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ON UNIQUE RANGE SETS OF
MEROMORPHIC FUNCTIONS IN \mathbb{C}^m

XIAO-TIAN BAI AND QI HAN

ABSTRACT. By considering a question proposed by F. Gross concerning unique range sets of entire functions in \mathbb{C} , we study the unicity of meromorphic functions in \mathbb{C}^m that share three distinct finite sets CM and obtain some results which reduce $5 \leq c_3(\mathcal{M}(\mathbb{C}^m)) \leq 9$ to $5 \leq c_3(\mathcal{M}(\mathbb{C}^m)) \leq 6$.

1. INTRODUCTION AND MAIN RESULTS

Let f be a non-constant meromorphic function in \mathbb{C} , and let $a \in \mathbb{C}$ be a finite value. Define $E_f(a)$ to be the set of zeros of $f - a = 0$, each one counted according to its multiplicity. For $a = \infty$, we define $E_f(\infty) := E_{1/f}(0)$. Let $\mathcal{S} \subset \mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ be a non-empty set with distinct elements. Set $E_f(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} E_f(a)$. If, for another non-constant meromorphic function g in \mathbb{C} , we have $E_f(\mathcal{S}) = E_g(\mathcal{S})$, then we say that f and g share the set \mathcal{S} CM. In particular, when \mathcal{S} contains only one element, it coincides with the usual definition of CM shared values. We refer the reader to books [7] or [11] for more details on *Nevanlinna's value distribution theory* of meromorphic functions with single variable and its applications.

In 1968, it was F. Gross who first studied the uniqueness problem of meromorphic functions in \mathbb{C} that share distinct sets instead of values in [5]. From then on, he, as well as many other mathematicians, has studied and obtained a lot of results on this topic and its related problems (see, e.g., [8] or [11]).

In 1976, F. Gross asked the following two questions.

Question 1 (see [6] or [12]). *Can one find two distinct finite sets \mathcal{S}_1 and \mathcal{S}_2 such that any two non-constant entire functions f and g in \mathbb{C} sharing them CM will be identically equal to each other?*

Question 2 (see [6] or [12]). *If the answer to Question 1 is affirmative, then it would be interesting to know how large both sets would have to be?*

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Questions 1 and 2 have been answered by H.-X. Yi completely in 1998. In fact, he proved the following two theorems.

Theorem A (see [12]). *Let f and g be two non-constant entire functions in \mathbb{C} , and let $\mathcal{S}_1 = \{0\}$ and $\mathcal{S}_2 = \{\omega|\omega^2(\omega + a) - b = 0\}$ be two sets, where a and b are two non-zero constants such that $\frac{4a^3}{27} \neq b$. If f and g share the sets \mathcal{S}_1 and \mathcal{S}_2 CM, then $f \equiv g$.*

Theorem B (see [12]). *If \mathcal{S}_1 and \mathcal{S}_2 are unique range sets of non-constant entire functions in \mathbb{C} , then $\max\{\iota(\mathcal{S}_1), \iota(\mathcal{S}_2)\} \geq 3$ and $\min\{\iota(\mathcal{S}_1), \iota(\mathcal{S}_2)\} \geq 1$, where $\iota(\mathcal{S}_j)$ denotes the cardinality of the set \mathcal{S}_j for $j = 1, 2$.*

Here, we say that $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ are unique range sets of entire or meromorphic functions, if the condition that any two non-constant entire or meromorphic functions f and g sharing $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ CM implies that $f \equiv g$. Also, examples are given in [12] to show that the conclusions of Theorem B is sharp.

P.-C. Hu and C.-C. Yang generalized the above two theorems to holomorphic functions in \mathbb{C}^m , and obtained the following result.

Theorem C (see [8, Theorem 3.42]). *Let f and g be two non-constant holomorphic functions in \mathbb{C}^m , and let $\mathcal{S}_1 = \{0\}$ and $\mathcal{S}_2 = \{\omega|\omega^n + a\omega^p - b = 0\}$ be two sets, where n and p are two relatively prime integers such that $n > p \geq 2$ and $2p > n$, and a and b are two non-zero constants such that $\frac{a^n}{b^{n-p}} \neq \frac{(-1)^p n^n}{p^p(n-p)^{n-p}}$. If f and g share the sets \mathcal{S}_1 and \mathcal{S}_2 CM, then $f \equiv g$. Obviously, $\min\{\iota(\mathcal{S}_2)\} = 3$.*

Also, they studied unique range sets of meromorphic functions in \mathbb{C}^m , and obtained the following extension of Theorems A-C.

Theorem D (see [8, Theorem 3.43]). *Let f and g be two non-constant meromorphic functions in \mathbb{C}^m , and let $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \{\omega|\omega^n + a\omega^p - b = 0\}$ and $\mathcal{S}_3 = \{\infty\}$ be three sets, where n and p are two relatively prime integers such that $n > p + 1 \geq 3$ and $2p > n + 2$, and a and b are two non-zero constants such that $\frac{a^n}{b^{n-p}} \neq \frac{(-1)^p n^n}{p^p(n-p)^{n-p}}$. If f and g share the sets $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S}_3 CM, then $f \equiv g$. Obviously, $\min\{\iota(\mathcal{S}_2)\} = 7$.*

Remark. Please see Section 2 for the definition of meromorphic functions of several variables and that of the corresponding CM shared sets.

Example. Let f and g be two non-constant distinct meromorphic functions in \mathbb{C}^m with the following expressions

$$f = -\frac{ae^\alpha(e^{n\alpha} - 1)}{e^{(n+1)\alpha} - 1} \quad \text{and} \quad g = -\frac{a(e^{n\alpha} - 1)}{e^{(n+1)\alpha} - 1}.$$

Then $f/g = e^\alpha$, where α is a non-constant entire function in \mathbb{C}^m . So, f and g share the values $0, \infty$ CM. Also, $f^n(f + a) \equiv g^n(g + a)$, which means f and g sharing the set $\mathcal{S} = \{\omega|\omega^n(\omega + a) - b = 0\}$ CM for any $n \in \mathbb{N}$ and any $a(\neq 0), b \in \mathbb{C}$.

Hence, the above example shows that the assumption “ $n > p + 1$ ” in Theorem D is sharp. Further, it also shows that, in order to reduce the cardinality of the set \mathcal{S}_2 , we may have to increase the cardinalities of the sets \mathcal{S}_1 or \mathcal{S}_3 .

Define $(\mathcal{S})_n := \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n\}$, where the non-empty sets $\mathcal{S}_j \subset \mathbb{P}^1$ are of distinct elements for $j = 1, 2, \dots, n$, and $\mathcal{S}_j \cap \mathcal{S}_k = \emptyset$ whenever $j \neq k$. Define $\iota((\mathcal{S})_n) := \sum_{j=1}^n \iota(\mathcal{S}_j)$ to be the *total cardinality* of the sets \mathcal{S}_j for $1, 2, \dots, n$. If the sets \mathcal{S}_j ($j = 1, 2, \dots, n$) are unique range sets of meromorphic functions in \mathbb{C}^m , then we define $c_n(\mathcal{M}(\mathbb{C}^m)) := \min\{\iota((\mathcal{S})_n)\}$, where $\mathcal{M}(\mathbb{C}^m)$ denotes the set of meromorphic functions in \mathbb{C}^m and obviously, is a field.

Apparently, $c_3(\mathcal{M}(\mathbb{C}^m)) \geq 5$, since, under some trivial transformation, any three non-intersecting sets containing four pairwise distinct elements totally will assume the form $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \{\omega | \omega(\omega + a) - b = 0\}$ and $\mathcal{S}_3 = \{\infty\}$ for two constants a and $b(\neq 0)$ such that $\frac{a^2}{4} + b \neq 0$. If $a = 0$, then $f = -g$ will be in our benefit, while if $a \neq 0$, then our aforesaid *Example* will help to this purpose.

In this paper, we shall reduce the upper bound $5 \leq c_3(\mathcal{M}(\mathbb{C}^m)) \leq 9$ to $5 \leq c_3(\mathcal{M}(\mathbb{C}^m)) \leq 6$. For admissible meromorphic functions in the unit disc $\Delta \subset \mathbb{C}$, the corresponding result has been obtained by M.-L. Fang in [2], where he stated that the same conclusion holds well for meromorphic functions in \mathbb{C} . In this paper, by employing his main ideas, we shall prove the following two theorems.

Theorem 1. *Let f and g be two non-constant meromorphic functions in \mathbb{C}^m , and let $\mathcal{S}_1 = \{0, c\}$, $\mathcal{S}_2 = \{\omega | \omega^n(\omega + a) - b = 0\}$ and $\mathcal{S}_3 = \{\infty\}$ be three sets, where n is a positive integer such that $n \geq 2$, and a, b and c are three non-zero constants such that $c = -\frac{na}{n+1}$, $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \neq b, 2b$, and $\frac{(-1)^n n^n (n+2)a^{n+1}}{2^{n+1}(n+1)^{n+1}} \neq b$. If f and g share the sets $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S}_3 CM, then $f \equiv g$. Obviously, $\min\{\iota(\mathcal{S}_2)\} = 3$.*

Theorem 2. *Let f and g be two non-constant meromorphic functions in \mathbb{C}^m , and let $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \{\omega | \omega^n(\omega + a) - b = 0\}$ and $\mathcal{S}_3 = \{\infty, c\}$ be three sets, where n is a positive integer such that $n \geq 2$, and a, b and c are three non-zero constants such that $c = \frac{(n+1)b}{na}$, $\frac{n^n a^{n+1}}{(n+1)^{n+1} b^n} \neq -1, -2$, and $\frac{n^n (n+2)a^{n+1}}{2^{n+1}(n+1)^{n+1} b^n} \neq -1$. If f and g share the sets $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S}_3 CM, then $f \equiv g$. Obviously, $\min\{\iota(\mathcal{S}_2)\} = 3$.*

2. PRELIMINARIES AND SOME LEMMAS

If f is a holomorphic function on an open connected neighborhood of $\mathfrak{z}_0 \in \mathbb{C}^m$ and $f \neq 0$, then a series

$$f(\mathfrak{z}) = \sum_{j=\nu}^{\infty} \mathcal{P}_j(\mathfrak{z} - \mathfrak{z}_0)$$

converges uniformly on some neighborhood of \mathfrak{z}_0 and represents f on this neighborhood. Here, \mathcal{P}_j denotes a homogeneous polynomial of degree j and $\mathcal{P}_\nu \neq 0$. The non-negative integer ν , uniquely determined by f and \mathfrak{z}_0 , is called the *zero multiplicity* (or *order*) of f at \mathfrak{z}_0 and denoted by $\mathfrak{D}_f^0(\mathfrak{z}_0)$.

Let f be a non-constant meromorphic function in \mathbb{C}^m . Then, for each $\mathfrak{z} \in \mathbb{C}^m$, there exists an open connected neighborhood $U_{\mathfrak{z}}$ of \mathfrak{z} and two holomorphic functions

$g \not\equiv 0$ and $h \not\equiv 0$ on $U_{\mathfrak{z}}$, coprime at \mathfrak{z} (i.e., the germs of g and h have no common factors in the local ring of germs of holomorphic functions at \mathfrak{z}), such that $hf \equiv g$ on $U_{\mathfrak{z}}$. Then, in $U_{\mathfrak{z}}$ and for $a \in \mathbb{P}^1$,

$$\begin{aligned} \mathfrak{D}_f^a(\mathfrak{z}) &:= \mathfrak{D}_{g-ah}^0(\mathfrak{z}) & (a \in \mathbb{C}), \\ \mathfrak{D}_f^\infty(\mathfrak{z}) &:= \mathfrak{D}_h^0(\mathfrak{z}) & (a = \infty) \end{aligned}$$

is well defined and called the a -multiplicity of f . The function

$$\mathfrak{D}_f^a : \mathbb{C}^m \rightarrow \mathbb{Z}^+$$

is called the a -divisor of f , where \mathbb{Z}^+ denotes the set of non-negative integers. If f is a meromorphic function in \mathbb{C}^m , then it is considered as a holomorphic map into the Riemann sphere \mathbb{P}^1 outside its set of *indeterminacy* that is usually denoted by \mathcal{I}_f . For $a \in \mathbb{P}^1$, we define

$$f^{-1}(a) := \text{supp}(\mathfrak{D}_f^a),$$

where $\text{supp}(\mathfrak{D}_f^a)$ is the support of \mathfrak{D}_f^a , defined as the closed set $\overline{(\mathfrak{D}_f^a)^{-1}(\mathbb{Z}^+ \setminus \{0\})}$.

Define the *differential form*

$$\eta := dd^c |\mathfrak{z}|^2,$$

where $d := \partial + \bar{\partial}$ and $d^c := \frac{1}{4\pi i}(\partial - \bar{\partial})$. For a meromorphic function f in \mathbb{C}^m and $a \in \mathbb{P}^1$, we define the *counting function* of the a -divisor of f as

$$n_f^a(r) := \sum_{|\mathfrak{z}| \leq r} \mathfrak{D}_f^a(\mathfrak{z}) \quad \text{for } m = 1,$$

and

$$n_f^a(r) := r^{2-2m} \int_{|\mathfrak{z}| \leq r} \mathfrak{D}_f^a(\mathfrak{z}) \eta^{m-1} \quad \text{for } m > 1.$$

Write $n(r, \frac{1}{f-a}) = n_f^a(r)$ for $a \in \mathbb{C}$, and $n(r, f) = n_f^\infty(r)$ for $a = \infty$. Define the *valence function* of the a -divisor of f to be

$$\begin{aligned} N\left(r, \frac{1}{f-a}\right) &:= \int_{r_0}^r \frac{n(r, \frac{1}{f-a})}{t} dt, & a \in \mathbb{C}, r \geq r_0 > 0; \\ N(r, f) &:= \int_{r_0}^r \frac{n(r, f)}{t} dt, & a = \infty, r \geq r_0 > 0. \end{aligned}$$

The *compensation function* of $f - a$ for $a \in \mathbb{P}^1$ is defined as

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &:= \frac{1}{V_m(r)} \int_{S_m(r)} \log \frac{1}{\|f(\mathfrak{z}), a\|} d\sigma_r, & a \in \mathbb{C}, \\ m(r, f) &:= \frac{1}{V_m(r)} \int_{S_m(r)} \log \sqrt{1 + |f(\mathfrak{z})|^2} d\sigma_r, & a = \infty, \end{aligned}$$

where $\|f(\mathfrak{z}), a\|$ is the chordal distance between $f(\mathfrak{z})$ and a in the Riemann sphere \mathbb{P}^1 (for $a = \infty$, it is $\frac{1}{\sqrt{1+|f(\mathfrak{z})|^2}}$), $V_m(r) = \frac{2\pi^m r^{2m-1}}{(m-1)!}$, $S_m(r) = \{\mathfrak{z} \in \mathbb{C}^m \mid |\mathfrak{z}| = r\}$, and $d\sigma_r$ is the positive element of volume on $S_m(r)$ such that $\int_{S_m(r)} d\sigma_r = V_m(r)$.

As for the sphere $S_m(r)$, it is considered as a $(2m - 1)$ -dimensional real manifold that orients to the exterior of the ball $B_m(r) = \{\mathfrak{z} \in \mathbb{C}^m \mid |\mathfrak{z}| < r\}$.

The *Nevanlinna characteristic function* of f is defined as

$$T(r, f) := m(r, f) + N(r, f).$$

In particular, when $m = 1$, the difference between this definition and that in [7] or [11] is an $O(1)$, i.e., a bounded quantity.

Nevanlinna's first main theorem states that for any $a \in \mathbb{C}$,

$$T(r, f) - m(r_0, f) = T\left(r, \frac{1}{f - a}\right) - m\left(r_0, \frac{1}{f - a}\right),$$

i.e., $T(r, f) = T\left(r, \frac{1}{f - a}\right) + O(1)$.

For $k \in \mathbb{Z}^+ \setminus \{0\}$, define the *truncated a -divisor* of f as

$$\mathfrak{D}_{f,k}^a(\mathfrak{z}) := \min\{\mathfrak{D}_f^a(\mathfrak{z}), k\},$$

and define the *truncated counting function* $n_k\left(r, \frac{1}{f - a}\right)$ ($a \in \mathbb{C}$), $n_k(r, f)$ ($a = \infty$), and the *truncated valence function* $N_k\left(r, \frac{1}{f - a}\right)$ ($a \in \mathbb{C}$), $N_k(r, f)$ ($a = \infty$) generated by $\mathfrak{D}_{f,k}^a(\mathfrak{z})$ similarly.

Nevanlinna's second main theorem states that for pairwise distinct values $a_j \in \mathbb{C}$ ($j = 1, 2, \dots, q$),

$$\begin{aligned} (q - 1)T(r, f) &\leq \sum_{j=1}^q N\left(r, \frac{1}{f - a_j}\right) + N(r, f) - N_{\text{Ram}}(r, f) \\ &\quad + O\left(\log \frac{\rho^{2m-1}T(R, f)}{r^{2m-1}(\rho - r)}\right) \\ &\leq \sum_{j=1}^q N_1\left(r, \frac{1}{f - a_j}\right) + N_1(r, f) + O\left(\log \frac{\rho^{2m-1}T(R, f)}{r^{2m-1}(\rho - r)}\right), \end{aligned}$$

where $N_{\text{Ram}}(r, f)$ is called the *ramification term* and $r_0 < r < \rho < R < +\infty$.

Let f and g be two non-constant meromorphic functions in \mathbb{C}^m , and let a be a value in \mathbb{P}^1 . If $\mathfrak{D}_f^a(\mathfrak{z}) \equiv \mathfrak{D}_g^a(\mathfrak{z})$ for all $\mathfrak{z} \in \mathbb{C}^m \setminus \mathcal{I}_f \cup \mathcal{I}_g$, then we say that f and g share the value a CM. For a non-empty set $\mathcal{S} \subset \mathbb{P}^1$, define

$$\mathfrak{D}_f^{\mathcal{S}}(\mathfrak{z}) := \sum_{a \in \mathcal{S}} \mathfrak{D}_f^a(\mathfrak{z}).$$

If $\mathfrak{D}_f^{\mathcal{S}}(\mathfrak{z}) \equiv \mathfrak{D}_g^{\mathcal{S}}(\mathfrak{z})$ for all $\mathfrak{z} \in \mathbb{C}^m \setminus \mathcal{I}_f \cup \mathcal{I}_g$, then we say that f and g share the set \mathcal{S} CM.

We refer the reader to [1] or [8] for details on *Nevanlinna's value distribution theory* of meromorphic functions with several variables.

Now, let's introduce several lemmas.

Lemma 1 (see [4] or [8, Theorem 1.26]). *Let f be a non-constant meromorphic function in \mathbb{C}^m , and let $P(z)$ and $Q(z)$ be two coprime polynomials with constant*

coefficients and of degrees p and q , respectively. Write

$$R(f) := \frac{P(f)}{Q(f)} = \frac{a_p f^p + a_{p-1} f^{p-1} + \dots + a_0}{b_q f^q + b_{q-1} f^{q-1} + \dots + b_0} \quad a_p b_q \neq 0.$$

Then, we have

$$T(r, R(f)) = \max\{p, q\}T(r, f) + O(1).$$

Lemma 2 (see [8, Lemma 1.39] or [10, Equation (5.1)]). *Let f_0, f_1, \dots, f_n be $n+1$ meromorphic functions in \mathbb{C}^m such that they are linearly independent. Write $f := (f_0, f_1, \dots, f_n)$. Then, there are multi-indices $\nu_j \in \mathbb{Z}_+^m$ ($j = 1, 2, \dots, n$) such that $0 < |\nu_j| \leq j$ and $f, \partial^{\nu_1} f, \partial^{\nu_2} f, \dots, \partial^{\nu_n} f$ are linearly independent over \mathbb{C}^m , where \mathbb{Z}_+^m denotes the m -th Descartes' product of \mathbb{Z}^+ .*

Lemma 3 (see [3] or [9]). *Let f and g be two non-constant meromorphic functions in \mathbb{C}^m , and let $a_j \in \mathbb{P}^1$ ($j = 1, 2, 3, 4$) be four distinct values. If f and g share a_j ($j = 1, 2, 3, 4$) CM, then f is some Möbius transformation of g .*

Remark. Since only Borel's Lemma was involved in [3] and [9] for the proof of their main results, it's straightforward to get the conclusions of our Lemma 3.

Lemma 4 (see [8, Lemma 3.36]). *Let f and g be two non-constant meromorphic functions in \mathbb{C}^m such that they share the value 1 CM. If there exists a real number $\lambda \in [0, \frac{1}{2})$ such that*

$$\| N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r, f) + N_2(r, g) \leq (\lambda + o(1))(T(r, f) + T(r, g)),$$

then we have either $f \equiv g$ or $fg \equiv 1$.

Here and in the following, the notation “ $\|$ ” denotes that an (in)equality holds as $r \rightarrow +\infty$ outside a possible set of finite linear measure.

3. PROOF OF THEOREM 1

Define

$$F := \frac{f^n(f+a)}{b} \quad \text{and} \quad G := \frac{g^n(g+a)}{b}.$$

Then, from the assumptions of Theorem 1, we know that F and G share the values 1 and ∞ CM. By Lemma 1, we have

$$T(r, F) = (n+1)T(r, f) + O(1)$$

and

$$(3.1) \quad T(r, G) = (n+1)T(r, g) + O(1).$$

We now distinguish the following two cases for discussions.

Case 1. $F - 1$ and $G - 1$ are linearly dependent. Then, there exists a non-zero constant $k \in \mathbb{C}$ such that

$$F - 1 \equiv k(G - 1),$$

which implies that

$$(3.2) \quad f^n(f+a) - b \equiv k(g^n(g+a) - b).$$

Set $E(f, g) = \{z \mid \text{either } f(z) = g(z) = 0 \text{ or } f(z) = g(z) = c\} \subset \mathbb{C}^m \setminus \mathcal{I}_f \cup \mathcal{I}_g$.

Subcase 1.1. $E(f, g) \neq \emptyset$.

By (3.2), noting $b \neq 0$ and $b \neq \frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}}$, we have $k = 1$. Thus,

$$(3.3) \quad f^n(f+a) \equiv g^n(g+a),$$

which means that f and g share the values $0, -a$ and c CM, since we assume that f and g share the set \mathcal{S}_1 CM. Also, by assumption, f and g share the value ∞ CM. By the conclusions of Lemma 3, f is some Möbius transformation of g , say,

$$(3.4) \quad f = \frac{Ag+B}{Cg+D} \quad AD - BC \neq 0.$$

Substituting (3.4) into (3.3) yields

$$g^{n+1} + ag^n \equiv \frac{(Ag+B)^n((A+aC)g+(B+aD))}{(Cg+D)^{n+1}}.$$

Obviously, $C = 0$ and thus $AD \neq 0$. Noting $a \neq 0$ and $n \geq 2$, a routine calculation on the like terms of g leads to $B = 0$ and $A = D$. Hence, $f \equiv g$.

Subcase 1.2. $E(f, g) = \emptyset$.

Since f and g share the set \mathcal{S}_1 CM, then $\mathfrak{D}_f^0(z) \equiv \mathfrak{D}_g^c(z)$ and $\mathfrak{D}_f^c(z) \equiv \mathfrak{D}_g^0(z)$.

Subcase 1.2.1. Both $f^{-1}(0) = g^{-1}(c) = \emptyset$ and $f^{-1}(c) = g^{-1}(0) = \emptyset$.

Since f and g share the value ∞ CM, we know that

$$\frac{f}{f-c} \quad \text{and} \quad \frac{g-c}{g}$$

are non-vanishing holomorphic functions and share the value 1 CM. According to the conclusions of Lemma 4, we have either $\frac{f}{f-c} \equiv \frac{g-c}{g}$ or $\frac{f}{f-c} \frac{g-c}{g} \equiv 1$, which implies that either $f+g \equiv c$ or $f \equiv g$. However, if $f+g \equiv c$, then by (3.2) and the fact that $a \neq -c$, $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \neq 2b$, it is self-contradicted. So, $f \equiv g$.

Subcase 1.2.2. Either $f^{-1}(0) = g^{-1}(c) \neq \emptyset$ or $f^{-1}(c) = g^{-1}(0) \neq \emptyset$.

Without loss of generality, we may assume that $f^{-1}(0) = g^{-1}(c) \neq \emptyset$ while $f^{-1}(c) = g^{-1}(0) = \emptyset$. Hence, from (3.2), we get $k = \frac{b}{b-ac^n-c^{n+1}}$.

Obviously, c is the only double root of the equation $z^{n+1} + az^n - c^{n+1} - ac^n = 0$ while the remaining $n-1$ roots, say, a_j ($j = 1, 2, \dots, n-1$), are all simple.

Let b_k ($k = 1, 2, \dots, n+1$) be the $n+1$ roots of the following equation

$$(3.5) \quad z^{n+1} + az^n - b = -\frac{b-ac^n-c^{n+1}}{k} = -\frac{(b-ac^n-c^{n+1})^2}{b}.$$

By our hypothesis that $a \neq -c$, $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \neq b$ and $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \neq 2b$, equation (3.5) has no multiple roots at all, since neither 0 nor c is a root of it. Hence, b_k

($k = 1, 2, \dots, n + 1$) are pairwise distinct such that $\prod_{k=1}^{n+1} b_k \neq 0$. By (3.2), noting that $f^{-1}(c) = \emptyset$, we have $\sum_{j=1}^{n-1} \mathfrak{D}_{f,1}^{a_j}(\mathfrak{z}) \equiv \sum_{k=1}^{n+1} \mathfrak{D}_{g,1}^{b_k}(\mathfrak{z})$.

Applying *Nevanlinna's second main theorem* to g , and noting that $f^{-1}(c) = g^{-1}(0) = \emptyset$, and f and g share the value ∞ CM, we conclude that

$$\begin{aligned}
 (3.6) \quad \| (n + 1)T(r, g) &\leq N_1(r, g) + N_1\left(r, \frac{1}{g}\right) + \sum_{k=1}^{n+1} N_1\left(r, \frac{1}{g - b_k}\right) + o(T(r, g)) \\
 &\leq N_1(r, f) + N_1\left(r, \frac{1}{f - c}\right) + \sum_{j=1}^{n-1} N_1\left(r, \frac{1}{f - a_j}\right) + o(T(r, g)) \\
 &\leq nT(r, f) + o(T(r, g)).
 \end{aligned}$$

However, by (3.2), we have $T(r, f) = T(r, g) + O(1)$. Combining it with (3.6) yields $\| T(r, g) = o(T(r, g))$, a contradiction.

If $f^{-1}(0) = g^{-1}(c) = \emptyset$ and $f^{-1}(c) = g^{-1}(0) \neq \emptyset$, interchanging the positions of f and g yields a contradiction, too. Hence, Subcase 1.2.2 can be ruled out.

Subcase 1.2.3. Neither $f^{-1}(0) = g^{-1}(c) = \emptyset$ nor $f^{-1}(c) = g^{-1}(0) = \emptyset$.

By a similar way as above, we have $k = \frac{b}{b - ac^n - c^{n+1}}$ and $k = \frac{b - ac^n - c^{n+1}}{b}$, which yields $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} = 2b$ since we assume that $a \neq -c$, and a contradiction against our hypothesis follows immediately.

Case 2. $F - 1$ and $G - 1$ are linearly independent. In this case, we have $F \neq G$.

From the conclusions of Lemma 2, there exists an integer $j_0 \in \{1, 2, \dots, m\}$ such that $(F - 1, G - 1)$ and $(\partial_{z_{j_0}} F, \partial_{z_{j_0}} G)$ are linearly independent, i.e.,

$$W = \begin{vmatrix} F - 1 & G - 1 \\ \partial_{z_{j_0}} F & \partial_{z_{j_0}} G \end{vmatrix} \neq 0.$$

Define

$$(3.7) \quad H := \frac{W}{(F - 1)(G - 1)} = \frac{\partial_{z_{j_0}} G}{G - 1} - \frac{\partial_{z_{j_0}} F}{F - 1}.$$

By the *lemma of the logarithmic derivative* (see [8, Lemma 1.34] or [10]),

$$\| m(r, H) = o(T(r, f) + T(r, g)).$$

Define \mathcal{I}_{F-1} to be the set of *indeterminacy* of $F - 1$. For each $\mathfrak{z} \in \mathbb{C}^m$, there exists an open connected neighborhood $U_{\mathfrak{z}}$ of \mathfrak{z} and two holomorphic functions $F_1 \neq 0$ and $F_2 \neq 0$ on $U_{\mathfrak{z}}$, coprime at \mathfrak{z} , such that $F_1(F - 1) \equiv F_2$,

$$\dim_{\mathfrak{z}} F_1^{-1}(0) \cap F_2^{-1}(0) \leq m - 2$$

and

$$\mathcal{I}_{F-1} \cap U_{\mathfrak{z}} \equiv F_1^{-1}(0) \cap F_2^{-1}(0).$$

Define $\mathcal{E}_1 := \{\text{supp}(\mathfrak{D}_{F-1}^0)\}_s$ to be the set of *singular points* of the analytic set $\text{supp}(\mathfrak{D}_{F-1}^0)$. Then, \mathcal{E}_1 is of co-dimension at least 2. Define $\mathcal{E}_2 := \{\text{supp}(\mathfrak{D}_{F-1}^\infty)\}_s$, \mathcal{I}_{G-1} , $\mathcal{E}_3 := \{\text{supp}(\mathfrak{D}_{G-1}^0)\}_s$ and $\mathcal{E}_4 := \{\text{supp}(\mathfrak{D}_{G-1}^\infty)\}_s$ similarly. Write $\mathcal{E} := \mathcal{I}_{F-1} \cup \mathcal{I}_{G-1} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$. Then, $\dim_3 \mathcal{E} \leq m - 2$.

Take $\mathfrak{z}_0 \in \mathbb{C}^m \setminus \mathcal{E}$ to be a point such that $\mathfrak{D}_{F-1}^0(\mathfrak{z}_0) = p \in \mathbb{Z}^+ \setminus \{0\}$. Hence, $\mathfrak{D}_{G-1}^0(\mathfrak{z}_0) = p$ by our hypothesis that F and G share the value 1 CM. By a result of Bernstein-Chang-Li [1, Lemma 2.3], there exists a holomorphic coordinate system $\mathbf{u} = (u_1, u_2, \dots, u_m)$ of \mathfrak{z}_0 on $U_{\mathfrak{z}_0} \subset \mathbb{C}^m \setminus \mathcal{E}$ such that

$$U_{\mathfrak{z}_0} \cap \text{supp}(\mathfrak{D}_{F-1}^0) = \{\mathfrak{z} \in U_{\mathfrak{z}_0} \mid u_1(\mathfrak{z}) = 0\}$$

and

$$\mathbf{u}(\mathfrak{z}_0) = (u_1(\mathfrak{z}_0), u_2(\mathfrak{z}_0), \dots, u_m(\mathfrak{z}_0)) = \mathfrak{o} \in \mathbb{C}^m.$$

Hence, there exists a biholomorphic coordinate transformation

$$z_j = z_j(u_1, u_2, \dots, u_m) \quad j = 1, 2, \dots, m$$

around $\mathfrak{o} \in \mathbb{C}^m$ such that $\mathfrak{z}_0 = \mathfrak{z}(\mathfrak{o}) = (z_1(\mathfrak{o}), z_2(\mathfrak{o}), \dots, z_m(\mathfrak{o}))$. So, we can write

$$F(\mathfrak{z}) - 1 = u_1^p F^*(u_1, u_2, \dots, u_m)$$

and

$$G(\mathfrak{z}) - 1 = u_1^p G^*(u_1, u_2, \dots, u_m),$$

where F^* and G^* are holomorphic functions around $\mathfrak{o} \in \mathbb{C}^m$ and do not vanish along the analytic set $U_{\mathfrak{z}_0} \cap \text{supp}(\mathfrak{D}_{F-1}^0)$. A routine calculation leads to

$$(3.8) \quad \left. \frac{\partial_{z_{j_0}} F}{F-1} \right|_{\mathfrak{z}_0} = \left. \frac{p}{u_1} \frac{\partial u_1}{\partial z_{j_0}} \right|_{\mathfrak{o}} + \frac{1}{F^*} \sum_{t=1}^m \left. \frac{\partial F^*}{\partial u_t} \frac{\partial u_t}{\partial z_{j_0}} \right|_{\mathfrak{o}}$$

and

$$(3.9) \quad \left. \frac{\partial_{z_{j_0}} G}{G-1} \right|_{\mathfrak{z}_0} = \left. \frac{p}{u_1} \frac{\partial u_1}{\partial z_{j_0}} \right|_{\mathfrak{o}} + \frac{1}{G^*} \sum_{t=1}^m \left. \frac{\partial G^*}{\partial u_t} \frac{\partial u_t}{\partial z_{j_0}} \right|_{\mathfrak{o}}.$$

Hence, $H|_{\mathfrak{z}_0} = O(1)$, i.e., $\mathfrak{D}_H^\infty(\mathfrak{z}_0) = 0$.

Take $\mathfrak{z}_\infty \in \mathbb{C}^m \setminus \mathcal{E}$ to be a point such that $\mathfrak{D}_{F-1}^\infty(\mathfrak{z}_\infty) = q \in \mathbb{Z}^+ \setminus \{0\}$. Similarly, $\mathfrak{D}_{G-1}^\infty(\mathfrak{z}_\infty) = q$ and $\mathfrak{D}_H^\infty(\mathfrak{z}_\infty) = 0$. Hence, H is holomorphic on \mathbb{C}^m and

$$\| N(r, H) = o(T(r, f) + T(r, g)).$$

Therefore, we obtain

$$(3.10) \quad \| T(r, H) = o(T(r, f) + T(r, g)).$$

It is not difficult to show that

$$\begin{aligned} \| N\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{1}{H}\right) + o(T(r, f) + T(r, g)) \\ &\leq T(r, H) + o(T(r, f) + T(r, g)) \leq o(T(r, f) + T(r, g)). \end{aligned}$$

Analogously, $\| N\left(r, \frac{1}{g}\right) \leq o(T(r, f) + T(r, g))$, $\| N\left(r, \frac{1}{f-c}\right) \leq o(T(r, f) + T(r, g))$ and $\| N\left(r, \frac{1}{g-c}\right) \leq o(T(r, f) + T(r, g))$.

Combining the method used in Subcase 1.2.1, the estimate that $N_2(r, \cdot) \leq N(r, \cdot) + O(1)$ on valence functions, and the conclusions of Lemma 4 yields either $f + g \equiv c$ or $f \equiv g$. The former case implies that $f^{-1}(0) = g^{-1}(c)$ and $f^{-1}(c) = g^{-1}(0)$. Since f and g share the set \mathcal{S}_2 CM, and 0 and c are the only two Picard values of both f and g , and $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \neq b$ (then, all the zeros, say, $\omega_1, \omega_2, \dots, \omega_{n+1}$, of the equation $\omega^n(\omega + a) - b = 0$ are simple and distinct from 0, c) and $\frac{(-1)^n n^n (n+2)a^{n+1}}{2^{n+1}(n+1)^{n+1}} \neq b$ (then, $\omega_j \neq \frac{c}{2}$ for $j = 1, 2, \dots, n+1$), so, without loss of generality, we might assume that $\omega_1 + \omega_2 = c$, $\omega_2 + \omega_3 = c$, \dots , $\omega_n + \omega_{n+1} = c$, $\omega_{n+1} + \omega_1 = c$. Noting $n \geq 2$, we derive that $\omega_2 = \omega_{n+1}$, a contradiction. On the other hand, the latter case yields $F \equiv G$, a contradiction, too. The proof of Theorem 1 finishes here completely. \square

4. PROOF OF THEOREM 2

Define $f^* := 1/f$ and $g^* := 1/g$. By the assumptions of Theorem 2 and the conclusions of Theorem 1, we have $f^* \equiv g^*$. Hence, $f \equiv g$. \square

Final Note. From a recent discussion with M. Shirotsaki in a conference at Hiroshima University, the second author was informed that any three non-intersecting sets of the form $\mathcal{S}_1 = \{a_1, a_2\}$, $\mathcal{S}_2 = \{b_1, b_2\}$ and $\mathcal{S}_3 = \{c\}$ are necessarily not unique range sets, see also H.-X. Yi's paper [12, Examples 3 and 4] with $\mathcal{S}_3 = \{\infty\}$. On the other hand, our aforementioned several examples show that, under some trivial transformation, the only possible triple unique range sets of five elements might be $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \{\omega|\omega^3 + a\omega^2 + b\omega + c = 0\}$ and $\mathcal{S}_3 = \{\infty\}$ such that the polynomial in \mathcal{S}_2 has no multiple zeros and $bc \neq 0$. Further, we think new method, say, that of algebraic curve, might be involved to completely solve this problem. Also see related works of A. Boutabaa and A. Escassut on p -adic fields.

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