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ON CERTAIN PROPERTIES CHARACTERIZING LOCALLY SEPARABLE METRIC SPACES

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A metric space (X, ϱ) is called locally separable (see [1]) if to each $p \in X$ there exists $\delta > 0$ such that $(S(p, \delta), \varrho)$ is a separable metric space ($S(p, \delta) = \{x \in X; \varrho(p, x) < \delta\}$). Each separable space is locally separable. The space (P, ϱ), where P is an uncountable set and ϱ the trivial metric is an example of a locally separable and non separable space.

If A is a subset of a metric space, then A^c will denote the set of all condensation points of the set A. Further $A^{cc} = (A^c)^c$. A point x is called a point of condensation of the set A if the intersection of each neighbourhood of the point x with the set A is uncountable. If (X, ϱ) is a separable space, then (see [2] p. 79) $A^c = A^{cc}$ holds. It will be shown that the last property characterises locally separable spaces (among all metric spaces). A set $A \subset X$ will be called ε-isolated ($\varepsilon > 0$) in X, if $A \cap S(p, \varepsilon) = \{p\}$ for each $p \in A$.

Theorem 1. *A metric space (X, ϱ) is locally separable if and only if $A^c = A^{cc}$ for each set $A \subset X$.*

Lemma. *Let the metric space (P, ϱ) not be separable. Then there exists $\varepsilon_1 > 0$ and an uncountable set $B \subset P$ such that B is ε_1-isolated in P.*

Proof. It is known (see [2] p. 80) that (P, ϱ) is separable if and only if corresponding to each $\varepsilon > 0$ there exists a countable set $A \subset P$ such that

$$\operatorname{dist}(P, A) = \sup_{x \in P} \varrho(x, A) < \varepsilon .$$

Hence if (P, ϱ) is not separable there exists $\varepsilon > 0$ such that

$$\operatorname{dist}(P, A) = \sup_{x \in P} \varrho(x, A) \geqq \varepsilon , \tag{1}$$

where $A \subset P$ is an arbitrary countable set.

Let us choose $x_0 \in P$ and put $A = \{x_0\}$. In view of (1) there exists $x_1 \in P$ such that $\varrho(x_1 x_0) > \varepsilon/2$. Let ω_1 denote the first uncountable ordinal number and let $\xi < \omega_1$. Let us suppose that the points x_η, $\eta < \xi$ were constucted so that for $\eta', \eta'' < \xi$, $\eta' \neq \eta''$, $\varrho(x_{\eta'}, x_{\eta''}) > \varepsilon/2$ holds. Then $A = \{x_0, x_1, \ldots, x_\eta, \ldots\}$, $\eta < \xi$ is countable and in view of (1) there exists $x_\xi \in P$ such that $\varrho(x_\xi, x_\eta) > \varepsilon/2$ for each $\eta < \xi$. Thus, by means of transfinite induction a set $B = \{x_0, x_1, \ldots, x_\xi, \ldots\}$, $\xi < \omega_1$ is obtained which is evidently uncountable and ε_1-isolated if we put $\varepsilon_1 = \varepsilon/2$.

Proof of theorem 1. a) Let (X, ϱ) be a locally separable metric space and let $A \subset X$. It is sufficient to proove the inclusion $A^c \subset A^{cc}$. The other inclusion follows from the fact that A^c is closed.

Let $x_0 \in A^c$. From the theorem of Sierpinski concerning the structure of locally separable spaces (see [1]), we have $X = \bigcup_{t \in T} G_t$ where G_t $(t \in T)$ are pariwise disjoint open-closed sets in X and (G_t, ϱ) for each $t \in T$ is a locally separable space. Hence $x_0 \in G_{t_0}$ and there exists $\delta > 0$ such that $S(x_0, \delta) \subset G_{t_0}$. As the point x_0 is a condensation point of the set A, the set $S(x_0, \delta) \cap A$, and what is more, the set $A_1 = A \cap G_{t_0}$ is uncountable and $x_0 \in A_1^c$. A_1^c denotes the set of all condensation points of the set A_1 in X or, (which is the same in view of the closedness of the set G_{t_0}) the set of all condensation points of A_1 in G_{t_0}. The symbol A_1^{cc} has a similar meaning. (G_{t_0}, ϱ) is separable, $A_1 \subset G_{t_0}$, hence $A_1^c = A_1^{cc}$ and consequently $x_0 \in A_1^{cc} \subset A^{cc}$. The inclusion $A^c \subset A^{cc}$ is proved.

b) Let (X, ϱ) not be locally separable. We shall prove the existence of a set $A \subset X$ such that $A^c \neq A^{cc}$. There exists a point $p \in X$ such that $(S(p, \delta), \varrho)$ is not separable for each $\delta > 0$. In particular $(S(p, \frac{1}{2}), \varrho$ is not separable. In view of our lemma there exists an uncountable set B_1 which is ε_1-isolated in $S(p, \frac{1}{2})$, $\varepsilon_1 > 0$.

As it is easily seen the set B_1 as an isolated set has not a condensation point, hence there exists $n_1 > 2$ such that $B_1 \cap S(p, 1/n_1)$ is countable and consequently:

$$A_1 = B_1 \cap (S(p, \tfrac{1}{2}) - S(p, 1/n_1)) = \{x \in B_1, 1/n_1 \leqq \varrho(p, x) < \tfrac{1}{2}\}$$

is uncountable and ε_1-isolated set in $S(p, 1)$. So we have $A_1^c = \emptyset$ (A_1^c denotes the set of all condensation points of the set A_1 in X or in $S(p, 1)$). Since $(S(p, 1/n_1), \varrho)$ is not separable, there exists on the base of the above lemma an uncountable set B_2 which is ε_2-isolated in $S(p, 1/n_1)$ $(\varepsilon_2 > 0)$. Quite a similar procedure to the above one leads to the number $n_2 > n_1$ such that the set $A_2 = \{x \in B_2;\ 1/n_2 \leqq \varrho(p, x) < 1/n_1\}$ is uncountable. Evidently A_2 is ε_2-isolated in $S(p, 1)$ and $A_2^c = \emptyset$. Using induction we construct a sequence of natural numbers

$$2 = n_0 < n_1 < \ldots n_k < \ldots$$

and a sequence $\{A_k\}$ of uncountable ε_k-isolated $(\varepsilon_k > 0)$ sets in $S(p, 1)$ such that $A_k^c = \emptyset$. Let us put $A = \bigcup_{k=1}^{\infty} A_k$ Evidently $p \in A^c$. If $q \in X$, $q \neq p$, let us put $\varrho(p, q) = 2\eta$ and let us take the spaces $S(p, \eta)$, $S(q, \eta)$. Since for all k, begining from certain

k_0, $1/n_{k-1} < \eta$ holds, we have $\bigcup_{k=k_0+1}^{\infty} A_k \subset S(p, \eta)$ and consequently

$$A \cap S(q, \eta) = (A_1 \cup \ldots \cup A_{k_0}) \cap S(q, \eta). \tag{2}$$

We shall show that $q \notin A^c$.

The case $q \in A^c$ leads to the inclusion

$$\{q\} \subset (A \cap S(q, \eta))^c . \tag{3}$$

From (2) on the base of the known properties of condensation points (see [3] p. 140) we get

$$(A \cap S(q, \eta))^c \subset (A_1 \cup \ldots \cup A_{k_0})^c \cap (S(q, \eta))^c = (A_1^c \cup \ldots \cup A_{k_0}^c) \cap (S(q, \eta))^c$$

and since $A_k^c = \emptyset$ $(k = 1, 2, \ldots)$ we have $(A \cap S(q, \eta))^c = \emptyset$ and this is a contradiction with (3). Consequently $q \notin A^c$ and we have $A^c = \{p\}$, $A^{cc} = \emptyset \neq A^c$. The proof is finished.

It is not difficult to construct examples of metric spaces (which are in view of theorem 1 not separable) in which there exist sets A such that $A^c \neq A^{cc}$. We shall show some such examples.

Example 1. Let m denote the space of all bounded sequences of real numbers with the metric

$$\varrho(x, y) = \sup_{n=1,2,\ldots} |\xi_n - \eta_n| , \quad x = \{\xi_n\}_1^{\infty} , \quad y = \{\eta_n\}_1^{\infty} \in m .$$

Let A_n be the set of all sequences of the form $\{\varepsilon_k/n\}_{k=1}^{\infty}$ where $\varepsilon_k = 1$ or -1 $(k = 1, 2, \ldots)$. Let us put $A = \bigcup_{n=1}^{\infty} A_u$. Then $A^c = \{\{0\}_{k=1}^{\infty}\}$ and $A^{cc} = \emptyset \neq A^c$.

Example 2. Let a be some symbol, let X denote the set of all triples (a, φ, r), where $0 \leqq \varphi < 2\pi$, $r \geqq 0$, r, φ are real numbers. If $r = 0$ then we shall identify the triple $(a, \varphi, 0)$ with a. If $\xi_1 = (a, \varphi_1, r_1)$, $\xi_2 = (a, \varphi_2, r_2)$ we define $\varrho(\xi_1, \xi_2) = r_1 + r_2$, if $\varphi_1 \neq \varphi_2$ and $\varrho(\xi_1, \xi_2) = |r_1 - r_2|$ if $\varphi_1 = \varphi_2$.

It is easily seen that ϱ is a metric on X (see [4]). Let A_k denote the set of all triples $\xi = (a, \varphi, 1/k)$. Let us put $A = \bigcup_{k=1}^{\infty} A_k$. Then $A^c = \{(a, \varphi, 0)\} = a$. Hence $A^{cc} = \emptyset \neq A^c$.

Example 3. Let X be the set of all real numbers. Let us put $\varrho(x, x) = 0$ and $\varrho(x, y) = |x| + |y|$ if $x \neq y$. Then $X^c = \{0\}$ and $X^{cc} = \emptyset \neq X^c$.

In a separable metric space there may not exist an uncountable isolated set. In a locally separable space an uncountable isolated set may evidently exist. As an example it suffices to take an uncountable set with trivial metric. Whe shall show that if A is isolated in a locally separable space then $A^c = \emptyset$ holds. The last property characteristes the locally separable spaces.

Theorem 2. *A metric space (X, ϱ) is locally separable if and only if $A^c = \emptyset$ for each isolated set A in X.*

Proof. a) Let (X, ϱ) not be locally separable. Then there exists a point $p \in X$ such that for each $\delta > 0$ $(S(p, \delta), \varrho)$ is not separable. In $S(p, 1)$ we shall construct the sets A_k $(k = 1, 2, \ldots)$ in the same way as in the proof of theorem 1. Let us put again $A = \bigcup_{k=1}^{\infty} A_k$. From the construction of the sets A_i it follows that A is an isolated set in X. In fact, if $q \in A = \bigcup_{k=1}^{\infty} A_k$, then $q \neq p$ and there exists k such that $q \in A_k$. Let us put $\varrho(p, q) = 2\eta > 0$ and let us take $S(p, \eta)$, $S(q, \eta)$. Then there exists $m > k$ such that, $\bigcup_{s=m+1}^{\infty} A_s \subset S(p, \eta)$. Consequently

$$S(q, \eta) \cap \bigcup_{s=1}^{\infty} A_s = S(q, \eta) \cap \bigcup_{s=1}^{m} A_s . \tag{4}$$

Each of the sets A_s is ε_s-isolated $(\varepsilon_s > 0)$, so if we put $\varepsilon = \min(\eta, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$, we have

$$S(q, \varepsilon) \cap \bigcup_{s=1}^{m} A_s = \bigcup_{s=1}^{m} (A_s \cap S(q, \varepsilon)) = \{q\} . \tag{5}$$

From (4) and (5) immediately follows that q is an isolated point of the set $A = \bigcup_{s=1}^{\infty} A_s$. From the proof of theorem 1 we have that $p \in A^c$. Hence A is isolated (in X) with the property $A^c \neq \emptyset$.

b) Let $A \subset X$, A isolated in X and $A^c \neq \emptyset$. Let $p \in A^c$. Then for each $\delta > 0$, $B = A \cap S(p, \delta)$ is uncountable and isolated, $B \subset S(p, \delta)$. Let us put

$$B_n = \{x \in B;\ \varrho(x, B - \{x\}) > 1/n\} .$$

Evidently $B = \bigcup_{n=1}^{\infty} B_n$, hence a natural number n exists such that B_n is uncountable. B_n is $1/n$-isolated and $B_n \subset S(p, \delta)$. From these facts it follows that $(S(p, \delta), \varrho)$ is not separable. Since $\delta > 0$ was arbitrary chosen (X, ϱ) is not locally separable.

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Výťah

O ISTÝCH VLASTNOSTIACH, KTORÉ CHARAKTERIZUJÚ LOKÁLNE SEPARABILNÉ METRICKÉ PRIESTORY

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Nech A^c je množina všetkých kondenzačných bodov množiny A v metrickom priestore (X, ϱ). Nasledujúce vlastnosti sú ekvivalentné:

a) (X, ϱ) je lokálne separabilný.
b) pre každú množinu $A \subset X$ je $(A^c)^c = A^c$.
c) pre každú izolovanú množinu $A \subset X$ je $A^c = \emptyset$.

Резюме

О НЕКОТОРЫХ СВОЙСТВАХ, ХАРАКТЕРИЗУЮЩИХ ЛОКАЛЬНО СЕПАРАБЕЛЬНЫЕ МЕТРИЧЕСКИЕ ПРОСТРАНСТВА

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Пусть A^c – множество всех точек кондензации множества A в метрическом пространстве (X, ϱ). Следующие свойства равносильны:

(а) (X, ϱ) – локально сепарабельное пространство.
(б) $(A^c)^c = A^c$ для всякого множества $A \subset X$.
(в) $A^c = \emptyset$ для всякого изолированного множества $A \subset X$.