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Archivum Mathematicum, Vol. 44 (2008), No. 1, 77--88

Persistent URL: http://dml.cz/dmlcz/108098

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 77–88

NON-DEGENERATE HYPERSURFACES OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We derive the equations of Gauss and Weingarten for a non-degenerate hypersurface of a semi-Riemannian manifold admitting a semi-symmetric metric connection, and give some corollaries of these equations. In addition, we obtain the equations of Gauss curvature and Codazzi-Mainardi for this non-degenerate hypersurface and give a relation between the Ricci and the scalar curvatures of a semi-Riemannian manifold and of its a non-degenerate hypersurface with respect to a semi-symmetric metric connection. Eventually, we establish conformal equations of Gauss curvature and Codazzi-Mainardi.

1. Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced for the first time by Friedmann and Schouten [4] in 1924. In 1932, Hayden [5] introduced a semi-symmetric metric connection on a Riemannian manifold. Yano [10], in 1970, proved the theorem: *In order that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat.* Some topics relative to this theorem were studied by Imai [7] in 1972. Imai [6] gave basic properties of a hypersurface of a Riemannian manifold with the semi-symmetric metric connection and got the conformal equations of Gauss curvature and Codazzi-Mainardi.

In 1986, Duggal and Sharma [3] studied semi-symmetric metric connection in a semi-Riemannian manifold. In this work, they gave some properties of Ricci tensor, affine conformal motions, geodesics and group manifolds with respect to a semi-symmetric metric connection.

In 2001, A. Konar and B. Biswas [8] considered a semi-symmetric metric connection on a Lorentz manifold. They showed that the perfect fluid spacetime with a non-vanishing constant scalar curvature admits a semi-symmetric metric connection whose Ricci tensor vanishes and that it has vanishing speed vector.

In the present paper, we defined a semi-symmetric metric connection on a non-degenerate hypersurface of a semi-Riemannian manifold similar to the hypersurface

2000 *Mathematics Subject Classification:* Primary: 53B15; Secondary: 53B30, 53C05, 53C50.
Key words and phrases: semi-symmetric metric connection, Levi-Civita connection, mean curvature, Ricci tensor, conformally flat.
Received September 9, 2007. Editor P. W. Michor.

of a Riemannian manifold (see [9] for the terminologies of semi-Riemannian manifolds). And we gave the equations of Gauss and Weingarten for a non-degenerate hypersurface of a semi-Riemannian manifold admitting a semi-symmetric metric connection. After having stated these, we derived the equations of Gauss curvature and Codazzi-Mainardi. We obtained a relation between the Ricci and the scalar curvatures of a semi-Riemannian manifold and of its a non-degenerate hypersurface. Then we had a condition under which the Ricci tensor of a non-degenerate hypersurface with respect to the semi-symmetric metric connection is symmetric. Finally, we established the conformal equations of Gauss curvature and Codazzi-Mainardi for this type of a hypersurface.

The semi-symmetric metric connection is one of the three basic types of metric connections, as already described by E. Cartan in [2], and this connection is also called a metric connection with vectorial torsion. Connections with vectorial torsion on spin manifolds may also play a role in superstring theory (see [1]), but this aspect was not discussed in the present paper.

2. Preliminaries

Let $\widetilde{M}$ be an $(n+1)$-dimensional differentiable manifold of class C^∞ and M an n-dimensional differentiable manifold immersed in $\widetilde{M}$ by a differentiable immersion

$$i\colon M \to \widetilde{M}\,.$$

$i(M)$, identical to M, is said to be a hypersurface of $\widetilde{M}$. The differential di of the immersion i will be denoted by B so that a vector field X in M corresponds to a vector field BX in $\widetilde{M}$. We now suppose that the manifold $\widetilde{M}$ is a semi-Riemannian manifold with the semi-Riemannian metric $\widetilde{g}$ of index $0 \leq \nu \leq n+1$. Hence the index of $\widetilde{M}$ is the ν and we will denote with $\operatorname{ind}\widetilde{M} = \nu$. If the induced metric tensor $g = \widetilde{g}_{|M}$ defined by

$$g(X,Y) = \widetilde{g}(BX,BY)\,, \quad \forall\, X,Y \in \chi(M)$$

is non-degenerate, the hypersurface M is called a *non-degenerate hypersurface*. Also, M is a semi-Riemannian manifold with the induced semi-Riemannian metric g (see [9]). If the semi-Riemannian manifolds $\widetilde{M}$ and M are both orientable, we can choose a unit vector field N defined along M such that

$$\widetilde{g}(BX,N) = 0\,, \quad \widetilde{g}(N,N) = \varepsilon = \begin{cases} +1\,, & \text{for spacelike } \ N \\ -1\,, & \text{for timelike } \ N \end{cases}$$

for $\forall\, X \in \chi(M)$, which is called the unit normal vector field to M, and it should be noted that $\operatorname{ind} M = \operatorname{ind}\widetilde{M}$ if $\varepsilon = 1$, but $\operatorname{ind} M = \operatorname{ind}\widetilde{M} - 1$ if $\varepsilon = -1$.

3. Semi-symmetric metric connection

Let $\widetilde{M}$ be an $(n+1)$-dimensional differentiable manifold of class C^∞ and $\widetilde{\nabla}$ a linear connection in $\widetilde{M}$. Then the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ is given by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}]\,, \quad \forall\, \widetilde{X},\widetilde{Y} \in \chi(\widetilde{M})$$

and is of type $(1,2)$. When the torsion tensor $\widetilde{T}$ satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\pi}(\widetilde{Y})\widetilde{X} - \widetilde{\pi}(\widetilde{X})\widetilde{Y}$$

for a 1-form $\widetilde{\pi}$, the connection $\widetilde{\nabla}$ is said to be semi-symmetric (see [10]).

Let there be given a semi-Riemannian metric $\widetilde{g}$ of index ν with $0 \le \nu \le n+1$ in $\widetilde{M}$ and $\widetilde{\nabla}$ satisfy

$$\widetilde{\nabla}\widetilde{g} = 0 .$$

Such a linear connection is called a metric connection (see [9]).

We now suppose that the semi-Riemannian manifold $\widetilde{M}$ admits a semi-symmetric metric connection given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\pi}(\widetilde{Y})\widetilde{X} - \widetilde{g}(\widetilde{X},\widetilde{Y})\widetilde{P} \tag{3.1}$$

for arbitrary vector fields $\widetilde{X}$ and $\widetilde{Y}$ of $\widetilde{M}$, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric $\widetilde{g}$, $\widetilde{\pi}$ a 1-form and $\widetilde{P}$ the vector field defined by

$$\widetilde{g}(\widetilde{P},\widetilde{X}) = \widetilde{\pi}(\widetilde{X})$$

for an arbitrary vector field $\widetilde{X}$ of $\widetilde{M}$ (see [3]). Since M is a non-degenerate hypersurface, we have

$$\chi(\widetilde{M}) = \chi(M) \oplus \chi(M)^{\perp} .$$

Hence we can write

$$\widetilde{P} = BP + \lambda N , \tag{3.2}$$

where P is a vector field and λ a function in M.

Denoting by $\overset{\circ}{\nabla}$ the Levi-Civita connection induced on the non-degenerate hypersurface from $\overset{\circ}{\widetilde{\nabla}}$ with respect to the unit spacelike or timelike normal vector field N, from [10] we have

$$\overset{\circ}{\widetilde{\nabla}}_{BX}BY = B(\overset{\circ}{\nabla}_X Y) + \overset{\circ}{h}(X,Y)N \tag{3.3}$$

for arbitrary vector fields X and Y of M, where $\overset{\circ}{h}$ is the second fundamental form of the non-degenerate hypersurface M. Denoting by ∇ the connection induced on the non-degenerate hypersurface from $\widetilde{\nabla}$ with respect to the unit spacelike or timelike normal vector field N, we have

$$\widetilde{\nabla}_{BX}BY = B(\nabla_X Y) + h(X,Y)N \tag{3.4}$$

for arbitrary vector fields X and Y of M, where h is the second fundamental form of the non-degenerate hypersurface M and we call (3.4) the *equation of Gauss* with respect to the induced connection ∇.

From (3.1), we obtain

$$\widetilde{\nabla}_{BX}BY = \overset{\circ}{\widetilde{\nabla}}_{BX}BY + \widetilde{\pi}(BY)BX - \widetilde{g}(BX,BY)\widetilde{P} ,$$

and hence, using (3.3) and (3.4), we have

$$B(\nabla_X Y) + h(X,Y)N = B(\overset{\circ}{\nabla}_X Y) + \overset{\circ}{h}(X,Y)N \\ + \widetilde{\pi}(BY)BX - \widetilde{g}(BX,BY)\widetilde{P}\,. \tag{3.5}$$

Substituting (3.2) into (3.5), we get

$$B(\nabla_X Y)+h(X,Y)N = B(\overset{\circ}{\nabla}_X Y+\pi(Y)X-g(X,Y)P)+\{\overset{\circ}{h}(X,Y)-\lambda g(X,Y)\}N\,,$$

from which

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X,Y)P\,, \tag{3.6}$$

where $\pi(X) = \widetilde{\pi}(BX)$ and

$$h(X,Y) = \overset{\circ}{h}(X,Y) - \lambda g(X,Y)\,. \tag{3.7}$$

Taking account of (3.6), we find

$$\nabla_X(g(Y,Z)) = (\nabla_X g)(Y,Z) + \overset{\circ}{\nabla}_X(g(Y,Z))\,,$$

from which

$$(\nabla_X g)(Y,Z) = 0\,. \tag{3.8}$$

We also have from (3.6)

$$T(X,Y) = \pi(Y)X - \pi(X)Y\,. \tag{3.9}$$

From (3.8) and (3.9), we have the following theorem:

Theorem 3.1. *The connection induced on a non-degenerate hypersurface of a semi-Riemannian manifold with a semi-symmetric metric connection with respect to the unit spacelike or timelike normal vector field is also a semi-symmetric metric connection.*

Now, the equation of Weingarten with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ is

$$\overset{\circ}{\widetilde{\nabla}}_{BX} N = -B(\overset{\circ}{A}_N X) \tag{3.10}$$

for any vector field X in M, where $\overset{\circ}{A}_N$ is a tensor field of type $(1,1)$ of M defined by

$$g(\overset{\circ}{A}_N X, Y) = \varepsilon \overset{\circ}{h}(X,Y)$$

(see [9]). On the other hand, using (3.1), we get

$$\widetilde{\nabla}_{BX} N = \overset{\circ}{\widetilde{\nabla}}_{BX} N + \varepsilon\lambda BX$$

since

$$\widetilde{\pi}(N) = \widetilde{g}(\widetilde{P},N) = \widetilde{g}(BP+\lambda N,N) = \lambda\widetilde{g}(N,N) = \varepsilon\lambda\,.$$

Thus using (3.10), we find the *equation of Weingarten* with respect to the semi-symmetric metric connection as

$$\widetilde{\nabla}_{BX}N = -B(\overset{\circ}{A}_N - \varepsilon\lambda I)X\,, \quad \varepsilon = \mp 1\,, \tag{3.11}$$

where I is the unit tensor. Defining A_N by

$$A_N = \overset{\circ}{A}_N - \varepsilon\lambda I\,, \tag{3.12}$$

then (3.11) can be written as

$$\widetilde{\nabla}_{BX}N = -B(A_N X) \tag{3.13}$$

for any vector field X in M. Then, we have the following corollary:

Corollary 3.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\widetilde{M}$. Then*

i) *If M has a spacelike normal vector field, the shape operator A_N with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ is*

$$A_N = \overset{\circ}{A}_N - \lambda I\,,$$

ii) *If M has a timelike normal vector field, the shape operator A_N with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ is*

$$A_N = \overset{\circ}{A}_N + \lambda I\,.$$

Now, let $E_1, E_2, \ldots, E_\nu, E_{\nu+1}, \ldots, E_n$ be principal vector fields corresponding to unit spacelike or timelike normal vector field N with respect to $\overset{\circ}{\widetilde{\nabla}}$. Then, by using (3.12), we have

$$A_N(E_i) = \overset{\circ}{A}_N(E_i) - \varepsilon\lambda E_i = \overset{\circ}{k}_i E_i - \varepsilon\lambda E_i = (\overset{\circ}{k}_i - \varepsilon\lambda)E_i\,, \quad 1 \le i \le n\,, \tag{3.14}$$

where $\overset{\circ}{k}_i$, $1 \le i \le n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. If we write

$$k_i = \overset{\circ}{k}_i - \varepsilon\lambda\,, \quad 1 \le i \le n\,, \tag{3.15}$$

we deduce that

$$A_N(E_i) = k_i E_i\,, \quad 1 \le i \le n\,, \tag{3.16}$$

where k_i, $1 \le i \le n$, are the principal curvatures corresponding to the normal vector field N (spacelike or timelike) with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Hence, it yields the following:

Corollary 3.3. *Let M be a non-degenerate hypersurface of the semi-Riemannian manifold $\widetilde{M}$. Then*

i) *If M has a spacelike normal vector field, the principal curvatures corresponding to unit spacelike normal N with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ are $k_i = \overset{\circ}{k}_i - \lambda$, $1 \leq i \leq n$,*

ii) *If M has a timelike normal vector field, the principal curvatures corresponding to unit timelike normal N with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ are $k_i = \overset{\circ}{k}_i + \lambda$, $1 \leq i \leq n$.*

The function $\frac{1}{n}\sum\limits_{i=1}^{n} \varepsilon_i \overset{\circ}{h}(E_i, E_i)$ is the *mean curvature* of M with respect to $\overset{\circ}{\nabla}$ and $\frac{1}{n}\sum\limits_{i=1}^{n} \varepsilon_i h(E_i, E_i)$ is called the *mean curvature* of M with respect to ∇, where

$$\varepsilon_i = \begin{cases} -1\,, & \text{for timelike } \; E_i \\ +1\,, & \text{for spacelike } \; E_i \end{cases}$$

If $\overset{\circ}{h}$ vanishes, then M is *totally geodesic* with respect to $\overset{\circ}{\nabla}$, and if $\overset{\circ}{h}$ is proportional to g, then M is *totally umbilical* with respect to $\overset{\circ}{\nabla}$ (see [9]). Similarly, if h vanishes, then M is said to be *totally geodesic* with respect to ∇. If h is proportional to g, then M is said to be *totally umbilical* with respect to ∇.

From (3.7), we have the following propositions:

Proposition 3.4. *In order that the mean curvature of M with respect to $\overset{\circ}{\nabla}$ coincides with that of M with respect to ∇, it is necessary and sufficient that the vector field $\widetilde{P}$ is tangent to M.*

Proposition 3.5. *A non-degenerate hypersurface is totally umbilical with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$ if and only if it is totally umbilical with respect to the semi-symmetric metric connection ∇.*

4. Equations of Gauss curvature and Codazzi-Mainardi

We denote by

$$\overset{\circ}{\widetilde{R}}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}} \overset{\circ}{\widetilde{\nabla}}_{\widetilde{Y}} \widetilde{Z} - \overset{\circ}{\widetilde{\nabla}}_{\widetilde{Y}} \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}} \widetilde{Z} - \overset{\circ}{\widetilde{\nabla}}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}$$

the curvature tensor of $\widetilde{M}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$ and by

$$\overset{\circ}{R}(X, Y)Z = \overset{\circ}{\nabla}_X \overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y \overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X,Y]} Z$$

that of M with respect to $\overset{\circ}{\nabla}$. Then *the equation of Gauss curvature* is given by

$$\overset{\circ}{R}(X, Y, Z, U) = \overset{\circ}{\widetilde{R}}(BX, BY, BZ, BU) + \varepsilon\big\{\overset{\circ}{h}(X, U)\overset{\circ}{h}(Y, Z) - \overset{\circ}{h}(Y, U)\overset{\circ}{h}(X, Z)\big\}\,,$$

where

$$\overset{\circ}{\widetilde{R}}(BX, BY, BZ, BU) = \widetilde{g}\big(\overset{\circ}{\widetilde{R}}(BX, BY)BZ, BU\big)\,,$$

$$\overset{\circ}{R}(X, Y, Z, U) = g\big(\overset{\circ}{R}(X, Y)Z, U\big)\,,$$

and *the equation of Codazzi-Mainardi* is given by

$$\overset{\circ}{\widetilde{R}}(BX, BY, BZ, N) = \varepsilon\big\{(\overset{\circ}{\nabla}_X \overset{\circ}{h})(Y, Z) - (\overset{\circ}{\nabla}_Y \overset{\circ}{h})(X, Z)\big\}$$

(see [9]).

Now, we shall find the equation of Gauss curvature and Codazzi-Mainardi with respect to the semi-symmetric metric connection. The curvature tensor of the semi-symmetric metric connection $\widetilde{\nabla}$ of $\widetilde{M}$ is, by definition,

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}} \widetilde{\nabla}_{\widetilde{Y}} \widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{\nabla}_{\widetilde{X}} \widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}\,.$$

Putting $\widetilde{X} = BX$, $\widetilde{Y} = BY$, $\widetilde{Z} = BZ$, we get

$$\widetilde{R}(BX, BY)BZ = \widetilde{\nabla}_{BX} \widetilde{\nabla}_{BY} BZ - \widetilde{\nabla}_{BY} \widetilde{\nabla}_{BX} BZ - \widetilde{\nabla}_{B[X,Y]} BZ\,.$$

Thus, using (3.4) and (3.13), we have

$$\begin{aligned} \widetilde{R}(BX, BY)BZ = {} & B\big(R(X, Y)Z + h(X, Z)A_N Y - h(Y, Z)A_N X\big) \\ & + \big\{(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ & + h(\pi(Y)X - \pi(X)Y, Z)\big\}N\,, \end{aligned} \tag{4.1}$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the curvature tensor of the semi-symmetric metric connection ∇. Putting now

$$\widetilde{R}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) = \widetilde{g}\big(\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{U}\big)\,, \quad R(X, Y, Z, U) = g\big(R(X, Y)Z, U\big)\,,$$

we obtain, from (4.1),

$$\begin{aligned} \widetilde{R}(BX, BY, BZ, BU) = {} & R(X, Y, Z, U) \\ & + \varepsilon\big\{h(X, Z)h(Y, U) - h(Y, Z)h(X, U)\big\}\,, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \widetilde{R}(BX, BY, BZ, N) = {} & \varepsilon\big\{(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ & + h(\pi(Y)X - \pi(X)Y, Z)\big\}\,. \end{aligned} \tag{4.3}$$

Equations (4.2) and (4.3) are called respectively *the equations of Gauss curvature and Codazzi-Mainardi* with respect to the semi-symmetric metric connection.

5. The Ricci and scalar curvatures

We denote by R the Riemannian curvature tensor of a non-degenerate hypersurface M with respect to the semi-symmetric metric connection ∇ and by $\overset{\circ}{R}$ that of M with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$. Then, by a straightforward computation, we find

$$\begin{aligned} R(X,Y)Z = \overset{\circ}{R}(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y \\ - g(Y,Z)\gamma(X) + g(X,Z)\gamma(Y)\,, \end{aligned} \tag{5.1}$$

where

$$\alpha(Y,Z) = (\overset{\circ}{\nabla}_Y \pi)Z - \pi(Y)\pi(Z) + \frac{1}{2}g(Y,Z)\pi(P) \tag{5.2}$$

and

$$\gamma(Y) = \overset{\circ}{\nabla}_Y P - \pi(Y)P + \frac{1}{2}\pi(P)Y \tag{5.3}$$

such that

$$g(\gamma(Y),Z) = \alpha(Y,Z)\,.$$

Theorem 5.1. *The Ricci tensor of a non-degenerate hypersurface M with respect to the semi-symmetric metric connection is symmetric if and only if π is closed.*

Proof. The Ricci tensor of a non-degenerate hypersurface M with respect to semi-symmetric metric connection is given by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(R(E_i,X)Y,E_i)\,.$$

Then, from (5.1) we get

$$\mathrm{Ric}(Y,Z) = \overset{\circ}{\mathrm{Ric}}(Y,Z) - (n-2)\alpha(Y,Z) + a g(Y,Z)$$

where $\overset{\circ}{\mathrm{Ric}}$ denotes the Ricci tensor of M with respect to the Levi-Civita connection and $a =$ trace of γ given by (5.3). Since $\overset{\circ}{\mathrm{Ric}}$ is symmetric, we obtain

$$\begin{aligned} \mathrm{Ric}(Y,Z) - \mathrm{Ric}(Z,Y) &= (n-2)\{\alpha(Z,Y) - \alpha(Y,Z)\} \\ &= 2(n-2)d\pi(Y,Z)\,. \end{aligned} \tag{5.4}$$

Hence, from (5.4) we find that the Ricci tensor of M with respect to the semi-symmetric connection is symmetric if and only if $d\pi = 0$, where d denotes exterior differentiation. That is, π is closed. □

Theorem 5.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold $\widetilde{M}$. If $\widetilde{\mathrm{Ric}}$ and Ric are the Ricci tensors of $\widetilde{M}$ and M with respect to the* *semi-symmetric metric connection, respectively, then for* $\forall X, Y \in \chi(M)$

$$\begin{aligned} \widetilde{\mathrm{Ric}}(BX, BY) &= \mathrm{Ric}(X, Y) - f\, h(X, Y) \\ &+ \varepsilon\Big\{ \sum_{i=1}^{n} \varepsilon_i k_i^2 g(X, E_i) g(Y, E_i) + \widetilde{g}\big(\widetilde{R}(N, BX)BY, N\big)\Big\} \end{aligned} \tag{5.5}$$

where $\varepsilon_i = g(E_i, E_i)$, $\varepsilon_i = 1$, *if* E_i *is spacelike or* $\varepsilon_i = -1$, *if* E_i *is timelike, and* $f =$ *trace of* A_N.

Proof. Suppose that $\{BE_1, \ldots, BE_\nu, BE_{\nu+1}, \ldots, BE_n, N\}$ is an orthonormal basis of $\chi(\widetilde{M})$, then the Ricci curvature of $\widetilde{M}$ with respect to the semi-symmetric metric connection is

$$\widetilde{\mathrm{Ric}}(BX, BY) = \sum_{i=1}^{n} \varepsilon_i \widetilde{g}(\widetilde{R}(BE_i, BX)BY, BE_i) + \varepsilon \widetilde{g}\big(\widetilde{R}(N, BX)BY, N\big) \tag{5.6}$$

for all $X, Y \in \chi(M)$. By using the equation of Gauss curvature (4.2) and (3.16), and considering the symmetry of shape operator we get

$$\begin{aligned} g(\widetilde{R}(BE_i, BX)BY, BE_i) &= g(R(E_i, X)Y, E_i) \\ &+ \varepsilon g(A_N E_i, Y) g(A_N E_i, X) - h(X, Y) g(A_N E_i, E_i)\,. \end{aligned} \tag{5.7}$$

Hence, inserting (5.7) into (5.6) yields to (5.5). □

Theorem 5.3. *Let* M *be a non-degenerate hypersurface of a semi-Riemannian manifold* $\widetilde{M}$. *If* $\widetilde{\rho}$ *and* ρ *are the scalar curvatures of* $\widetilde{M}$ *and* M *with respect to the semi-symmetric metric connection, respectively, then*

$$\widetilde{\rho} = \rho - \varepsilon f^2 + f^* + 2\varepsilon \widetilde{\mathrm{Ric}}(N, N) \tag{5.8}$$

where $f =$ trace *of* A_N *and* $f^* =$ trace *of* A_N^2.

Proof. Assume that $\{BE_1, \ldots, BE_\nu, BE_{\nu+1}, \ldots, BE_n, N\}$ is an orthonormal basis of $\chi(\widetilde{M})$, then the scalar curvature of $\widetilde{M}$ with respect to the semi-symmetric metric connection is

$$\widetilde{\rho} = \sum_{i=1}^{n} \varepsilon_i \widetilde{\mathrm{Ric}}(E_i, E_i) + \varepsilon \widetilde{\mathrm{Ric}}(N, N)\,. \tag{5.9}$$

As (5.5) is considered, we get

$$\widetilde{\mathrm{Ric}}(E_i, E_i) = \mathrm{Ric}(E_i, E_i) + \varepsilon\big\{g\big(\widetilde{R}(N, e_i)e_i, N\big) + 2\varepsilon_i k_i^2\big\}$$

Hence, we obtain

$$\widetilde{\rho} = \rho - \varepsilon f^2 + f^* + 2\varepsilon \widetilde{\mathrm{Ric}}(N, N)\,.$$

□

6. The conformal equations of Gauss curvature and Codazzi-Mainardi

Denoting the conformal curvature tensors of type $(0,4)$ of the semi-symmetric metric connections $\widetilde{\nabla}$ and ∇, respectively, by $\widetilde{C}$ and C we have

$$\begin{aligned}\widetilde{C}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) &= \widetilde{R}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) + \widetilde{g}(\widetilde{X},\widetilde{U})\widetilde{L}(\widetilde{Y},\widetilde{Z}) - \widetilde{g}(\widetilde{Y},\widetilde{U})\widetilde{L}(\widetilde{X},\widetilde{Z}) \\ &\quad + \widetilde{g}(\widetilde{Y},\widetilde{Z})\widetilde{L}(\widetilde{X},\widetilde{U}) - \widetilde{g}(\widetilde{X},\widetilde{Z})\widetilde{L}(\widetilde{Y},\widetilde{U})\,, \end{aligned} \tag{6.1}$$

where

$$\widetilde{L}(\widetilde{Y},\widetilde{Z}) = -\frac{1}{n-1}\widetilde{\mathrm{Ric}}(\widetilde{Y},\widetilde{Z}) + \frac{\widetilde{\rho}}{2n(n-1)}\widetilde{g}(\widetilde{Y},\widetilde{Z})$$

and $\widetilde{\mathrm{Ric}}$ is the Ricci tensor and $\widetilde{\rho}$ is the scalar curvature of $\widetilde{M}$ with respect to the connection $\widetilde{\nabla}$. Similarly, we get

$$\begin{aligned}C(X,Y,Z,U) &= R(X,Y,Z,U) + g(X,U)L(Y,Z) - g(Y,U)L(X,Z) \\ &\quad + g(Y,Z)L(X,U) - g(X,Z)L(Y,U)\,, \end{aligned} \tag{6.2}$$

where

$$L(Y,Z) = -\frac{1}{n-2}\,\mathrm{Ric}(Y,Z) + \frac{\rho}{2(n-1)(n-2)}g(Y,Z)$$

and Ric is the Ricci tensor and ρ is the scalar curvature of M with respect to the connection ∇. From (4.2), we have

$$\widetilde{\mathrm{Ric}}(BY,BZ) - \varepsilon\widetilde{R}(N,BY,BZ,N) = \mathrm{Ric}(Y,Z) - \varepsilon f h(Y,Z) + h(A_N Y,Z)\,, \tag{6.3}$$

where $f=$ trace of A_N. On the other hand, from (6.1), we find

$$\begin{aligned}\widetilde{C}(N,BY,BZ,N) &= \widetilde{R}(N,BY,BZ,N) + \varepsilon\frac{\widetilde{\rho}}{n(n-1)}g(Y,Z) \\ &\quad - \frac{1}{n-1}\big\{\varepsilon\widetilde{\mathrm{Ric}}(BY,BZ) + \widetilde{\mathrm{Ric}}(N,N)g(Y,Z)\big\}\,. \end{aligned} \tag{6.4}$$

Substituting (6.4) into (6.3), we get

$$\begin{aligned}\mathrm{Ric}(Y,Z) &= \frac{n-2}{n-1}\widetilde{\mathrm{Ric}}(BY,BZ) - \varepsilon\widetilde{C}(N,BY,BZ,N) \\ &\quad - \Big\{\frac{1}{n-1}\varepsilon\widetilde{\mathrm{Ric}}(N,N) - \frac{1}{n(n-1)}\widetilde{\rho}\Big\}g(Y,Z) \\ &\quad + \varepsilon f h(Y,Z) - h(A_N Y,Z)\,. \end{aligned} \tag{6.5}$$

From (6.5) and (5.8), we have

$$\begin{aligned}L(Y,Z) &= \widetilde{L}(BY,BZ) + \frac{1}{n-2}\big\{\varepsilon\widetilde{C}(N,BY,BZ,N) - \varepsilon f h(Y,Z) + h(A_N Y,Z)\big\} \\ &\quad + \frac{1}{2(n-1)(n-2)}(\varepsilon f^2 - f^*)g(Y,Z)\,, \end{aligned} \tag{6.6}$$

where $f^* =$ trace of A_N^2. Thus, from (6.1), we obtain

$$\begin{aligned}\widetilde{C}(BX,BY,BZ,BU) &= \widetilde{R}(BX,BY,BZ,BU) + g(X,U)\widetilde{L}(BY,BZ)\\ &\quad - g(Y,U)\widetilde{L}(BX,BZ) + g(Y,Z)\widetilde{L}(BX,BU)\\ &\quad - g(X,Z)\widetilde{L}(BY,BU)\,. \end{aligned} \tag{6.7}$$

Using (6.2), (6.6), (6.7) and (4.2), we get

$$\begin{aligned}C(X,Y,Z,U) &= \widetilde{C}(BX,BY,BZ,BU) + \varepsilon\big\{h(Y,Z)h(X,U) - h(X,Z)h(Y,U)\big\}\\ &\quad + \frac{\varepsilon}{n-2}\big\{\widetilde{C}(N,BY,BZ,N)g(X,U) - \widetilde{C}(N,BX,BZ,N)g(Y,U)\\ &\quad + \widetilde{C}(N,BX,BU,N)g(Y,Z) - \widetilde{C}(N,BY,BU,N)g(X,Z)\big\}\\ &\quad - \frac{1}{n-2}\big\{\big(\varepsilon f h(Y,Z) - h(A_NY,Z)\big)g(X,U) - \big(\varepsilon f h(X,Z)\\ &\quad - h(A_NX,Z)\big)g(Y,U) + \big(\varepsilon f h(X,U) - h(A_NX,U)\big)g(Y,Z)\\ &\quad - \big(\varepsilon f h(Y,U) - h(A_NY,U)\big)g(X,Z)\big\}\\ &\quad + \frac{(\varepsilon f^2 - f^*)}{(n-1)(n-2)}\big\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big\}\,. \end{aligned} \tag{6.8}$$

Equation (6.8) is *the conformal equation of Gauss curvature*. Hence, from (6.1), we have

$$\begin{aligned}\widetilde{C}(BX,BY,BZ,N) &= \widetilde{R}(BX,BY,BZ,N)\\ &\quad - \frac{1}{n-1}\big\{g(Y,Z)\widetilde{\mathrm{Ric}}(BX,N) - g(X,Z)\widetilde{\mathrm{Ric}}(BY,N)\big\}\,. \end{aligned} \tag{6.9}$$

Taking into consideration equation (4.3), we obtain

$$\begin{aligned}\widetilde{C}(BX,BY,BZ,N) &= \varepsilon\big\{(\nabla h)(X,Y,Z) - (\nabla h)(Y,X,Z) + h(\pi(Y)X - \pi(X)Y,Z)\big\}\\ &\quad - \frac{1}{n-1}\big\{g(Y,Z)\widetilde{\mathrm{Ric}}(BX,N) - g(X,Z)\widetilde{\mathrm{Ric}}(BY,N)\big\}\,. \end{aligned} \tag{6.10}$$

Equation (6.10) is *the conformal equation of Codazzi-Mainardi.*

We suppose that the semi-Riemannian manifold $\widetilde{M}$ is conformally flat ($\widetilde{C} = 0$) and that the $(n > 3)$-dimensional non-degenerate hypersurface M is totally umbilical, then we have $\widetilde{R} = 0$ (see [3]) and we also have $h = cg$, since M is totally umbilical with respect to ∇ by Proposition 3.5. Then from (6.8) we get the following theorem:

Theorem 6.1. *A totally umbilical non-degenerate hypersurface in a conformally flat semi-Riemannian manifold with a semi-symmetric metric connection is conformally flat.*

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