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Časopis pro pěstování matematiky, roč. 85 (1960), Praha

A NOTE ON THE SOLUTION OF THE TRANSPORTATION PROBLEM BY THE SIMPLEX METHOD

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The purpose of this note is to establish a connection between the general simplex method of linear programming and the algorithm for the solution of the transportation problem developed by F. NOŽIČKA [1], and reproved by J. BÍLÝ, M. FIEDLER, F. NOŽIČKA [2] by means of graph theory. By establishing this connection we avoid the necessity of building up a special theory for the algorithm described in [1]. The approach by means of the simplex method also suggests another method of treating degeneracy.

The transportation problem of linear programming may be stated as follows: To minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij} \tag{1}$$

under the conditions

$$\sum_{i=1}^{m} x_{ij} = a_j\,, \quad j = 1, 2, \ldots, n\,, \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = b_i\,, \quad i = 1, 2, \ldots, m\,, \tag{3}$$

$$x_{ij} \geqq 0\,, \quad i = 1, 2, \ldots, m,\; j = 1, 2, \ldots, n\,, \tag{4}$$

where a_j and b_i are given positive numbers, satisfying

$$\sum_{j=1}^{n} a_j = \sum_{i=1}^{m} b_i \tag{5}$$

and c_{ij} are given positive constants.

The condition (5) implies that one of the equations is superfluous and may be omitted. Suppose that it is the last one, viz. $\sum_{j=1}^{n} x_{mj} = b_m$.

Denote by $\boldsymbol{\delta}_k^{(1)}$ the column vector $\boldsymbol{\delta}_k^{(1)} = (\delta_{k1}, \delta_{k2}, \ldots, \delta_{kn})'$, $k = 1, 2, \ldots, m - 1$ (the prime denotes transposition), by $\boldsymbol{\delta}_k^{(2)}$ the column vector $\boldsymbol{\delta}_k^{(2)} = (\delta_{k1}, \delta_{k2}, \ldots, \delta_{k,m-1})'$, $k = 1, 2, \ldots, n$ where $\delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$. It is easily seen that the system (2), (3) may be written $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} P_{ij} = P_0$; where

$$P_{ij} = \begin{pmatrix} \boldsymbol{\delta}_j^{(1)} \\ \boldsymbol{\delta}_i^{(2)} \end{pmatrix} \quad \text{for} \quad i = 1, 2, \ldots, m - 1;\ j = 1, 2, \ldots, n$$

$$P_{mj} = \begin{pmatrix} \boldsymbol{\delta}_j^{(1)} \\ \mathbf{O} \end{pmatrix} \quad \text{for} \quad j = 1, 2, \ldots, n\,,$$

(here $\mathbf{O}$ denotes a column of $m - 1$ zeros) and P_0 is the transpose of the row $(a_1, \ldots, a_n, b_1, \ldots, b_{m-1})$.

Among the columns P_{ij} we may find $m + n - 1$ linearly independent ones. Thus we may select from among them a basis of the $(m + n - 1)$-dimensional space whose elements are columns of $m + n - 1$ real numbers.

Suppose first that the problem is non-degenerate, i. e. that the column $P_0 = (a_1, \ldots, a_n, b_1, \ldots, b_{m-1})'$ depends linearly on not less than $m + n - 1$ columns P_{ij}. The case in which this assumption is not satisfied will be treated later. The simplex method applied to this problem consists in, first, obtaining a solution $\mathbf{x}^{(1)} = (x_{11}^{(1)}, x_{12}^{(1)}, \ldots, x_{1n}^{(1)}, \ldots, x_{mn}^{(1)})$ with the property

"the system $\{P_{ij} : x_{ij}^{(1)} > 0\} = \mathscr{P}$ is a basis of the space of columns with $m + n - 1$ components",

and then in repeated application of the following optimality criterion:

"let $\xi_{ij}^{pq_1}, \xi_{ij}^{pq_2}, \ldots, \xi_{ij}^{pq_{m+n-1}}$ be the coordinates of P_{ij} in the basis $\mathscr{P}$; for every (i, j) form the difference

$$\Delta_{ij} = c_{ij} - \sum_{\nu=1}^{m+n-1} c_{ij} \xi_{ij}^{pq_\nu}\,.$$

If all Δ_{ij}'s are non-negative, then the solution is optimal; if, for some (i, j), Δ_{ij} is negative, the solution may be improved by giving the corresponding $x_{ij}^{(1)}$ a positive value and correspondingly changing the remaining positive components of the solution $\mathbf{x}^{(1)}$."

Due to the special form of the columns P_{ij} the coordinates ξ_{ij}^{pq} are very simple, equal to 1 or 0 or -1 only. Accordingly the optimality criterion takes on a simple form, so that it is even unnecessary to arrange the computation in the usual simplex tableaux.

Let us express every P_{ij} in terms of the basis vectors from $\mathscr{P}$. It is clear that any value of the first subscript, i, must appear among the subscripts of the elements of $\mathscr{P}$. The same holds for the second subscripts, j. Thus for our column P_{ij} there must be an element with the same value of the first subscript, i, say P_{ik_1}, in the basis, and an element with the same value of the second subscript, say P_{l_1j}. If now $x^{(1)}_{l_1k_1} > 0$, we have $P_{l_1k_1} \in \mathscr{P}$ and thus P_{ij} may be expressed

$$P_{ij} = P_{ik_1} + P_{l_1j} - P_{l_1k_1} .$$

This is easily seen by recalling the expression for P_{ij} in terms of $\boldsymbol{\delta}^{(1)}_j$ and $\boldsymbol{\delta}^{(2)}_i$. If $P_{l_1k_1}$ is not in the basis, then there must be some $P_{l_1k_2}$ and $P_{l_2k_1}$ in it; if $P_{l_2k_2}$ is in the basis, we may write

$$P_{ij} = P_{ik_1} + P_{l_1j} - P_{l_1k_2} - P_{l_2k_1} + P_{l_2k_2} ;$$

if $P_{l_2k_2}$ is not in the basis, we add $P_{l_2k_3}$, $P_{l_3k_2}$ and $P_{l_3k_3}$, etc. The general expression for P_{ij} is

$$P_{ij} = P_{ik_1} + P_{l_1j} - P_{l_1k_2} - P_{l_2k_1} + P_{l_2k_3} + P_{l_3k_2} - \dots$$

Not more than $m + n - 1$ columns are involved. Thus the optimality criterion assumes the form

$$\Delta_{ij} = c_{ij} - c_{ik_1} - c_{l_1j} + c_{l_1k_2} + c_{l_2k_1} - \dots$$

In the special case when P_{ij} is expressed by means of three columns,

(6) $$\Delta_{ij} = c_{ij} - c_{ik_1} + c_{l_1k_1} - c_{l_1j} ;$$

when five columns are needed for the expression of P_{ij},

(7) $$\Delta_{ij} = c_{ij} - c_{ik_1} - c_{l_1j} + c_{l_1k_2} + c_{l_2k_1} - c_{l_2k_2} ,$$

etc.

The basis elements are easily recognised; if we arrange the solution as a matrix the rows of which are numbered by i (they correspond to the production centres) and the columns by j (they correspond to the consumption centres) then the positions of positive numbers $x^{(1)}_{ij}$ indicate the subscripts of the basis elements. Thus e. g. if $x^{(1)}_{11} > 0$, i. e., if the first element of the matrix is positive, P_{11} is in the basis. Since x_{ij}, x_{ik_1} stand in the same row, x_{ij}, x_{l_1j} in the same column, etc., we see that for P_{ij} to be expressed by means of three vectors only it is necessary to be able to find a "rectangle" whose one "corner point" is (i, j) and the three other ones are occupied by positive $x^{(1)}_{st}$'s. For other P_{ij}'s (not expressible by three basis vectors only) more complicated paths breaking at right angles must be constructed, with positive numbers at the corner points and $x^{(1)}_{ij} = 0$.

Thus we may apply the optimality criterion without ever constructing the simplex tableaux, using only the table of the transportation costs c_{ij} and the table of the first solution $\mathbf{x}^{(1)}$; this first solution is obtained by assigning largest possible shipments to cheapest possible routes as in the index method; assume for the moment that by this procedure a solution with precisely $m + n - 1$ positive components is obtained. For any $x^{(1)}_{ij} = 0$ we seek the rectangular path connecting it with positive $x^{(1)}_{st}$'s and form the corresponding differences of the type (6), (7) etc. The organization of the computation is not discussed here — the procedure may be faciliated by special aids or short-cuts. Having picked out some negative Δ_{ij}, say $\Delta_{\lambda\mu}$, we form a new solution, $\mathbf{x}^{(2)} = (x_{11}^{(2)}, \ldots, x_{mn}^{(2)})$ with

$$x_{\lambda\mu}^{(2)} = \min \{x_{ij}^{(1)} : x_{ij}^{(1)} > 0, \xi_{\lambda\mu}^{ij} > 0\} ,$$

$$x_{ij}^{(2)} = x_{ij}^{(1)} - \xi_{\lambda\mu}^{ij} x_{\lambda\mu} \quad \text{for} \quad \{(i, j) : x_{ij}^{(1)} > 0\} , \quad x_{ij}^{(1)} = 0 \quad \text{otherwise} .$$

At length, $x_{\lambda\mu}^{(2)}$ is equal to the least positive $x_{pq}^{(1)}$ for which P_{pq} appeared in the expression of $P_{\lambda\mu}$ with a coefficient of $+1$ or, alternatively, the laest positive $x_{pq}^{(1)}$ for which c_{pq} appeared in $\Delta_{\lambda\mu}$ with a negative coefficient. Other $x_{ij}^{(2)}$'s are obtained from the positive $x_{ij}^{(1)}$'s by simply adding or subtracting $x_{\lambda\mu}^{(2)}$ from them where necessary to keep the marginal totals a_j and b_i unchanged. Due to the choice of $x_{\lambda\mu}^{(2)}$ precisely one of the $x_{ij}^{(1)}$'s vanishes; so that the system $\{P_{ij} : x_{ij}^{(2)} > 0\} = \mathscr{P}^{(2)}$ again forms a basis of the space of columns with $m + n - 1$ components (it is known that the system thus obtained is linearly independent). The whole procedure is then repeated with $\mathbf{x}^{(2)}$. Passing to a new solution and to a new basis we need not compute the new coefficients ξ_{ij}^{pq}; the next stage of computation is carried out similarly to the preceding one, using only the table of transportation costs and the solution $\mathbf{x}^{(2)}$. If at some stage of computation all the Δ_{ij}'s are nonnegative, this signalises that an optimal solution has been reached. This is the algorithm of [1].

In the case of degeneracy, when at some step of the iterative procedure, say the k-th one, the minimum of $\{x_{ij}^{(k)} : x_{ij}^{(k)} > 0, \xi_{\lambda\mu}^{ij} = 1\}$ is attained at two or more combinations of subscripts, so that the system $\{P_{ij} : x_{ij}^{(k+1)} > 0\}$ no longer forms a basis, CHARNES's method [3] of resolving degeneracy results in the following rule: From among the possible candidates for replacement, P_{ij}, only that P_{uv} is deleted from the basis and replaced by $P_{\lambda\mu}$, which first has the smallest value of $\xi_{\lambda\mu}^{uv}$. (It is clear what this means graphically — in terms of the paths denoting expressions for P_{ij} in terms of basis columns.) The remaining columns P_{ij} are kept in the basis. The fact that P_{ij} remains in the basis must be expressed by marking the (i, j)-th field in the table of solutions. In further computations this field is treated as though it were occupied by a positive component of solution, althought actually $x_{ij}^{(k+1)} = 0$. Thus in constructing the paths this field may form a corner. A practical rule for deciding which P_{ij} should be kept in the basis (i. e. which fields should be marked) gives J. HABR in [4].

If degeneracy occurs immediately at the beginning of the computation, when the initial solution is being sought by the index method (so that we do not obtain $m + n - 1$ positive components) we must add the number of columns necessary; i. e. to mark several fields in the table of solutions occupied by $x = 0$, so that to any other "empty" field there may be found the rectangular figure referred to above. These additional columns P_{ij} may be found by two procedures. Either by picking out those "empty" fields for which the path connecting them with "occupied" fields does not exist. Or alternatively we may use the fact that the values a_j, b_i are not relevant to the dependence or independence of the columns P_{ij}, and find the initial solution not for the original problem but for one in which the numbers a_j, b_i have been slightly modified. This modification may consist — as suggested in [1] — in adding a sufficiently small number ε to every b_i and $m\varepsilon$ to one of the a_j's. When the first solution has been found, ε is put equal to zero, i. e. we return back to the original problem. The columns P_{ij} corresponding to $x_{ij}^{(1)} = \varepsilon$ in the modified problem are kept in the basis, even though ε is put equal to zero, i. e. the fields occupied by ε in the first solution are marked. This method of resolving degeneracy, however, seems to be less convenient computationally than keeping the ε's throughout the whole procedure and putting them equal to zero afterwards.

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Výtah

POZNÁMKA K ŘEŠENÍ DOPRAVNÍHO PROBLÉMU LINEÁRNÍHO PROGRAMOVÁNÍ SIMPLEXOVOU METODOU

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V této poznámce je z obecné simplexové metody lineárního programování odvozen algoritmus pro řešení tzv. dopravního problému, ke kterému dospěl F. NOŽIČKA v [1] geometrickou cestou.

Dopravní problém lineárního programování ve své nejjednodušší formě zní: Jest minimalisovat lineární formu v mn proměnných x_{ij} (1) na množině nezáporných řešení soustavy lineárních rovnic (2), (3), kde čísla a_j, b_i jsou vázána podmínkou (5). Soustavu (2), (3) lze zapsat

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} P_{ij} = P_0 ,$$

kde P_{ij} je sloupec s $m + n - 1$ složkami, $P'_{ij} = (\delta_{j1}, \ldots, \delta_{jn}, \delta_{i1}, \ldots, \ldots, \delta_{im-1})$ $(i = 1, 2, \ldots, m - 1, \; j = 1, 2, \ldots, n)$, $P'_{mj} = (\delta_{j1}, \ldots, \delta_{jn}, 0, \ldots, 0)$ (čárka značí transposici), $P'_0 = (a_1, \ldots, a_n, b_1, \ldots, b_{m-1})$, $\delta_{st} = 0$ pro $s \neq t$, $\delta_{st} = 1$ pro $s = t$.

V článku je ukázáno, že v nedegerovaném případě (kdy P_0 tvoří s kterýmikoliv $m + n - 2$ vektory P_{ij} lineárně nezávislý systém) se kriterium optimálnosti řešení $\mathbf{x} = (x_{11}, \ldots, x_{mn})$ s $m + n - 1$ kladnými složkami redukuje vzhledem ke zvláštním vlastnostem sloupců P_{ij} na vyšetření rozdílů Δ_{ij} typu (6), (7) atd. Zde jsou ik_1, l_1j, l_1k_1, l_1k_2 atd. indexy u prvků P_{ik_1} atd. base odpovídající kladným složkám řešení $\mathbf{x}$, pomocí nichž je vyjádřen P_{ij}. Pro vyjádření prvků P_{ij} pomocí prvků base není třeba sestavovat simplexovou tabulku, protože dvojice i_1k_1, l_1j, l_1k_1 atd. lze vyhledat z tabulky řešení $\mathbf{x}$ s m řádky a n sloupci. Graficky lze si postup představit jako vyhledání cesty, vycházející z pole (i, j) tabulky a opět se do něj vracející, která se lomí v pravých úhlech jen na polích tabulky, v nichž jsou zapsány kladné složky řešení. Do kriteria Δ_{ij} vcházejí se střídavými znaménky hodnoty c_{st} odpovídající vrcholům této cesty. To je vlastně algoritmus F. Nožičky popsaný v [1].

V degenerovaném případě, kdy na některém kroku můžeme dostat řešení s méně než $m + n - 1$ kladnými složkami, je třeba doplnit basi některým prvkem P_{st}, byť mu odpovídala nulová složka řešení. I pro takové doplnění base dává simplexová metoda pravidlo, jež je v článku rovněž pro případ dopravního problému modifikováno.

Резюме

ЗАМЕТКА К РЕШЕНИЮ ТРАНСПОРТНОЙ ПРОБЛЕМЫ ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ МЕТОДОМ СИМПЛЕКСОВ

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В настоящей заметке из общего метода симплексов выведен алгорифм для решения так называемой транспортной проблемы, к которому подошел Ф. Ножичка ([1]) совсем другим (геометрическим) путем.

Транспортная проблема в своей найболее простой форме такова: минимизовать на множестве всех неотрицательных решений системы (2), (3) линейную форму (1). (Числа в скобках относятся к уравнениям в главном тексте.) Систему (2), (3) можно записать в форме $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} P_{ij} = P_0$, где P_{ij} — столбцевой вектор возникший транспонированием строки $(\delta_{j1}, \ldots,$

$\ldots, \delta_{jn}, \delta_{i1}, \ldots, \delta_{im-1}), \delta_{st}$ — символ Кронэкера, $= 0$ для $s \neq t$, $= 1$ для $s = t$, и P_0 — транспозиция строки $(a_1, \ldots, a_n, b_1, \ldots, b_{m-1})$.

В статье показано, что для невырожденной проблемы, (когда столбец P_0 линейно независим на любых $m + n - 2$ столбцах P_{ij}) применение метода симплексов сводится к рассмотрению разностей Δ_{ij} вида (6), (7), итд. Здесь ik_1, b_1j, l_1k_1, итд. субскрипты элементов P_{ik_1} итд. базиса отвечающего положительным компонентам начального решения $\mathbf{x} = (x_{11}, \ldots, x_{mn})$, при помощи которых выражается P_{ij}. Для выражения P_{ij} в элементах базиса не надо конструировать симплексную таблицу, так как пары ik_1, l_1j, итд. можно отыскать в таблице решения $\mathbf{x}$ (с m строками и n столбцами). Процедуру можно представить себе как нахождение дороги, начинающей на поли таблицы и опять возвращающейся на i, j, которая может уклоняться в правых углах на полях, в которых записаны положительные компоненты решения; в критерий Δ_{ij} входят с чередующими знаками значения c_{st}, отвечающие поворотам этой дороги. Это в действительности алгорифм из [1].

В вырожденном случае, когда после некоторого шага можно получить решение с менее чем $m + n - 1$ положительными компонентами, надо базис дополнить, некоторым столбцем P_{ij}, хотя он отвечает нулевой компоненте x_{ij}. Для такого дополнения в общем методе симплексов тоже имеется правило. В настоящей статье оно приспособлено для транспортной проблемы.