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THE DIRICHLET PROBLEM AND WEIGHTED SPACES II

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5. STRONG SINGULARITIES AND STRONG DEGENERATION INSIDE Ω

5.1. Introductory remarks. In [1], we introduced the Sobolev weighted spaces $W^{k,p}(\Omega; S)$ and $W_0^{k,p}(\Omega; S)$ and used them to solve the Dirichlet problem for certain degenerate or singularly elliptic linear equations. Let us recall that

(i) $\mathscr{W}(\Omega)$ denotes the set of all weight functions on the open set $\Omega \subset \mathbb{R}^N$, i.e. the set of all measurable, a.e. in Ω positive and finite functions;

(ii) $\boldsymbol{M}(N, k)$ is the set of all N-dimensional multiindices $\alpha \in \mathbb{N}_0^N$ such that $|\alpha| \leqq k$;

(iii) $\boldsymbol{M}$ is a fixed subset of $\boldsymbol{M}(N, k)$ containing $\theta = (0, 0, \ldots, 0)$ and at least one other multiindex of the length k;

(iv) S is a collection of weight functions,

$$S = \{w_\alpha = w_\alpha(x),\ w_\alpha \in \mathscr{W}(\Omega),\ \alpha \in \boldsymbol{M}\}\,. \tag{5.1}$$

Assuming that, for $p > 1$,

$$w_\alpha^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for all} \quad \alpha \in \boldsymbol{M}\,, \tag{5.2}$$

we defined the space $W^{k,p}(\Omega; S)$ as the set of all measurable functions $u = u(x)$, $x \in \Omega$, which have on Ω distributional derivatives $D^\alpha u$ with $\alpha \in \boldsymbol{M}$ such that $D^\alpha u \in L^p(\Omega; w_\alpha)$, i.e.

$$\|D^\alpha u\|_{p,w_\alpha}^p = \int_\Omega |D^\alpha u(x)|^p\, w_\alpha(x)\, \mathrm{d}x < \infty\,. \tag{5.3}$$

Conditions (5.2) guarantee that this space is well defined and that, moreover, it is a Banach space under the norm

$$\|u\|_{k,p,S} = \Big(\sum_{\alpha \in \boldsymbol{M}} \|D^\alpha u\|_{p,w_\alpha}^p\Big)^{1/p}\,. \tag{5.4}$$

If the conditions (5.2) are violated then the above definition is meaningless, since it is not guaranteed that a function $u \in L^p(\Omega; w_\theta)$ has distributional derivatives on Ω at all (see the counterexample 1.7 in [2]) and that, even if these derivatives exist, they are regular distributions (see the counterexample 2.5 in [2]). These difficulties can be avoided if we assume that $D^\alpha u \in L^p(\Omega; w_\alpha) \cap L^1_{loc}(\Omega)$; in this case the above definition is meaningful but it is not guaranteed that the resulting linear normed space is complete (see the counterexample 1.12 in [2]).

5.2. A modified definition of the space $W^{k,p}(\Omega; S)$. Let us assume that some of the conditions (5.2) are violated, and let us denote, for $w \in \mathscr{W}(\Omega)$,

$$(5.5) \qquad M_p(w) = \left\{ x \in \Omega; \int_{\Omega \cap \mathscr{U}(x)} w^{-1/(p-1)}(y)\, \mathrm{d}y = \infty \text{ for every neighbourhood } \mathscr{U}(x) \text{ of } x \right\}.$$

As follows from examples in [2], the set

$$(5.6) \qquad \mathscr{B} = \bigcup_{\alpha \in \boldsymbol{M}} M_p(w_\alpha)$$

is the "bad" set which causes the noncompleteness of the corresponding weighted space $W^{k,p}(\Omega; S)$.

(Let us note that, obviously, $M_p(w_\alpha) = \emptyset$ if w_α satisfies condition (5.2).)

Let us denote

$$(5.7) \qquad \Omega_1 = \Omega - \mathscr{B}.$$

Since $\mathscr{B}$ is closed in Ω (see Lemma 3.2 in [2]), Ω_1 is an open set in $\mathbb{R}^N$ and it follows from the definition that

$$w_\alpha^{-1/(p-1)} \in L^1_{loc}(\Omega_1).$$

Therefore, the space $W^{k,p}(\Omega_1; S)$ is meaningful and, moreover, it is a Banach space. Therefore, *we introduce the space*

$$W^{k,p}(\Omega; S)$$

as the space $W^{k,p}(\Omega_1; S)$.

Obviously, this "new" space coincides with the "old" one if conditions (5.2) are satisfied for all $\alpha \in \boldsymbol{M}$.

5.3. The space $W_0^{k,p}(\Omega; S)$ and its modification. In [1], we introduced the space $W_0^{k,p}(\Omega; S)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (5.4), assuming that, in addition to conditions (5.2), the following condition is fulfilled:

$$(5.8) \qquad w_\alpha \in L^1_{loc}(\Omega) \quad \text{for all} \quad \alpha \in \boldsymbol{M}.$$

This last condition guarantees that

$$(5.9) \qquad C_0^\infty(\Omega) \subset W^{k,p}(\Omega; S).$$

Obviously, $W_0^{k,p}(\Omega; S)$ is again a Banach space under the norm (5.4).

If (5.8) is violated, then inclusion (5.9) is meaningless (see, e.g., Lemma 4.4 in [2]), and therefore the space $W_0^{k,p}(\Omega; S)$ cannot be introduced. Then we proceed as follows.

We denote, for $w \in \mathscr{W}(\Omega)$,

$$M_0(w) = \left\{x \in \Omega; \int_{\Omega \cap \mathscr{U}(x)} w(y)\, dy = \infty \right. \tag{5.10}$$

$$\left. \text{for every neighbourhood } \mathscr{U}(x) \text{ of } x \right\}$$

(formally we obtain this set by setting $p = 0$ in (5.5)). Obviously $M_0(w) = \emptyset$ if $w \in L^1_{\text{loc}}(\Omega)$. Let us further introduce the set

$$\mathscr{C} = \bigcup_{\alpha \in \boldsymbol{M}} M_0(w_\alpha)\, ; \tag{5.11}$$

then $\mathscr{C}$ is closed in Ω and $w_\alpha \in L^1_{\text{loc}}(\Omega - \mathscr{C})$ for every $\alpha \in \boldsymbol{M}$.

If Ω_1 is the set from (5.7) (i.e. $\Omega_1 = \Omega - \mathscr{B}$) and we denote

$$\Omega_2 = \Omega - \mathscr{C}\,, \tag{5.12}$$

then we introduce the space

$$W_0^{k,p}(\Omega; S)$$

as the closure of the set

$$V = \{f; f = g|_{\Omega_1},\ g \in C_0^\infty(\Omega_2)\} \tag{5.13}$$

with respect to the norm (5.4).

Again, $W_0^{k,p}(\Omega; S)$ is a Banach space: the assumption $f = g|_{\Omega_1}$ with $g \in C_0^\infty(\Omega_2)$ guarantees that $V \subset W^{k,p}(\Omega; S)$, so that the closure is meaningful, and since $W^{k,p}(\Omega; S)$ is $W^{k,p}(\Omega_1; S)$ by definition (see Section 5.2), the completeness of $W_0^{k,p}(\Omega; S)$ as a closed set in a Banach space is guaranteed as well.

5.4. The Dirichlet problem. Having now introduced the spaces $W^{k,p}(\Omega; S)$ and $W_0^{k,p}(\Omega; S)$ without any further assumptions on the functions $w_\alpha \in \mathscr{W}(\Omega)$, we can proceed in complete analogy with [1], Chapters 2–4. Naturally, we work with the spaces $W^{k,2}(\Omega; S)$ and $W_0^{k,2}(\Omega; S)$; it is only necessary to point out that we have in mind the new spaces just introduced.

5.5. Remark. The foregoing considerations show that in fact we are considering – in this new setting – a boundary value problem not on Ω but on $\Omega_1 = \Omega - \mathscr{B}$. If w_α are continuous a.e. in Ω then the set $\mathscr{B}$ from (5.6) as well as the set $\mathscr{C}$ from (5.11) are of measure zero (see [2], Theorem 3.3 and Lemma 4.6); in this case, we can consider $\mathscr{B}$ and $\mathscr{C}$ as parts of the boundary of the domain of definition. All will be seen more clearly from the following examples, in which we shall work with the plane domain $\Omega = (-1, 1) \times (-1, 1)$ and with the operator

$$Au = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u}{\partial x_i}\right) + a(x)\, u\,. \tag{5.14}$$

In this case, the natural space (in the sense of [1]) is $W^{1,2}(\Omega; S)$ with $S = \{a, a, a\}$.

We denote

$$\Omega_+ = \{x \in \Omega;\ x_1 > 0\}, \quad \Omega_- = \{x \in \Omega;\ x_1 < 0\}, \qquad (5.15)$$
$$\Gamma = \{(x_1, 0);\ 0 < x_1 < 1\}.$$

5.6. Example (strong singularity). Let us take

$$a(x) = a(x_1, x_2) = \begin{cases} x_2^{-2} & \text{if } x \in \Omega_+, \\ |x_2|^{-\lambda} & \text{if } x \in \Omega_- \end{cases} \qquad (5.16)$$

with $0 < \lambda < 1$ in (5.14). Since $a^{-1/(p-1)} = a^{-1}$ $(p = 2)$ belongs to $L^1_{loc}(\Omega)$, the set $\mathscr{B}$ from (5.6) is empty (i.e. $\Omega_1 = \Omega$) and the space $W^{1,2}(\Omega; S)$ is well defined and complete. On the other hand, $a \notin L^1_{loc}(\Omega)$ and the set $\mathscr{C}$ is the segment $\bar{\Gamma}$ from (5.15). We say that on Γ a *strong singularity* of the coefficient a appears. In accordance with Section 5.3, we define $W_0^{1,2}(\Omega; S)$ as the closure of $C_0^\infty(\Omega - \bar{\Gamma})$. The weak solution of the Dirichlet problem for the operator A from (5.14) is a function $u \in W^{1,2}(\Omega; S)$ for which

$$\int_\Omega a(x) \left[\frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} + uv \right] \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \qquad (5.17)$$

for all $v \in C_0^\infty(\Omega - \bar{\Gamma})$. Since v vanishes in a neighbourhood of $\bar{\Gamma}$, we can consider the identity (5.17) on $\Omega_2 = \Omega - \bar{\Gamma}$ instead on Ω. Further, the boundary condition is expressed by the requirement

$$u - u_0 \in W_0^{1,2}(\Omega; S)$$

with a prescribed $u_0 \in W^{1,2}(\Omega; S)$ $(= W^{1,2}(\Omega_2, S)$, since meas $\bar{\Gamma}$ = meas $(\Omega - \Omega_2) = 0)$. These facts suggest the idea that we have to prescribe a boundary condition not only on $\partial\Omega$, but also on $\bar{\Gamma}$ since $\partial\Omega_2 = \partial\Omega \cup \bar{\Gamma}$. But in fact we automatically have $u|_\Gamma = 0$ (in the sense of a trace) since also u_0 has to have a zero trace on Γ as a consequence of the fact that the weight $a(x)$ is of the form $[\operatorname{dist}(x, \Gamma)]^\varepsilon$ with $\varepsilon = -2 < -1$ (see Section 4.3 in [1]).

Let us mention that a singularity of $a(x)$ appears on the whole segment

$$P = \{(x_1, 0),\ -1 < x_1 < 1\}, \qquad (5.18)$$

but on $P - \bar{\Gamma}$ the singularity is weak.

5.7. Example (strong degeneration). Let us take

$$a(x) = a(x_1, x_2) = \begin{cases} x_2^2 & \text{if } x \in \Omega_+, \\ |x_2|^\lambda & \text{if } x \in \Omega_- \end{cases} \qquad (5.19)$$

with $0 < \lambda < 1$ in (5.14) (this function is the reciprocal of the function from (5.16)). Here we have a *strong degeneration* on the segment Γ (and a weak one on $P - \bar{\Gamma}$) since the condition $a^{-1/(p-1)} = a^{-1} \in L^1_{loc}(\Omega)$ is not fulfilled. Therefore, the space

$W^{1,2}(\Omega; S)$ is in fact the space $W^{1,2}(\Omega - \bar{\Gamma}; S)$ and $W_0^{1,2}(\Omega; S)$ is the closure of the restriction of functions from $C_0^\infty(\Omega)$ to $\Omega_1 = \Omega - \bar{\Gamma}$.

5.8. Remark. A comparison of the last two examples shows that the behaviour of the solutions $u \in W^{1,2}(\Omega; S)$ differs on Γ: In Example 5.6 we necessarily have $u|_\Gamma = 0$; in Example 5.7, we have no information and no requirement – the "trace from above" (for $x_2 \to 0+$) can be completely different from the "trace from below" (for $x_2 \to 0-$); and in the case of no strong singularity or degeneration the trace exists (possibly nonzero) on Γ since our space $W^{1,2}(\Omega; S)$ is imbedded into the Sobolev space $W^{1,1}(\Omega)$ (see also Section 4.3 in [1]).

5.9. Remark. Combining the considerations from Examples 5.6 and 5.7, we can construct examples in which strong singularities appear in one part of Ω (the set $\mathscr{B}$ from (5.6)) while strong degeneration appears on another part (the set $\mathscr{C}$ from (5.11)). Nevertheless, both phenomena can take place on the same set, as the following example shows.

5.10. Example (strong singularity together with strong degeneration). Let us take

$$a(x) = a(x_1, x_2) = \begin{cases} e^{-1/x_2} & \text{if } x \in \Omega_+ , \\ 1 & \text{if } x \in \Omega_- \end{cases} \tag{5.20}$$

in (5.14). In this case we have $a^{-1/(p-1)} = a^{-1} \notin L^1_{\mathrm{loc}}(\Omega)$ and $a \notin L^1_{\mathrm{loc}}(\Omega)$. The sets $\mathscr{B}$ and $\mathscr{C}$ coincide with the segment $\bar{\Gamma}$ and therefore, we define $W^{1,2}(\Omega; S)$ as the space $W^{1,2}(\Omega - \bar{\Gamma}; S)$, $W_0^{1,2}(\Omega; S) = \overline{C_0^\infty(\Omega - \bar{\Gamma})}$.

In this case, we have a strong singularity on Γ from below (i.e. if $x_2 \to 0-$) and a strong degeneration on Γ from above (i.e. if $x_2 \to 0+$). In view of the definition of the space $W^{1,2}(\Omega; S)$, we should consider Γ as part of the boundary of the domain of definition $\Omega_2 = \Omega - \mathscr{C} = \Omega - \bar{\Gamma}$. But in this case, for $u \in W^{1,2}(\Omega; S)$ we automatically have a zero "trace on Γ from below" and no condition for a "trace from above". The arguments are analogous to those of Examples 5.6 and 5.7.

6. NONLINEAR EQUATIONS

6.1. Introductory remarks. In [1] and in the foregoing sections, we considered a linear differential operator

$$(Au)(x) = \sum_{\alpha,\beta \in \boldsymbol{M}} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u)$$

and constructed a suitable Sobolev weighted space $W^{k,p}(\Omega; S)$ (with $p = 2$) in which the Dirichlet problem for A was uniquely solvable. The set $S = \{w_\alpha \in \mathscr{W}(\Omega), \alpha \in \boldsymbol{M}\}$ was determined by the operator A or, more precisely, by its coefficients $a_{\alpha\beta}$.

Now, we shall consider nonlinear operators of the form

$$(6.1) \qquad (Au)(x) = \sum_{\alpha \in \mathbf{M}} (-1)^{|\alpha|} D^\alpha a_\alpha(x; \delta_{\mathbf{M}} u(x))$$

with

$$(6.2) \qquad \delta_{\mathbf{M}} u = \{D^\beta u;\ \beta \in \mathbf{M}\}\ .$$

We shall proceed in the reversed way: We shall assume that the spaces $W^{k,p}(\Omega; S)$ and $W_0^{k,p}(\Omega; S)$ are given (i.e., that the set S of weight functions w_α is prescribed, $\alpha \in \mathbf{M}$) and show, *what* operators A (i.e., *what* functions $a_\alpha(x; \xi)$) are suitable for the Dirichlet problem to be solved (in a weak sense) in these spaces.

Therefore, let us assume that the set $\mathbf{M} \subset \mathbf{M}(N, k)$ and the family $S = \{w_\alpha \in \mathscr{W}(\Omega), \alpha \in \mathbf{M}\}$ are given (according to points (i)–(iii) from Section 5.1) and that $W^{k,p}(\Omega; S)$ and $W_0^{k,p}(\Omega; S)$ with $p > 1$ are the corresponding weighted Sobolev spaces from Sections 5.2 and 5.3 (i.e., the modified spaces, if conditions (5.2) and/or (5.8) are not fulfilled). So, we have Banach spaces at our disposal, which are obviously reflexive.

6.2. Formulation of the Dirichlet problem. Let m be the number of elements of the set $\mathbf{M}$, i.e., the number of components of the vector function $\delta_{\mathbf{M}} u$ from (6.2). We shall write $\xi \in \mathbb{R}^m$ in the form $\xi = \{\xi_\beta, \beta \in \mathbf{M}\}$.

Consider the operator A from (6.1) and suppose that the functions $a_\alpha = a_\alpha(x; \xi)$ (the "coefficients" of A) satisfy the following conditions:

(i) they are defined for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^m$ and satisfy the *Carathéodory condition* (see e.g. [3], or [7], Sec. 12.2);

(ii) they satisfy the (*weighted*) *growth conditions*

$$(6.3) \qquad |a_\alpha(x; \xi)| \leqq w_\alpha^{1/p}(x) \left[g_\alpha(x) + c_\alpha \sum_{\beta \in \mathbf{M}} |\xi_\beta|^{p-1} w_\beta^{1/q}(x)\right]$$

where $g_\alpha \in L^q(\Omega)$ with $q = p/(p-1)$ and $c_\alpha \geqq 0$ are certain functions and constants, respectively, while w_α are elements of S.

The class of all such functions a_α will be denoted by

$$\mathrm{CAR}(p, S)\ .$$

Further, let $u_0 \in W^{k,p}(\Omega; S)$ and $f \in [W_0^{k,p}(\Omega; S)]^*$ be given. We shall say that a function $\hat{u} = u + u_0$ is a weak solution of the Dirichlet problem for the operator A (with the right hand side f and boundary data u_0) if

$$(6.4) \qquad u = \hat{u} - u_0 \in W_0^{k,p}(\Omega; S)$$

and

$$(6.5) \qquad \sum_{\alpha \in \mathbf{M}} \int_\Omega a_\alpha(x; \delta_{\mathbf{M}} u(x) + \delta_{\mathbf{M}} u_0(x))\, D^\alpha v(x)\, \mathrm{d}x = \langle f, v \rangle$$

for all $v \in W_0^{k,p}(\Omega; S)$.

6.3. Existence theorem. *Let Ω be an open set in $\mathbb{R}^N$, $p > 1$, $\boldsymbol{M}$ and S the sets from Section* 5.1 (iii), (iv), *$W^{k,p}(\Omega; S)$ and $W_0^{k,p}(\Omega; S)$ the corresponding Sobolev weighted spaces. Let the coefficients $a_\alpha = a_\alpha(x; \xi)$ of the differential operator A from* (6.1) *satisfy the following conditions*:

$$a_\alpha \in \mathrm{CAR}(p, S)\,; \tag{6.6}$$

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^m$ the inequalities

$$\sum_{\alpha \in \boldsymbol{M}} [a_\alpha(x; \xi) - a_\alpha(x, \eta)]\,(\xi_\alpha - \eta_\alpha) \geqq 0\,, \tag{6.7}$$

$$\sum_{\alpha \in \boldsymbol{M}} a_\alpha(x; \xi)\, \xi_\alpha \geqq c_1 \sum_{\alpha \in \boldsymbol{M}} |\xi_\alpha|^p\, w_\alpha(x) \tag{6.8}$$

hold with $c_1 > 0$.

Then there exists at least one weak solution $\hat{u} \in W^{k,p}(\Omega; S)$ of the Dirichlet problem from Section 6.2.

If the inequality in (6.7) *is strict, then the solution $\hat{u}$ is uniquely determined.*

6.4. Remark. The reader familiar with elements of the theory of monotone operators has certainly observed that (6.7) is the usual *monotonicity condition* and (6.8) the coercivity condition except for the factor $w_\alpha(x)$ at $|\xi_\alpha|^p$ on the right hand side (we shall call (6.8) a *weighted coercivity condition*). Consequently, Theorem 6.3 asserts that the weighted growth condition (6.3) together with monotonicity and weighted coercivity guarantee the existence of a weak solution. The proof follows by standard methods of the theory of monotone operators, and conditions (6.3), (6.7) and (6.8) can be generalized as usual (monotonicity of the main part of the operator together with some compact imbeddings etc.). We shall give the proof of Theorem 6.3 after presenting an example and some auxiliary results.

6.5. Example. The operator

$$(Au)(x) = \sum_{\alpha \in \boldsymbol{M}} (-1)^{|\alpha|}\, D^\alpha[|D^\alpha u(x)|^{p-1} \operatorname{sgn} D^\alpha u(x)\, w_\alpha(x)]$$

with $w_\alpha \in \mathscr{W}(\Omega)$ is a typical representative of operators involving solutions in the weighted space with weights w_α. The functions

$$a_\alpha(x; \xi) = |\xi_\alpha|^{p-1} \operatorname{sgn} \xi_\alpha\, w_\alpha(x)$$

obviously satisfy condition (6.3) (with $g_\alpha \equiv 0$, $c_\alpha = 1$), (6.7) and (6.8) (with $c_1 = 1$). Analogously as in the linear case, the coefficients w_α express a certain degeneration or singularity on those parts of $\bar{\Omega}$ on which w_α tend to zero or to infinity, respectively.

6.6. Two auxiliary assertions. It is easily seen that the mapping $\Phi\colon L^p(\Omega) \to L^p(\Omega; \varrho)$ defined by

$$\Phi(u) = u\varrho^{-1/p}\; (\varrho \in \mathscr{W}(\Omega))$$

is an isometric isomorphism between the spaces considered and connects weighted and nonweighted spaces. Using this mapping one can prove the following two assertions about continuous linear functionals on $W^{k,p}(\Omega; S)$ and about Nemyckij operators on weighted spaces $L^p(\Omega; \varrho)$, modifying in an obvious way the proof of the corresponding assertion for the nonweighted case (see, e.g., [5], Theorem 3.8, or [4], Sec. 5.9, for the first assertion and [3] for the second).

(i) *Let F be a functional from the dual space $[W^{k,p}(\Omega; S)]^*$. Then there exists an m-tuple*

$$Q = \{f_\alpha \in L^q(\Omega; w_\alpha^{-1});\ \alpha \in \mathbf{M}\}\,, \quad q = \frac{p}{p-1}\,, \tag{6.9}$$

such that

$$\langle F, v\rangle = \sum_{\alpha\in \mathbf{M}} \int_\Omega f_\alpha\, D^\alpha v\, w_\alpha^{1/p-1/q}\, dx \tag{6.10}$$

and

$$\|F\| = \inf\{\sum_{\alpha\in\mathbf{M}} \|f_\alpha\|^q_{q,w_\alpha^{-1}}\}^{1/q} \tag{6.11}$$

where the infimum is taken over all m-tuples Q of the form (6.9) *such that the representation* (6.10) *takes place.*

(ii) *Let $\Omega \subset \mathbb{R}^N$, $m \in \mathbb{N}$, $p > 1$. Let $h(x, \xi)$ be a function defined for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^m$ which satisfies the Carathéodory condition. Let $\mathscr{H}(u_1, \ldots, u_m)$ be the Nemyckij operator generated by the function h, i.e.*

$$\mathscr{H}(u_1, \ldots, u_m)(x) = h(x; u_1(x), \ldots, u_m(x))\,, \quad x \in \Omega\,.$$

Let ϱ, $w_j \in \mathscr{W}(\Omega)$, $j = 1, 2, \ldots, m$.

If $(u_1, \ldots, u_m) \in \prod_{j=1}^m L^p(\Omega; w_j)$, then

$$\mathscr{H}(u_1, \ldots, u_m) \in L^q(\Omega; \varrho^{-1}) \quad \left(q = \frac{p}{q-1}\right)$$

if and only if the following condition is fulfilled: There exist a function $g \in L^q(\Omega)$ and a constant $c \geqq 0$ such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^m$ we have

$$|h(x; \xi_1, \ldots, \xi_m)| \leqq \varrho^{1/q}(x)\,[g(x) + c\sum_{j=1}^m |\xi_j|^{p-1}\, w_j^{1/q}(x)]\,. \tag{6.12}$$

If condition (6.12) *is fulfilled, then the Nemyckij operator $\mathscr{H}$ is a continuous mapping from $\prod_{j=1}^m L^p(\Omega; w_j)$ into $L^q(\Omega, \varrho^{-1})$.*

6.7. Proof of Theorem 6.3. (i) Let us consider the form

$$(6.13)\qquad a(u, v) = \sum_{\alpha\in M} \int_\Omega a_\alpha(x; \delta_M u(x))\, D^\alpha v\, dx$$

associated with the differential operator A from (6.1), and define functions h_α by the formulae

$$(6.14)\qquad h_\alpha(x; \xi) = a_\alpha(x; \xi)\, [w_\alpha(x)]^{1/q-1/p}\,, \quad \alpha \in M\,.$$

Since $a_\alpha \in \mathrm{CAR}\,(p, S)$, we have that h_α satisfy the Carathéodory condition and, in view of (6.3),

$$(6.15)\qquad |h_\alpha(x; \xi)| \leqq w_\alpha^{1/q}(x)\, [g_\alpha(x) + c_\alpha \sum_{\beta\in M} |\xi_\beta|^{p-1}\, w_\beta^{1/q}(x)]\,.$$

Assertion (ii) from Section 6.6 implies – see (6.12) with $\varrho = w_\alpha$ – that the operator $\mathscr{H}_\alpha(u)\,(x) = h_\alpha(x; \{u_\beta(x)\}_{\beta\in M})$ is a continuous Nemyckij operator from $\prod_{\beta\in M} L^p(\Omega; w_\beta)$ into $L^q(\Omega; w_\alpha^{-1})$. Particularly, the function $f_\alpha(x) = h_\alpha(x; \delta_M u(x))$ belongs to $L^q(\Omega; w_\alpha^{-1})$ for $u \in W^{k,p}(\Omega; S)$.

Since

$$a(u, v) = \sum_{\alpha\in M} \int_\Omega h_\alpha(x; \delta_M u(x))\, D^\alpha v(x)\, w_\alpha^{1/p-1/q}(x)\, dx =$$

$$= \sum_{\alpha\in M} \int_\Omega f_\alpha(x)\, D^\alpha v(x)\, w_\alpha^{1/p-1/q}(x)\, dx\,,$$

we obtain from assertion (i) of Section 6.6 that $a(u, v)$ is (for u fixed) the value of a continuous linear functional on $W^{k,p}(\Omega; S)$. We denote this functional by $\hat{T}u$, since it depends on u, and so we have

$$a(u, v) = \langle \hat{T}u, v\rangle \quad \text{for} \quad u, v \in W^{k,p}(\Omega; S)\,.$$

Since u was fixed but arbitrary, we have constructed an operator

$$(6.16)\qquad \hat{T}\colon W^{k,p}(\Omega; S) \to [W^{k,p}(\Omega; S)]^*\,.$$

From (6.11) we have

$$(6.17)\qquad \|\hat{T}u\| \leqq \{\sum_{\alpha\in M} \|f_\alpha\|^q_{q,w_\alpha^{-1}}\}^{1/q} \leqq c_3(1 + \|u\|^p_{k,p,S})^{1/q}$$

since inequality (6.15) implies that

$$\|f_\alpha\|^q_{q,w_\alpha^{-1}} = \int_\Omega |h_\alpha(x; \delta_M u(x))|^q\, w_\alpha^{-1}(x)\, dx \leqq$$

$$\leqq \int_\Omega |w_\alpha^{1/q}(x)\, [g_\alpha(x) + c_\alpha \sum_{\beta\in M} |D^\beta u(x)|^{p-1}\, w_\beta^{1/q}(x)]|^q\, w_\alpha^{-1}(x)\, dx \leqq$$

$$\leqq (m + 1)^{q-1}\, \{\|g_\alpha\|^q_q + c_\alpha^q \sum_{\beta\in M} \|D^\beta u\|^p_{p,w_\beta}\} \leqq c_2(1 + \|u\|^p_{k,p,S})$$

where c_2 is a fixed constant depending on the c_α's and on the L^q-norms of the functions g_α.

(ii) According to formula (6.5), to find a solution $\hat{u}$ of the Dirichlet problem means to find a function $u \in W_0^{k,p}(\Omega; S)$ such that $a(u + u_0, v) = \langle f, v \rangle$, i.e. $\langle \hat{T}(u + u_0), v \rangle = \langle f, v \rangle$ for every $v \in W_0^{k,p}(\Omega; S)$. If we denote

$$Tu = \hat{T}(u + u_0)\,, \tag{6.18}$$

then obviously T is an operator from $X = W_0^{k,p}(\Omega; S)$ into its dual X^*. Consequently, our problem reduces to the problem of finding $u \in X$ such that $\langle Tu, v \rangle = \langle f, v \rangle$ for every $v \in X$, i.e. to the equation

$$Tu = f \quad \text{on} \quad X \tag{6.19}$$

with a given $f \in X^*$.

Equation (6.19) will be solved by Browder's theorem (see, e.g., [6], Chap. 2, Theorem 2.1, or [7], Theorem 29.5), which guarantees the existence of a solution $u \in X$ if T is bounded, demicontinuous, monotone and coercive.

(iii) Boundedness of T follows immediately from formula (6.17).

(iv) Demicontinuity of T is a consequence of the continuity of the Nemyckij operators $\mathscr{H}_\alpha(u)$ from part (i) of this proof. Indeed, if $u_n \to u$ in X, then $\langle Tu_n - Tu, v \rangle \to 0$ for every $v \in X$ since by Hölder's inequality we have

$$|\langle Tu_n - Tu, v \rangle| =$$

$$= \left| \sum_{\alpha \in M} \int_\Omega [h_\alpha(x; \delta_M u_n + \delta_M u_0) - h_\alpha(x; \delta_M u + \delta_M u_0)]\, D^\alpha v\, w_\alpha^{1/p - 1/q}\, dx \right| \leqq$$

$$\leqq \sum_{\alpha \in M} \|h_\alpha(\cdot\,; \delta_M u_n + \delta_M u_0) - h_\alpha(\cdot\,; \delta_M u + \delta_M u_0)\|_{q, w_\alpha^{-1}}\, \|D^\alpha v\|_{p, w_\alpha}$$

and the first norms in the last expression tend to zero for $n \to \infty$.

(v) Monotonicity of T follows from condition (6.7), where we take $\xi = D^\alpha u(x) + D^\alpha u_0(x)$, $\eta = D^\alpha v(x) + D^\alpha u_0(x)$:

$$\langle Tu - Tv, u - v \rangle = \sum_{\alpha \in M} \int_\Omega [a_\alpha(x; \delta_M u + \delta_M u_0) - \tag{6.20}$$

$$- a_\alpha(x; \delta_M v + \delta_M u_0)]\,(D^\alpha u - D^\alpha v)\, dx \geqq 0\,.$$

(vi) Coercivity of T follows from condition (6.8): If we take $\xi = \delta_M u(x) + \delta_M u_0(x)$, then

$$\langle Tu, u + u_0 \rangle = \sum_{\alpha \in M} \int_\Omega a_\alpha(x; \delta_M u + \delta_M u_0)\,(D^\alpha u + D^\alpha u_0)\, dx \geqq \tag{6.21}$$

$$\geqq c_1 \sum_{\alpha \in M} \int_\Omega |D^\alpha u(x) + D^\alpha u_0(x)|^p\, w_\alpha(x)\, dx =$$

$$= c_1\|u + u_0\|^p_{k,p,S} \geqq c_1\big|\,\|u\|_{k,p,S} - \|u_0\|_{k,p,S}\big|^p = c_1\|u\|^p_{k,p,S}\left|1 - \frac{\|u_0\|_{k,p,S}}{\|u\|_{k,p,S}}\right|^p .$$

Further, from (6.17) we have

$$(6.22)\quad |\langle Tu, u_0\rangle| \leqq \|Tu\| \,.\, \|u_0\|_{k,p,S} \leqq c_3(1 + \|u + u_0\|^p_{k,p,S})^{1/q}\,\|u_0\|_{k,p,S} \leqq$$

$$\leqq c_4(1 + \|u\|^{p-1}_{k,p,S} + \|u_0\|^{p-1}_{k,p,S})\,\|u_0\|_{k,p,S} = c_5 + c_6\|u\|^{p-1}_{k,p,S}$$

(note that u_0 is given and $q = p/(p - 1)$). Formulae (6.21) and (6.22) yield

$$\langle Tu, u\rangle = \langle Tu, u + u_0\rangle - \langle Tu, u_0\rangle \geqq$$

$$\geqq c_1\|u\|^p_{k,p,S}\left|1 - \frac{\|u_0\|_{k,p,S}}{\|u\|_{k,p,S}}\right|^p - \{c_5 + c_6\|u\|^{p-1}_{k,p,S}\} =$$

$$= \|u\|^p_{k,p,S}\left\{c_1\left|1 - \frac{\|u_0\|_{k,p,S}}{\|u\|_{k,p,S}}\right|^p - \frac{c_5}{\|u\|^p_{k,p,S}} - \frac{c_6}{\|u\|_{k,p,S}}\right\}$$

and consequently

$$\frac{\langle Tu, u\rangle}{\|u\|_{k,p,S}} \to \infty \quad \text{for} \quad \|u\|_{k,p,S} \to \infty\,,$$

i.e., T is coercive.

So, the existence of at least one solution of the Dirichlet problem is proved. Uniqueness follows by contradiction if we assume that the inequality in (6.7) is strict: Analogously as in (6.20), for two solutions u, u^* we obtain the inequality $\langle Tu - Tu^*, u - u^*\rangle > 0$ while $Tu = Tu^* = f$.

6.8. Concluding remarks. Analogously as in the linear case we can weaken some of our assumptions in Theorem 6.3 provided we have more information about the structure of the spaces considered. Let us mention two of such generalizations:

(i) If there is a subset $\boldsymbol{M}_1 \subset \boldsymbol{M}$ such that $\|u\|_{k,p,S} \leqq c_0\{\sum_{\alpha\in\boldsymbol{M}_1} \|D^\alpha u\|^p_{p,w_\alpha}\}^{1/p}$ for every $u \in W_0^{k,p}(\Omega; S)$, then we can modify condition (6.8) summing only over $\boldsymbol{M}_1$ (instead of $\boldsymbol{M}$) on the right hand side.

(ii) If Ω has finite measure, then (6.8) can be replaced by

$$\sum_{\alpha\in\boldsymbol{M}} a_\alpha(x; \xi)\,\xi_\alpha \geqq c_1 \sum_{\alpha\in\boldsymbol{M}} |\xi_\alpha|^p\, w_\alpha(x) - c_2$$

with $c_1 > 0$, $c_2 \geqq 0$.

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