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THE INVESTIGATION OF THE EXISTENCE OF MAXIMAL SUBGROUPS OF SOME SIMPLE GROUPS

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Since the existence of a simple group without maximal subgroups is an unsolved problem (compare e.g. [3]), it is interesting to examine from this point of view some of the known simple groups. As concerned with the question of simplicity of mentioned groups see detailed description in [1].

The investigation was carried out by using appropriate systems of generators. The systems of generators of the groups $Sp_n(k)$, O_n^+ used here – as far as I know – are new.

I.

In this paragraph we denote by $\mathfrak{M}$ an arbitrary set of at least five elements. The symmetric group $\mathfrak{S}(\mathfrak{M})$ of the set $\mathfrak{M}$ is the group of all one-to-one mappings of the set $\mathfrak{M}$ onto itself, which change only a finite number of elements. The alternating group $\mathfrak{A}(\mathfrak{M})$ is defined as the subgroup of $\mathfrak{S}(\mathfrak{M})$ of index 2 in the usual way. Let us denote H_a a subgroup of $\mathfrak{A}(\mathfrak{M})$ which consists of all permutations s which leave the element a unchanged: $s(a) = a$.

We have

1) Every permutation of $\mathfrak{A}(\mathfrak{M})$ is a product of a finite number of cycles containing three elements.
2) If t is a permutation such that $t(a) = b$, then $tH_at^{-1} = H_b$.

Theorem 1.1. *The group H_a is a maximal subgroup of $\mathfrak{A}(\mathfrak{M})$.*

Proof. Obviously $H_a \subsetneq \mathfrak{A}(\mathfrak{M})$ and if we assume for some subgroup G

$$H_a \subsetneq G \subset \mathfrak{A}(\mathfrak{M}),$$

then there is an element $s \in G$, $s \notin H_a$. Write $s(a) = b$, $b \neq a$. Let us choose arbitrarily

$m_1, m_2 \in \mathfrak{M}$, such that $m_1 \neq m_2, m_i \neq a, m_i \neq b$ for $i = 1, 2$. Then $(m_1, m_2, k) \in H_a$ for $k \neq a$, $(m_1, m_2, k) \in H_b$ for $k \neq b$. But $G \supset sH_a s^{-1} = H_b$ which implies $(m_1, m_2, a) \in G$ and therefore $G = \mathfrak{A}(\mathfrak{M})$.

II.

Definition 2.1. The group of all the square matrices of n rows with elements in a skew field k and having the determinant equal to $\bar{1}$ [1]), will be denoted by $SL_n(k)$, where $n \geqq 2$.

This group — as it is well known — is generated by all the matrices $B_{ij}(\lambda)$, where $B_{ij}(\lambda)$ is a matrix, which has the identity element in the principal diagonal and in the ij position $(i \neq j)$ the element λ, the others elements being zeros.

The center Z of this group consists of all matrices L of the form

$$L = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix},$$

where α belongs to the center of the group k^* (k^* is a multiplicative group of k) and $\det L = 1$.

Definition 2.2. The factor group $SL_n(k)/Z$ will be denoted by $PSL_n(k)$.

Theorem 2.3. *All the matrices of the form*

$$\begin{pmatrix} \lambda & N_{1,n-1} \\ C_{n-1,1} & A_{n-1,n-1} \end{pmatrix} \tag{1}$$

(where the indices indicate the type of the matrix, N is the zero matrix), which belong to $SL_n(k)$, form a maximal subgroup of the group $SL_n(k)$. We shall denote it by $MSL_n(k)$.

The proof is carried out by investigation of a subgroup G for which $MSL_n(k) \subsetneq G \subset SL_n(k)$. This subgroup necessarily contains a matrix

$$\begin{pmatrix} \lambda & a_2 & \dots & a_n \\ B_{n-1,1} & A_{n-1,n-1} \end{pmatrix}, \quad \underset{k}{\exists}(a_k \neq 0,\ 2 \leqq k \leqq n) \tag{2}$$

and also all generators $B_{ij}(\lambda)$ which belong to $MSL_n(k)$. Using them, (2) can be modified by suitable multiplication on the right and on the left into an arbitrary matrix of the form $B_{1n}(\nu)$ and $B_{1n}(\nu)$ further to an arbitrary matrix $B_{1j}(\nu)$, $2 \leqq j \leqq n$, which means that G contains all generators uf $SL_n(k)$, and hence $G = SL_n(k)$.

[1]) For the definition of the determinant over a skew field see [1], Chap. IV.

From here, the statement about the maximal subgroup of the group $PSL_n(k)$ follows easily.

Theorem 2.4. *The group $MSL_n(k)/Z$ is a maximal subgroup of $PSL_n(k)$.*

III.

Definition 3.1. A matrix A is said to be *s-orthogonal* if $A'SA = S$, where A' denotes the transpose of the matrix A and where

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}.$$

The group $Sp_n(k)$ will be considered as a group of all square matrices of n rows which are s-orthogonal; n is even, $n \geqq 2$, k is a commutative field.

The center Z of $Sp_n(k)$ consists of $\pm E$.

Definition 3.2. The factor group $Sp_n(k)/Z$ will be denoted by $PSp_n(k)$.

Definition 3.3. Let i be an integer, $1 \leqq i \leqq n$. We associate the integer i with the integer $\bar{i}$ in the following manner:

$$\bar{i} = i - 1 \quad \text{for} \quad i \equiv 0 \bmod 2\,, \qquad \bar{i} = i + 1 \quad \text{for} \quad i \equiv 1 \bmod 2\,.$$

For $i \neq j$ we write $D_{ij}(\lambda) = (d_{rs})$, where

$$d_{rr} = 1 \quad \text{for} \quad 1 \leqq r \leqq n,\ d_{ij} = \lambda,\ d_{\bar{j}\bar{i}} = \varepsilon\lambda,\ \varepsilon = (-1)^{i-j+1}\,,$$

the other $d_{rs} = 0$.

It follows easily that $D_{ij}(\lambda) \in Sp_n(k)$.

Theorem 3.4. *The group $Sp_n(k)$ is generated by all matrices $D_{ij}(\lambda)$, $\lambda \in k$, $1 \leqq i \leqq$ $\leqq n$, $1 \leqq j \leqq n$.*

Proof. Let us consider an s-orthogonal matrix $A = (a_{ij})$. It can happen that the element a_{11} is the unique non-zero element in the first column. In this case let us multiply the matrix A on the left by matrix $D_{21}(1)$. Hence, we obtain a non-zero element a'_{21}. Then, it can be supposed that an m exists, $2 \leqq m \leqq n$, for which

$a_{m1} \neq 0$. Let us multiply the matrix A on the left by the matrix $D_{1m}((1 - a_{11})\, a_{m1}^{-1})$. In this way we get a matrix

$$\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}. \tag{3}$$

Here we should write more correctly a''_{ij} instead of a_{ij}; to simplify the notation we omit the commas. We shall often proceed similarly.

We multiply the matrix (3) on the left by matrices $D_{j1}(-a_{j1})$ for $2 < j \leqq n$. The matrix (3) will get the form of the matrix

$$\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a'_{21} & a'_{22} & \dots & a'_{2n} \\ 0 & a'_{32} & \dots & a'_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} \end{pmatrix}.$$

Then, we multiply also on the left by matrix $D_{21}(-a'_{21})$ and we obtain a matrix of the form

$$\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & a''_{22} & \dots & a''_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} \end{pmatrix}.$$

Now we shall use the induction. Let $j \geqq 2$ and suppose we have already adjusted the $(j - 1)$-th column of the matrix A and therefore for the $j - 1$ first columns the following equalities

$$a_{km} = \delta_{km} \quad \text{for} \quad 1 \leqq k \leqq n,\ 1 \leqq m \leqq j - 1$$

hold. Here is, as usual, $\delta_{kk} = 1$ and $\delta_{km} = 0$ for $k \neq m$.

We will show that also the j-th column can be brought to this form.

a) Let j be odd.

Then $a_{kj} = 0$ for $1 \leqq k \leqq j - 1$; for, the matrix being transformed is also s-orthogonal. The remaining a_{kj} ($i.e.$ $k > j - 1$) cannot be all zeros and if we proceed in a similar way as in the first column we get

$$a_{jj} = 1,\ a_{kj} = 0 \quad \text{for} \quad j < k \leqq n\,.$$

b) Let j be even.

The considered matrix is s-orthogonal; hence,

$$a_{kj} = 0 \quad \text{for} \quad 1 \leqq k < j - 1,\ a_{jj} = 1\,.$$

Now we multiply successively on the left by the matrices $D_{kj}(-a_{kj})$ for all $k, j < k \leqq$

$\leqq n$; in this way we obtain an element $a'_{\bar{j}-1,j}$ (for $\bar{j} = j - 1$); then we multiply on the left by the matrix $D_{j-1,j}(-a'_{j-1,j})$.

In both cases we continue till we arrive to the n-th column. In this manner we obtain the unit matrix. Taking into account the fact, that the matrix $D_{ij}(-\lambda)$ is inverse to the matrix $D_{ij}(\lambda)$, we obtain the statement of the theorem.

Theorem 3.5. *All matrices of the form*

$$\begin{pmatrix} \lambda & N_{1,n-1} \\ C_{n-1,1} & A_{n-1,n-1} \end{pmatrix}$$

(where $\lambda \in k$, $\lambda \neq 0$), which belong to $Sp_n(k)$, form a maximal subgroup of the group $Sp_n(k)$; we shall denote it by $MSp_n(k)$.

The proof is based on the same idea as the proof of the Theorem 2.3. We do not give it here, because it is too long. The large extent of the proof is caused – amongst other things – by the necessity to examine separatly the case of the field of the characteristic 2 (see [2]).

The result for $PSp_n(k)$ is obtained now easily.

Theorem 3.6. *The group $MSp_n(k)/Z$ is a maximal subgroup of $PSp_n(k)$.*

IV.

In this paragraph we turn our attention to the square matrices of n rows with elements in the field of real numbers. It is supposed $n \geqq 3$.

Definition 4.1. A matrix A is said to be orthogonal if $AA' = E$; it is said to be properly orthogonal if it is orthogonal and $\det A = 1$. The group of all the orthogonal matrices will be denoted O_n, the group of all the properly orthogonal matrices O_n^+.

The center of the group O_n^+ is E for n odd, $\pm E$ for n even.

Definition 4.2. We shall denote by $F_{ij}(x, y)$ $(i \neq j)$ the matrix (f_{rs}), where $f_{ii} = x$, $f_{jj} = x$, $f_{ij} = y$, $f_{ji} = -y$, $f_{rr} = 1$ for $r \neq i$, $r \neq j$, $f_{rs} = 0$ otherwise, and where, moreover, $x^2 + y^2 = 1$, x, y being real.

It is easily to be seen that $F_{ij} \in O_n^+$.

Theorem 4.3. *The group O_n^+ is generated by all matrices $F_{ij}(x, y)$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$.*

Proof. Let us take a properly orthogonal matrix A. If $a_{11} = 0$, then a $j > 1$ exists so that $a_{j1} \neq 0$. Let us multiply the matrix A by the matrix $F_{j1}(0, -1)$ on the left. In this way we shall get in the first place the element $a'_{11} = a_{1j} \neq 0$. Therefore we can suppose $a_{11} \neq 0$. Let us choose arbitrarily $j > 1$. We shall prove that multiplying by the matrices F_{ij} we can achieve $a_{j1} = 0$.

If we multiply the considered matrix on the left by the matrix $F_{j1}(x, y)$, we get $a'_{j1} = xa_{j1} + ya_{11}$. By a convenient choice of x, y we achieve $a'_{j1} = 0$ for $1 < j \leqq n$. Then, necessarily $a_{11} = \pm 1$. If $a_{11} = -1$, let us multiply the matrix on the left by $F_{21}(-1, 0)$; then $a'_{11} = 1$ (for $f > 1$ we have still $a'_{j1} = 0$). Since the considered matrix must be properly orthogonal, it is of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Suppose we have already adjusted in this way the first $j - 1$ columns of the original matrix so that

$$a_{lm} = \delta_{lm}, \quad 1 \leqq l \leqq n, \quad 1 \leqq m \leqq j - 1.$$

Here the elements a_{kj} for $1 \leqq k \leqq j - 1$ must be equal 0. In the same way as for the first column, we get first $a_{jj} \neq 0$ and then $a_{kj} = 0$ for $j < k \leqq n$ with $a_{jj} = \pm 1$; in the case $j < n$ we make $a_{jj} = +1$ if multiplying on the left by the matrix $F_{j+1,j}(-1, 0)$; in the case $j = n$ must be $a_{nn} = 1$ – it follows from the fact that $A \in O_n^+$, $F_{ij} \in O_n^+$. Therefore, it is possible by multiplying by the matrices $F_{ij}(x, y)$, to bring an arbitrary matrix A to the unit matrix. If doing so we have to use the fact that $F_{ij}(x, y)$ is an inverse matrix to $F_{ij}(x, -y)$. The theorem follows.

Theorem 4.4. *All the properly orthogonal matrices of the form*

$$\begin{pmatrix} \varepsilon & N_{1,n-1} \\ N_{n-1,1} & A_{n-1,n-1} \end{pmatrix}, \quad \varepsilon = \pm 1,$$

constitute a maximal subgroup of O_n^+; we denote this group by MO_n^+. Obviously $MO_n^+ \cong O_{n-1}$.

This theorem can be proved by reducing the case with general n to the case $n = 3$ which can be handled more easily. (See [2]). Hence we have

Theorem 4.5. *The group MO_n^+/Z is a maximal subgroup of the group O_n^+/Z.*

V.

Definition 5.1 Let us consider the groups $SL_2(k)$, $SL_3(k)$, ... and the isomorphisms φ_i mapping $SL_i(k)$ into $SL_{i+1}(k)$ so that

$$A \in SL_i(k) \Rightarrow \varphi_i(A) = \begin{pmatrix} A & N_{i,1} \\ N_{1,i} & 1 \end{pmatrix};$$

in this way a unique group is determined. It is the union of the increasing sequence of the groups $SL_i(k)$ $(i = 2, 3, \ldots)$ [2]. We shall denote it by $SL_\infty(k)$.

Further let us take into consideration the groups

$$(4) \qquad Sp_2(k),\ Sp_4(k), \ldots$$

and the isomorphisms φ_i mapping $Sp_i(k)$ into $Sp_{i+2}(k)$ defined by

$$A \in Sp_i(k) \Rightarrow \varphi_i(A) = \begin{pmatrix} A & N_{i,2} \\ N_{2,i} & E_{2,2} \end{pmatrix}.$$

Again, in this way a unique group is determined which is the union of the increasing sequence of the groups (4). We shall denote it by $Sp_\infty(k)$.

Finally, let us take into consideration the groups

$$(5) \qquad O_3^+,\ O_4^+, \ldots$$

and the isomorphisms φ_i mapping O_i^+ into O_{i+1}^+ defined by

$$A \in O_i^+ \Rightarrow \varphi_i(A) = \begin{pmatrix} A & N_{i,1} \\ N_{1,i} & 1 \end{pmatrix}.$$

In this way a unique group is determined which is the union of the increasing sequence of the groups (5). We shall denote it by O_∞^+.

Theorem 5.2. *The groups $SL_\infty(k)$, $Sp_\infty(k)$ and O_∞^+ are simple.*

Proof. We will show it *e.g.* for $SL_\infty(k)$. The proofs for the other groups are similar. If M belongs to a normal subgroup of $SL_\infty(k)$, then an n exists such that the matrix $M_{n,n}$ corresponding to M is an element of $SL_n(k)$ and we have

$$M_{n,n} = \begin{pmatrix} \alpha & 0 & \ldots & 0 \\ 0 & \alpha & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \alpha \end{pmatrix}, \quad \det M_{n,n} = 1\,;$$

for, $M_{n,n}$ is in a normal subgroup of $SL_n(k)$ and, hence, it belongs to the center. Similarly,

$$M_{n+1,n+1} = \begin{pmatrix} \beta & 0 & \ldots & 0 \\ 0 & \beta & \ldots & 0 \\ & & & \\ 0 & 0 & \ldots & \beta \end{pmatrix}, \quad \det M_{n+1,n+1} = 1\,,$$

where $\varphi_n(M_{n,n}) = M_{n+1,n+1}$. Therefore, $\alpha = \beta = 1$ and thus $M_{n,n} = E$.

[2]) See [4], pp. 67–68.

By means of the above mentioned isomorphisms we get from the subgroups $MSL_n(k)$, $MSp_n(k)$, MO_n^+ further groups which we denote by $MSL_\infty(k)$, $MSp_\infty(k)$, MO_∞^+.

The following theorem is then an immediate consequence of the theorems 2.3, 3.5, 4.4.

Theorem 5.4. *The groups* $MSL_\infty(k)$, $MSp_\infty(k)$, MO_∞^+ *are maximal subgroups of the groups* $SL_\infty(k)$, $Sp_\infty(k)$, O_∞^+, *respectively.*

In conclusion I wish to express my gratitude to Prof. VL. KOŘÍNEK for his advice and help during the work.

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Výtah

VYŠETŘENÍ EXISTENCE MAXIMÁLNÍCH PODGRUP JISTÝCH JEDNODUCHÝCH GRUP

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V článku jsou popsány maximální podgrupy H_a, $MSL_n(k)/Z$, $MSp_n(k)/Z$, MO_n^+/Z grup $\mathfrak{A}(\mathfrak{M})$, $PSL_n(k)$, $PSp_n(k)$, O_n^+/Z.

V grupách $SL_n(k)$, $Sp_n(k)$, O_n^+ jsou maximální například podgrupy matic tvaru (1), kde na λ, $C_{n-1,1}$, $A_{n-1,n-1}$ jsou kladeny odpovídající podmínky.

Z uvažovaných grup jsou metodou nítí konstruovány jiné jednoduché grupy, v nichž jsou opět udány jisté maximální podgrupy.

Резюме

ИССЛЕДОВАНИЕ СУЩЕСТВОВАНИЯ МАКСИМАЛЬНЫХ ПОДГРУПП НЕКОТОРЫХ ПРОСТЫХ ГРУПП

ЛАДИСЛАВ БЕРАН (Ladislav Beran), Прага

В статье описаны максимальные подгруппы H_a, $MSL_n(k)/Z$, $MSp_n(k)/Z$, MO_n^+/Z групп $\mathfrak{A}(\mathfrak{M})$, $PSL_n(k)$, $PSp_n(k)$, O_n^+/Z.

В группах $SL_n(k)$, $Sp_n(k)$, O_n^+ являются, например, максимальными подгруппы матриц вида (1), где на λ, $C_{n-1,1}$, $A_{n-1,n-1}$ наложены соответствующие условия.

Из рассматриваемых групп конструированы методом нитей другие простые группы, в которых снова определены некоторые максимальные подгруппы.