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ON ORDERING VERTICES OF INFINITE TREES

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1. By a finite (infinite) tree we understand a non-oriented connected graph without circles and with finitely (denumerably) many vertices. A tree is either a finite or an infinite tree. Denote by $\{T\}$ the set of the vertices of a tree T. The edge of the tree T connecting vertices a and b will be denoted by (a, b). A finite path of a tree T is a finite sequence of its edges (z_i, z_{i+1}), $i = 0, 1, 2, \ldots, n - 1$, such that $z_{i_1} \neq z_{i_2}$ for $i_1 \neq i_2$. Such a path is said to be of length n, it starts with the vertex z_0 and ends in z_n, and may be denoted by $(z_0, \ldots, z_n)$. The vertices z_0 and z_n are called the end-vertices of this path, the remaining vertices are termed inner. An infinite path of a tree T is an infinite sequence of its edges (z_i, z_{i+1}), $i = 0, 1, \ldots$, such that $z_{i_1} \neq z_{i_2}$ for $i_1 \neq i_2$. We say that this path begins in the vertex z_0 or that z_0 is its end-vertex, the other vertices are its inner vertices. Furthermore, a vertex x of a tree T is said to be between the vertices a and b of the same tree if there exists a finite path $(a, \ldots, b)$ of T such that x is its inner vertex. The distance $\mu(a, b)$ of distinct vertices a, b of a tree T is the length of the path $(a, \ldots, b)$ [1]; and for $x \in \{T\}$ set $\mu(x, x) = 0$. If τ_i denotes a finite sequence $t_1^i, t_2^i, \ldots, t_{n_i}^i$, then let $\tau_1, \tau_2, \tau_3, \ldots$ denote the sequence $t_1^1, t_2^1, \ldots, t_{n_1}^1, t_1^2, t_2^2, \ldots, t_{n_2}^2, t_1^3, t_2^3, \ldots, t_{n_3}^3, \ldots$.

Let T be a finite tree, $a, b \in \{T\}$, $a \neq b$. Then a 2-(a, b)-ordering of T is an ordering of the set $\{T\}$ into a simple finite sequence beginning with a, ending with b and such that the distance of consecutive members if this sequence is at most 2 (in the metric of T). If there exists a 2-(a, b)-ordering of T, we shall say that T can be 2-(a, b)-ordered; and otherwise that T cannot be 2-(a, b)-ordered. Furthermore, we shall say that it is possible to 2-order a finite tree T if there exist $x, y \in \{T\}$, $x \neq y$, such that T can be 2-(x, y)-ordered. Analogously, consider a tree U (finite or infinite), and an $a \in \{U\}$. A 2-a-ordering of the tree U is an ordering of the set $\{U\}$ into a simple sequence (finite or infinite), starting with a and such that the distance of consecutive members of this sequence is at most 2 in the metric of U. If there exists a 2-a-ordering of the tree U, we shall say that U can be 2-a-ordered.

In [2] there were obtained necessary and sufficient conditions under which it is

[1]) For distinct vertices a, b of a tree T the path $(a, \ldots, b)$ is uniquely determined [1, p. 165, theorem 1 (6)].

possible to 2-(a, b)-order or 2-order a finite tree. M. SEKANINA dealt with similar orderings of the vertices of a graph, [3], [4]. In this paper we shall give a necessary and sufficient condition under which it is possible to 2-a-order an infinite tree. This problem is due to G. A. DIRAC.

2. We shall recall the results of [2] needed in the sequel.

Theorem. *Let T be a finite tree, a, b two distinct vertices. It is possible to 2-(a, b)-order T if and only if for the subtree T_1 obtained from T by omitting all vertices of order 1 and all edges incident to these except for a, b there holds:*

1° *the degree of all vertices in T_1 is at most 4,*

2° *all vertices of degree 3 and 4 in T_1 are inner vertices of the path $(a, \ldots, b)$,*

3° *between every two vertices of degree 4 in T_1 there exists at least one vertex of degree 2 in T. If the degree of a is greater than 1 in T, then between it and the nearest vertex of degree 4 in T_1 there exists a vertex of degree 2 in T. Similarly for b. If the degree of both vertices a and b is greater than 1 in T, then there exists between them at least one vertex of degree 2 in T.*

Corollary. *A finite tree T can be 2-ordered if and only if for the subtree T_2 (the empty tree and the tree consisting of one vertex are now permitted) obtained from T by omitting all vertices of order 1 and incident edges there holds:*

1° *the degree of all vertices in T_2 is at most 4,*

2° *in T_2 there exists a path containing all vertices of degree 3 and 4 (in T_2),*

3° *between every two vertices of degree 4 in T_2 there exists at least one vertex of degree 2 in T.*

3. *Let T_0 be an infinite tree and let all its vertices be of a finite degree, $a_0 \in \{T_0\}$. Then there exists an infinite path of T_0 which begins with a_0.*

When we omit a_0 and all incident edges from T_0, we obtain a finite number of components, at least one of which is infinite. Denote one of these infinite components by T_1, an infinite tree. Let $a_1 \in \{T_1\}$, $\mu(a_0, a_1) = 1$, where μ is the metric in T_0. In general, after omitting the vertex a_j and all incident edges from an infinite tree T_j, we obtain a finite number of components, at least one of which is infinite. Denote by T_{j+1} one of these infinite components. Let $a_{j+1} \in \{T_{j+1}\}$, $\mu(a_j, a_{j+1}) = 1$ (in T_0). Evidently $a_0, a_1, a_2, a_3, \ldots$ and $(a_0, a_1), (a_1, a_2), (a_2, a_3), \ldots$ are the vertices and edges of an infinite path of T_0 starting with a_0.

4. *Let T be an infinite tree, $a \in \{T\}$. If two infinite components are obtained by omitting an edge of T, then T cannot be 2-a-ordered.*

Denote by (u, v) the omitted edge, and by T_u and T_v the two corresponding infinite components. Let there exist 2-a-ordering of T, and, without loss of generality, let u precede v. There is only a finite number of vertices in front of v; thus in the ordering there must necessarily occur, following v, a vertex from T_u and a vertex from T_v, distinct from u and v. Hence one can find two neighbouring vertices x, y in the 2-a-ordering following v such that $x \in \{T_u\}$ and $y \in \{T_v\}$. As $x \neq u$ and $y \neq v$, there is $\mu(x, y) \geqq 3$, which is a contradiction.

5. *If it is possible to 2-a-order a finite or an infinite tree T, then it is also possible to 2-a-order each of its subtrees U containing a. Moreover, if we omit, in an arbitrary 2-a-ordering of T, the vertices not belonging to U, we obtain a 2-a-ordering of the subtree U.*

The statement can be proved as in [4]. Let α be some 2-a-ordering of the tree T. From the sequence α form the subsequence β by omitting all vertices of α not belonging to the subtree U. In β there occur all vertices of U precisely once. Thus β is a 2-a-ordering of U if we show that the distance of every two neighbouring vertices in β is at most 2, in the metric of U (or T). Thus, let x and y be consequent vertices in β. There are two cases possible:

a. The vertices x and y are consequent in α; then $\mu(x, y) \leqq 2$ in T and also in U.

b. Vertices x and y are not neighbouring in α; thus the 2-a-ordering α of T has the form $a, \ldots, x, z_1, z_2, \ldots, z_m, y, \ldots, (m \geqq 1)$, where $z_i \bar{\in} \{U\}$ for $i = 1, 2, \ldots, m$. Suppose that $\mu(x, y) \geqq 3$ in U and hence also in T. Let $(x, v_1, v_2, \ldots, v_n, y)$ be the path of T. Evidently $n \geqq 2$ and all vertices $v_j, j = 1, 2, \ldots, n$, belong to U. By omitting the edge (v_1, v_2) from T there result two components X and Y, $x \in \{X\}$, $y \in \{Y\}$. Hence, between the vertices $x, z_1, z_2, \ldots, z_m, y$ there exists at least one pair b, c of consequent vertices in α such that $b \in \{X\}$ and $c \in \{Y\}$. As v_1 and v_2 do not coincide with any of the vertices $x, z_1, z_2, \ldots, z_m, y$, there is also $b \neq v_1$ and $c \neq v_2$. The path $(b, \ldots, c)$ necessarily contains v_1 and v_2 as inner vertices, therefore $\mu(b, c) \geqq 3$. But this is a contradiction, because b, c are consequent in the 2-a-ordering α, i.e. $\mu(b, c) \leqq 2$. Thus $\mu(x, y) \geqq 3$ cannot occur, and therefore $\mu(x, y) \leqq 2$.

6. *Let T be an infinite tree, $a \in \{T\}$. If T contains more than one vertex of infinite degree, then it cannot be 2-a-ordered.*

Let T be an infinite tree, $a \in \{T\}$. Let b be a vertex of denumerable degree of T. Denote by Z the set of all those vertices of T of degree 1 *(with the exception of a) which are connected by an edge to the vertex b. Then T can be 2-a-ordered if an only if Z is non-empty and the subtree U obtained from T by omitting vertices of the set $Z - \{z_1\}$ and all incident edges (where z_1 is an arbitrary vertex of Z) is a finite tree which can be 2-(a, z_1)-ordered.*

Let T be an infinite tree not containing a vertex of an infinite degree, $a \in \{T\}$. Then, T can be 2-a-ordered if and only if for the subtree V obtained from T by omitting the vertices of degree 1 *(except for the vertex a) and all incident edges, there holds:*

0° *there exists precisely one infinite path W of T starting with the vertex a,* 1° *the degree of all vertices of V is at most* 4,

2° *all vertices of degree* 3 *and* 4 *in V are inner vertices of the path W,*

3° *between every two vertices of degree* 4 *in V there lies at least one vertex of degree* 2 *in T; if a has degree greater than* 1 *in T, then between it and the nearest vertex of degree* 4 *in V there lies at least one vertex of degree* 2 *in T.*

Necessity. Let T be 2-a-orderable. Let T contain vertices of an infinite degree. According to item 4, T can contain at most one vertex of denumerable degree; denote it by b. Any infinite path cannot issue out of the vertex b (e.g. $(b, b_1, b_2, \ldots)$) because, again according to item 4, by omitting the edge (b, b_1) from T there would result two infinite components, and T could not be 2-a-orderable. From the vertex b there can issue only a finite number of disjoint paths of length greater than 1, since otherwise there would exist a subtree of T not 2-orderable (item 2 corollary, 1°), and thus, according to item 5, T itself could not be 2-a-orderable. As the vertex b is of denumerable degree, the set Z must be denumerable. Denote by $z_1, z_2, z_3, \ldots$ all vertices of Z. Hence, if we omit from T the vertices $z_2, z_3, \ldots$ and edges $(z_2, b), (z_3, b), \ldots$, we obtain a finite subtree U. Consider some 2-a-ordering of T. As $\{U\}$ is a finite set, some element of $\{U\} - \{z_1\}$, say u, is the final member of this 2-a-ordering. Thus the 2-a-ordering of T is of the form $a, \ldots, z_{i_1}, \ldots, z_{i_2}, \ldots, z_{i_s}, \ldots, u, z_{i_{s+1}}, z_{i_{s+2}}, z_{i_{s+3}}, \ldots$, where $i_1, i_2, \ldots$ is a suitable permutation of integers. Let $i_p = 1$. We assert that then $a, \ldots, u, z_1, z_{i_1}, z_{i_2}, \ldots, z_{i_{p-1}}, z_{i_{p+1}}, z_{i_{p+2}}, \ldots$ is also a 2-a-ordering of T. This follows from the following consideration:

From the notation of the z_j there follows $\mu(z_m, z_n) = 2$ for $m \neq n$, $m, n \geqq 1$. Furthermore, either $u = b$, or (u, b, z_j) is a path for all $j \geqq 1$. Therefore $\mu(u, z_j) \leqq 2$ for all $j \geqq 1$. Thus, finally, if the 2-a-ordering of T is of the form $a, \ldots, x, z_j, y, \ldots$, then $\mu(x, y) \leqq 2$ because $\mu(x, b) \leqq 1$ and $\mu(b, y) \leqq 1$.

Hence, the initial segment of the new 2-a-ordering of T, namely, $a, \ldots, u, z_1$, is a 2-(a, z_1)-ordering of its subtree U; therefore U can be 2-(a, z_1)-ordered.

Now assume T does not contain a vertex of infinite degree. Then, according to item 3, an infinite path W must issue from the vertex a of T. This path is unique, according to item 4. Indeed, consider another such path. As both belong T, the tree T decomposes into components after omitting the vertex a, and at least two of these (containing the considered infinite paths) would be infinite. Hence, according to item 4, T could not be 2-a-ordered. If, furthermore, condition 1° or 3° were not satisfied, there would exist a finite subtree of the tree T such that it would not satisfy the corresponding condition 1° or 3° in item 2. Thus, this subtree would not be 2-a-orderable, and thus according to item 5, even T would not be 2-a-orderable, a contradiction. It remains to show that condition 2° is necessary, which we again perform by means of a contradiction. Suppose that there exists some vertex x of degree 3 or 4 in V which is not an inner vertex of W. Choose a vertex $c \in \{W\}$ such that the path $(a, \ldots, x)$ does not contain c. Let T_0 be the maximal subtree of T containing the vertices a and c, in which the degree of c is 1. Furthermore, let τ be a 2-a-ordering of the tree T. According to item 5, if we omit from τ the vertices not belonging to T_0, we obtain a 2-a-ordering of T_0; denote this by τ_0. As $\{W\}$ is infinite, there exists an infinite sequence of distinct vertices $r_i \in \{W\}$, $i = 1, 2, \ldots$, following the vertex c. For $i = 1, 2, \ldots$ consider the maximal subtree T_i of T containing the vertices a and r_i, the degree of the vertex r_i being equal to 1 in T_i. By omitting the vertices not belonging to T_i, we obtain from τ a sequence τ_i which is a 2-a-ordering of the tree T_i. For $i < j$, τ_i is a proper subsequence of the sequence τ_j. Let the last vertex of the sequence τ_i be t_i, $i = 0, 1, 2, \ldots$. As $\{T_0\}$ is finite, there exists an m such that $t_m \overline{\in} \{T_0\}$, i.e. the path $(a, \ldots, t_m)$ contains the vertex c. In another words, at least one vertex of degree 3 or 4 in V is not an inner vertex of $(a, \ldots, t_m)$. On the other hand, T_m can be 2-(a, t_m)-ordered. But this is a contradiction to the theorem of item 2, condition 2°.

Sufficiency. Let there be given a finite tree U which can be 2-(a, z_1)-ordered, the degree of its vertex z_1 is 1 and (z_1, b) is its edge. Construct, from this tree U, an infinite tree T by adding denumerable many vertices $z_2, z_3, \ldots \overline{\in} \{U\}$ and edges (z_2, b), $(z_3, b), \ldots$. Let $a, \ldots, z_1$ be some 2-(a, z_1)-ordering of U. Then $a, \ldots, z_1, z_2, z_3, \ldots$ is a 2-a-ordering of T because $\mu(z_i, z_{i+1}) = 2$ for $i = 1, 2, 3, \ldots$.

Assume T does not contain a vertex of an infinite degree. By omitting all the vertices of degree 1 (except for a) and all incident edges from T, obtain the tree V for which there holds 0°–3°.

Let the infinite path W of the tree T contain an infinite number of vertices of degree 2 in T, say $p_1, p_2, \ldots$. Let P_0 denote the maximal subtree of T containing a and in which the vertex p_1 has degree 1. Let P_i denote the maximal subtree of T in which the vertices p_i and p_{i+1} are of degree 1. According to item 2, there exists a 2-(a, p_1)-ordering of the finite tree P_0 (denote it by π_0), and for every $i = 1, 2, \ldots$ there exists a 2-(p_i, p_{i+1})-ordering of the finite tree P_i (denote it by π_i). Therefore, if π_i^* is the sequence obtained from π_i by omitting the last member p_{i+1}, then $\pi_0^*, \pi_1^*, \pi_2^*, \ldots$ is a 2-a-ordering of T.

Let the path W contain only a finite number of vertices of degree 2 in T; hence it contains only a finite number of vertices of degree 4 in V. Thus, there exist denumerably many vertices of W with degree 2 or 3 in V. Choose an infinite sequence $v_1, v_2, \ldots$ from these, ordered by increasing distance from a, and sucht hat all vertices of degree 4 in V are between a and v_1. Let T_0 denote the maximal subtree of T such that it contains a and that v_1 is of the degree 1 in it. Let T_i denote the maximal subtree of T such that, with the exception of vertices of the path $(v_i, \ldots, v_{i+1})$, it does not contain any vertex of the path W, that it contains the vertices v_i, and v_{i+1}, and the degree of v_{i+1} is 1 in T_i. According to the theorem of item 2, T_0 can be 2-(a, v_1)-ordered; denote by τ_0^* one of these orderings in which we omit the last member, v_1. According to the same theorem, T_i can be 2-(v_i, v_{i+1})-ordered; denote by τ_i^* one of these orderings, in which we omit the last member, v_{i+1} $(i = 1, 2, \ldots)$. Evidently $\tau_0^*, \tau_1^*, \tau_2^*, \ldots$ is a 2-a-ordering of the tree T.

References

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Výtah

USPOŘÁDÁNÍ UZLŮ NEKONEČNÉHO STROMU

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Nekonečný strom T je neorientovaný souvislý graf bez kružnic se spočetně mnoha uzly. Množinu uzlů nekonečného stromu T označíme $\{T\}$. Nechť je $a \in \{T\}$. Říkáme, že takový strom lze 2-a-uspořádat, jestliže je možné seřadit uzly tohoto stromu v prostou posloupnost $a = t_1, t_2, t_3, \ldots$ takovou, že vzdálenost mezi sousedními uzly v této posloupnosti je nejvýše dvě (v metrice stromu T).

V práci je dokázána nutná a postačující podmínka pro to, aby bylo možné daný nekonečný strom 2-a-uspořádat:

Nechť T je nekonečný strom, $a \in \{T\}$. Existuje-li v T více než jeden uzel nekonečného stupně, pak T nelze 2-a-uspořádat.

Nechť T je nekonečný strom, $a \in \{T\}$. Nechť b je uzel stromu T, jehož stupeň je spočetný. Označme Z množinu všech uzlů stupně 1 stromu T s výjimkou uzlu a, které jsou spojeny hranou s uzlem b. Potom nekonečný strom T lze 2-a-uspořádat právě když je množina Z neprázdná a pro libovolné $z_1 \in Z$ je strom U, který dostaneme ze stromu T odstraněním uzlů $Z - \{z_1\}$ a s nimi incidentních hran, konečný a je možné jej 2-(a, z_1)-uspořádat. (Nechť je K konečný strom, a a b jeho různé uzly. Konečný strom K lze 2-(a, b)-uspořádat, jestliže je možné seřadit množinu jeho uzlů v prostou konečnou posloupnost $a = t_1, t_2, \ldots t_s = b$ takovou, že vzdálenost mezi sousedními uzly posloupnosti je nejvýše dvě (v metrice stromu K). Nutnou a postačující podmínku 2-(a, b)-uspořádatelnosti viz v [2].)

Nechť T je nekonečný strom, $a \in \{T\}$. Nechť v T neexistuje uzel nekonečného stupně. Pak T lze 2-a-uspořádat právě když strom V, který dostaneme ze stromu T odstraněním všech uzlů stupně 1, vyjma uzlu a, a všech s nimi incidentních hran, vyhovuje následujícím podmínkám:

0° existuje právě jedna nekonečná cesta W stromu T začínající uzlem a;

1° stupeň všech uzlů stromu V je nejvýše 4;

2° všechny uzly stupně 3 a 4 ve V jsou vnitřními uzly cesty W;

3° mezi každými dvěma uzly stupně 4 ve V leží alespoň jeden uzel stupně 2 v T; pokud uzel a je stupně většího než 1 v T pak mezi ním a nejbližším uzlem stupně 4 ve V leží alespoň jeden uzel stupně 2 v T.

Резюме

УПОРЯДОЧЕНИЕ ВЕРШИН БЕСКОНЕЧНОГО ДЕРЕВА

ФРАНТИШЕК НЕЙМАН (František Neuman), Брно

Бесконечное дерево T — это неориентированный связный граф без циклов со счетным множеством вершин. Множество вершин бесконечного дерева T мы обозначим через $\{T\}$. Пусть $a \in \{T\}$. Мы говорим, что такое дерево можно 2-a-упорядочить, если можно упорядочить все вершины этого дерева в простую последовательность $a = t_1, t_2, t_3, \ldots$ такую, что расстояние между соседними вершинами в этой последовательности не больше двух (в метрике дерева T).

В работе доказано необходимое и достаточное условие для того, чтобы данное бесконечное дерево было можно 2-a-упорядочить:

Пусть T — бесконечное дерево, $a \in \{T\}$. Если в T существует больше одной вершины бесконечной степени, то T нельзя 2-a-упорядочить.

Пусть T — бесконечное дерево, $a \in \{T\}$. Пусть b — вершина счетной степени дерева T. Мы обозначим через Z множество всех висячих вершин, за исключением вершины a, которые связаны ребром с вершиной b. Потом бесконечное дерево T можно 2-a-упорядочить тогда и только тогда, когда множество Z непусто и для произвольного $z_1 \in Z$ то дерево U, которое мы получим удалением из дерева T вершин множества $Z - \{z_1\}$ и с ними инцидентных ребер, есть конечное дерево, которое можно 2-(a, z_1)-упорядочить. (Пусть K — конечное дерево, a и b — его различные вершины. Конечное дерево K можно 2-(a, b)-упорядочить, если множество его вершин можно упорядочить в простую конечную последовательность $a = t_1, t_2, \ldots, t_s = b$, такую, что расстояние между соседними вершинами в этой последовательности не больше двух (в метрике дерева K). Необходимое и достаточное условие 2-(a, b)-упорядочения см. в [2].)

Пусть T — бесконечное дерево, $a \in \{T\}$. Пусть в T не существует вершина бесконечной степени. Потом T можно 2-a-упорядочить тогда и только тогда, когда дерево V — которое мы получим из дерева T удалением всех висячих вершин дерева T, за исключением вершины a, и с ними инцидентных ребер — удовлетворяет следующим условиям:

0° существует одна и только одна бесконечная цепь W дерева T, начинающаяся в вершине a,

1° степень всех вершин дерева V не больше 4,

2° все вершины степени 3 и 4 в V являются внутренними вершинами цепи W,

3° между каждыми двумя вершинами степени 4 в V существует по крайней мере одна вершина степени 2 в T; если степень вершины a больше 1 в T, то между a и ближайшей вершиной степени 4 в V существует по крайней мере одна вершина степени 2 в T.