i^{2'}(1) = u_{i+1}^{1'}(0) \quad (i = 1, \ldots, m).$$

Condition (29) will be changed correspondingly, giving necessary and sufficient conditions for A-6 to be fulfilled.

The set S (see Sec. 2) is defined by the equalities

$$u_1^{1'}(0) = \varphi'(\alpha), \quad u_{m+1}^{1'}(\delta) = \psi'(\beta).$$

By using (28) and deciphering the meaning of the notations introduced we transform these equalities to the form

$$\sum_{i=0}^{m} [\gamma_i^r \varphi(\alpha - i + a^r) + \delta_i^r \varphi'(\alpha - i + a^r) + \xi_i^r \psi(\beta + i + b^r) + \eta_i^r \psi'(\beta + i + b^r)] +$$
$$+ \int_0^m [\theta^r(x) \varphi(\alpha - x) + \varkappa^r(x) \psi(\beta + x)] \, dx = \mu^r \quad (r = 1, 2),$$

where $a^1 = 0$, $a^2 = \delta - 1$, $b^1 = -\delta$, $b^2 = 0$, the coefficients and (integrable) kernels on the left side are defined by the coefficients of equation (21) and the numbers μ^r also by the function f.

If $\beta - \alpha = m \in \mathbb{N}$ then the proof is simpler. Functions u_i^1 vanish, the system (24) reduces to its second part (with $\delta = 0$) and equalities (26) transform to

$$u_1^2(0) = \varphi(\alpha), \quad u_i^2(1) = u_{i+1}^2(0) \quad (i = 1, \ldots, m - 1), \quad u_m^2(1) = \psi(\beta),$$
$$\sum_{j=-i}^{m-i} a_j(\alpha + i) [u_{i+j+1}^{2'}(0) - u_{i+j}^{2'}(1)] = 0 \quad (i = 1, \ldots, m - 1)$$

while condition (29) transforms to

$$\det \begin{bmatrix} l_1^2[u^{12}] & \ldots & l_1^2[u^{2m,2}] \\ \ldots & \ldots & \ldots \\ l_{m+1}^2[u^{12}] & \ldots & l_{m+1}^2[u^{2m,2}] \\ \lambda_1^2[u^{12}] & \ldots & \lambda_1^2[u^{2m,2}] \\ \ldots & \ldots & \ldots \\ \lambda_{m-1}^2[u^{12}] & \ldots & \lambda_{m-1}^2[u^{2m,2}] \end{bmatrix} \neq 0 \qquad (32)$$

with the obvious meaning of notations. Further consideration are the same as in the case $\delta > 0$.

We formulate the main result as a theorem.

Theorem 9. *If for equation* (21) *with* $a_i \in W_1^1[\alpha, \beta]$, $b_i, c_i, f \in L_1[\alpha, \beta]$ *inegualities* (22) *and* (29) ((32) *if* $\beta - \alpha$ *is an integer*) *hold, then* A-1 *and* A-3 *are fulfilled. If in addition* $b_i \in W_1^1[\alpha, \beta]$, *then all conditions of Theorem* 4 *are fulfilled.*

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Authors' addresses: G. A. Kamenskii, MAI, Volokolamskoe 4, 125080 Moscow, A. D. D. Myhkis, A. L. Skubachevsii, MIIT, Obrazcova 15, 103055 Moscow, USSR.