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**THE NATURAL AFFINORS ON SOME
FIBER PRODUCT PRESERVING GAUGE
BUNDLE FUNCTORS OF VECTOR BUNDLES**

JAN KUREK AND WŁODZIMIERZ M. MIKULSKI

*Dedicated to Professor Ivan Kolář on the occasion
of his 70th birthday with respect and gratitude*

ABSTRACT. We classify all natural affinors on vertical fiber product preserving gauge bundle functors F on vector bundles. We explain this result for some more known such F . We present some applications. We remark a similar classification of all natural affinors on the gauge bundle functor F^* dual to F as above. We study also a similar problem for some (not all) not vertical fiber product preserving gauge bundle functors on vector bundles.

INTRODUCTION

Let m, n be fixed positive integers.

The category of vector bundles with m -dimensional bases and vector bundle maps with embeddings as base maps will be denoted by \mathcal{VB}_m .

The category of vector bundles with m -dimensional bases and n -dimensional fibers and vector bundle embeddings will be denoted by $\mathcal{VB}_{m,n}$.

Let $F : \mathcal{VB}_m \rightarrow \mathcal{FM}$ be a covariant functor. Let $B_{\mathcal{FM}} : \mathcal{FM} \rightarrow \mathcal{Mf}$ and $B_{\mathcal{VB}_m} : \mathcal{VB}_m \rightarrow \mathcal{Mf}$ be the base functors.

A gauge bundle functor on \mathcal{VB}_m is a functor F as above satisfying:

(i) (*Base preservation*) $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}_m}$. Hence the induced projections form a functor transformation $\pi : F \rightarrow B_{\mathcal{VB}_m}$.

(ii) (*Localization*) For every inclusion of an open vector subbundle $i_{E|U} : E|U \rightarrow E$, $F(E|U)$ is the restriction $\pi^{-1}(U)$ of $\pi : FE \rightarrow B_{\mathcal{VB}_m}(E)$ to U and $F i_{E|U}$ is the inclusion $\pi^{-1}(U) \rightarrow FE$.

(iii) (*Regularity*) F transforms smoothly parametrized systems of \mathcal{VB}_m -morphisms into smoothly parametrized families of \mathcal{FM} -morphisms.

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A gauge bundle functor $F : \mathcal{VB}_m \rightarrow \mathcal{FM}$ is of finite order r if from $j_x^r f = j_x^r g$ it follows $F_x f = F_x g$ for any \mathcal{VB}_m -objects $E_1 \rightarrow M$, $E_2 \rightarrow M$, any \mathcal{VB}_m -maps $f, g : E_1 \rightarrow E_2$ and any $x \in M_1$.

A gauge bundle functor F on \mathcal{VB}_m is fiber product preserving if for any fiber product projections

$$E_1 \xleftarrow{\text{pr}_1} E_1 \times_M E_2 \xrightarrow{\text{pr}_2} E_2$$

in the category \mathcal{VB}_m ,

$$FE_1 \xleftarrow{F \text{pr}_1} F(E_1 \times_M E_2) \xrightarrow{F \text{pr}_2} FE_2$$

are fiber product projections in the category \mathcal{FM} . In other words we have $F(E_1 \times_M E_2) = F(E_1) \times_M F(E_2)$.

A gauge bundle functor F on \mathcal{VB}_m is called vertical if for any \mathcal{VB}_m -objects $E \rightarrow M$ and $E_1 \rightarrow M$ with the same basis, any $x \in M$ and any \mathcal{VB}_m -map $f : E \rightarrow E_1$ covering the identity of M the fiber restriction $F_x f : F_x E \rightarrow F_x E_1$ depends only on $f_x : E_x \rightarrow (E_1)_x$.

From now on we are interested in vertical fiber product preserving gauge bundle functors on \mathcal{VB}_m .

The most known example of vertical fiber product preserving gauge bundle functor F on \mathcal{VB}_m is the so-called vertical r -jet prolongation functor $J_v^r : \mathcal{VB}_m \rightarrow \mathcal{FM}$, where for a \mathcal{VB}_m -object $p : E \rightarrow M$ we have a vector bundle $J_v^r E = \{j_x^r \gamma \mid \gamma \text{ is a local map } M \rightarrow E_x, x \in M\}$ and for a \mathcal{VB}_m -map $f : E_1 \rightarrow E_2$ covering $\underline{f} : M_1 \rightarrow M_2$ we have a vector bundle map $J_v^r f : J_v^r E_1 \rightarrow J_v^r E_2$, where $J_v^r f(j_x^r \gamma) = j_{\underline{f}(x)}^r(f \circ \gamma \circ \underline{f}^{-1})$ for $j_x^r \gamma \in J_v^r E_1$.

Another example is the vertical Weil functor V^A on \mathcal{VB}_m corresponding to a Weil algebra A , where for a \mathcal{VB}_m -object $p : E \rightarrow M$ we have $V^A E = \cup_{x \in M} T^A(E_x)$ and for a \mathcal{VB}_m -map $f : E_1 \rightarrow E_2$ we have $V^A f = \cup_{x \in M_1} T^A(f_x) : V^A E_1 \rightarrow V^A E_2$. The functor V^A is equivalent to $E \otimes A$.

The fiber product $F_1 \times_{B_{\mathcal{VB}_m}} F_2 : \mathcal{VB}_m \rightarrow \mathcal{FM}$ of vertical fiber product preserving gauge bundle functors $F_1, F_2 : \mathcal{VB}_m \rightarrow \mathcal{FM}$ is again a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m .

In [8], we proved that every fiber product preserving gauge bundle functor F on \mathcal{VB}_m has values in \mathcal{VB}_m . (More precisely, the fiber sum map $+$: $E \times_M E \rightarrow E$, the fiber scalar multiplication $\lambda_t : E \rightarrow E$ for $t \in \mathbf{R}$ and the zero map $0 : E \rightarrow E$ are \mathcal{VB}_m -map and we can apply F . We obtain $F(+)$: $FE \times_M FE \rightarrow FE$, $F(\lambda_t) : FE \rightarrow FE$ and $F(0) : FE \rightarrow FE$. Then $(F(+), F(\lambda_t), F(0))$ is a vector bundle structure on FE .) Then we can compose such functors. The composition of vertical fiber product preserving gauge bundle functors on \mathcal{VB}_m is again a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m .

If F is a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m , then $(F^*)^* : \mathcal{VB}_m \rightarrow \mathcal{FM}$, $(F^*)^*(E) = (FE^*)^*$, $(F^*)^*(f) = (Ff^*)^*$ is a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m (E^* denote the dual vector bundle of E).

In [8], we classified all fiber product preserving gauge bundle functors F on \mathcal{VB}_m of finite order r in terms of triples (V, H, t) , where V is a finite-dimensional vector space over \mathbf{R} , $H : G_m^r \rightarrow \text{GL}(V)$ is a smooth group homomorphism from $G_m^r = \text{inv}J_0^r(\mathbf{R}^m, \mathbf{R}^m)_0$ into $\text{GL}(V)$ and $t : \mathcal{D}_m^r \rightarrow \text{gl}(V)$ is a G_m^r -equivariant unity preserving associative algebra homomorphism from $\mathcal{D}_m^r = J_0^r(\mathbf{R}^m, \mathbf{R})$ into $\text{gl}(V)$. Moreover, we proved that all fiber product preserving gauge bundle functors F on \mathcal{VB}_m are of finite order. Analyzing the construction on (V, H, t) one can easily see that the triple (V, H, t) corresponding to a vertical F in question has trivial $t : \mathcal{D}_m^r \rightarrow \text{gl}(V)$, $t(j_x^r \gamma) = \gamma(0) \text{id}$, $j_0^r \gamma \in \mathcal{D}_m^r$. Then by Fact 5 and Theorem 2 in [8] it follows that all vertical fiber product preserving gauge bundle functors on \mathcal{VB}_m can be constructed (up to \mathcal{VB}_m -equivalence) as follows.

Let $V : \mathcal{M}f_m \rightarrow \mathcal{VB}$ be a vector natural bundle. For any \mathcal{VB}_m -object $p : E \rightarrow M$ we put $F^V E = E \otimes_M VM$ and for any \mathcal{VB}_m -map $f : E_1 \rightarrow E_2$ covering $\underline{f} : M_1 \rightarrow M_2$ we put $F^V f = f \otimes_{\underline{f}} V \underline{f} : F^V E_1 \rightarrow F^V E_2$. The correspondence $F^V : \mathcal{VB}_m \rightarrow \mathcal{FM}$ is a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . (For example, if $V : \mathcal{M}f_m \rightarrow \mathcal{VB}$ is the natural vector bundle corresponding to the standard G_m^r -space \mathcal{D}_m^r , then F^V is equivalent with J_v^r . If $V : \mathcal{M}f_m \rightarrow \mathcal{VB}$ is the trivial vector natural bundle with the standard fiber A , then F^V is equivalent to V^A .)

Let F be a gauge bundle functor on \mathcal{VB}_m . A $\mathcal{VB}_{m,n}$ -natural affinator B on F is a system of $\mathcal{VB}_{m,n}$ -invariant affinors $B : TFE \rightarrow TFE$ on FE for any $\mathcal{VB}_{m,n}$ -object E . The invariance means that $B \circ TFf = TFf \circ B$ for any $\mathcal{VB}_{m,n}$ -map f .

In the present paper we describe all $\mathcal{VB}_{m,n}$ -natural affinors B on vertical fiber product preserving gauge bundle functors F on \mathcal{VB}_m . We prove that $B : TFE \rightarrow TFE$ is of the form

$$B = \lambda \text{Id} + \text{Mod}(A)$$

for a real number λ and a fiber bilinear $\mathcal{VB}_{m,n}$ -natural transformation $A : TM \times_M FE \rightarrow FE$, where $\text{Mod}(A)$ is the $\mathcal{VB}_{m,n}$ -natural affinator corresponding to A (see Example 2) and Id is the identity affinator.

In Section 3, we explain this main result for some more known vertical fiber product preserving gauge bundle functors F on \mathcal{VB}_m . Thus for J_v^r we reobtain the result from [15] saying that the vector space of all $\mathcal{VB}_{m,n}$ -natural affinors on J_v^r is 2-dimensional.

In Section 4, we remark a similar classification of $\mathcal{VB}_{m,n}$ -natural affinors on a gauge bundle functor F^* dual to a vertical fiber product preserving gauge bundle functor F on \mathcal{VB}_m .

In Section 5, we remark a similar classification of $\mathcal{VB}_{m,n}$ -natural affinors for some (not all) not vertical fiber product preserving gauge bundle functors F on \mathcal{VB}_m (as the r -jet prolongation gauge bundle functor J^r on \mathcal{VB}_m and the vector r -tangent gauge bundle functor $T^{(r)\text{fl}}$ on \mathcal{VB}_m). Thus a similar result as the main result for not necessarily vertical F is very very probably.

Natural affinors can be used to study torsions of connections, see [5]. That is

why they have been classified in many papers, [1] – [4], [6], [8] – [16], e.t.c.

The trivial vector bundle $\mathbf{R}^m \times \mathbf{R}^n$ over \mathbf{R}^m with standard fiber \mathbf{R}^n will be denoted by $\mathbf{R}^{m,n}$. The coordinates on \mathbf{R}^m will be denoted by x^1, \dots, x^m . The fiber coordinates on $\mathbf{R}^{m,n}$ will be denoted by y^1, \dots, y^n .

All manifolds are assumed to be finite dimensional and smooth. Maps are assumed to be smooth, i.e. of class \mathcal{C}^∞ .

1. THE MAIN RESULT

Let F be a fiber product preserving gauge bundle functor on \mathcal{VB}_m . We are going to present examples of $\mathcal{VB}_{m,n}$ -natural affinors on F .

Example 1 (*The identity affinor*). For any $\mathcal{VB}_{m,n}$ -object E we have the identity map $\text{Id} : TFE \rightarrow TFE$. The family Id is a $\mathcal{VB}_{m,n}$ -natural affinor on FE .

Example 2. Suppose we have a family A of fiber bilinear maps $A : TM \times FE \rightarrow FE$ covering the identity of M for any $\mathcal{VB}_{m,n}$ -object $E \rightarrow M$ such that $Ff \circ A = A \circ (T\underline{f} \times_{\underline{f}} Ff)$ for any $\mathcal{VB}_{m,n}$ -map $f : E_1 \rightarrow E_2$ covering $\underline{f} : M_1 \rightarrow M_2$, i.e. we have a fiber bilinear $\mathcal{VB}_{m,n}$ -natural transformation $A : TM \times_M FE \rightarrow FE$, where TM is the tangent bundle of M and FE is the vector bundle as is explained in Introduction. For any $\mathcal{VB}_{m,n}$ -object $p : E \rightarrow M$ we define $\text{Mod}(A) : TFE \rightarrow TFE$ by

$$\text{Mod}(A)(v) = \frac{d}{dt}_0 (y + tA(T\pi(v), y)) \in T_y FY, \quad v \in T_y FE, \quad y \in FE,$$

where $\pi : FE \rightarrow M$ is the bundle projection. Then $\text{Mod}(A)$ is a $\mathcal{VB}_{m,n}$ -natural affinor on F . We call $\text{Mod}(A)$ the $\mathcal{VB}_{m,n}$ -natural affinor on F corresponding to A (the modification of A).

For example, in the case of $F = J_v^r$ we have a fiber bilinear $\mathcal{VB}_{m,n}$ -natural transformation $A_v^r : TM \times J_v^r E \rightarrow J_v^r E$, $A_v^r(w, j_x^r \sigma) = j_x^r(w\sigma)$, $w \in T_x M$, $x \in M$, $\sigma : M \rightarrow E_x$, $w\sigma \in E_x$ is the differential of σ with respect to w and $w\sigma : M \rightarrow E_x$ is the constant map.

The main result of the present paper is the following classification theorem.

Theorem 1. *Let F be a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . Any $\mathcal{VB}_{m,n}$ -natural affinor B on F is the form*

$$B = \lambda \text{Id} + \text{Mod}(A)$$

for some real number λ and some fiber bilinear $\mathcal{VB}_{m,n}$ -natural transformation $A : TM \times_M FE \rightarrow FE$.

Thus for $F = J_v^r$ we reobtain the result from [15] saying that any $\mathcal{VB}_{m,n}$ -natural affinor on J_v^r is a linear combination with real coefficients of the identity affinor and $\text{Mod}(A_v^r)$ (see Corollary 5 below).

We end this section by the following observation.

Let F be of the form F^V for some natural vector bundle $V : \mathcal{M}f_m \rightarrow \mathcal{V}\mathcal{B}$ (see Introduction). Let $C : TM \times_M VM \rightarrow VM$ be an $\mathcal{M}f_m$ -natural fiber bilinear transformation. Then we have a $\mathcal{V}\mathcal{B}_{m,n}$ -natural fiber bilinear transformation $A^C : TM \times_M F^V E \rightarrow F^V E$,

$$A^C(v, e \otimes y) = e \otimes C(v, y) ,$$

$y \in V_x M, e \in E_x, v \in T_x M, x \in M$.

Proposition 1. *Let $V : \mathcal{M}f_m \rightarrow \mathcal{V}\mathcal{B}$ be a natural vector bundle. Any $\mathcal{V}\mathcal{B}_{m,n}$ -natural fiber bilinear transformation $A : TM \times_M F^V E \rightarrow F^V E$ is of the form A^C for some $\mathcal{M}f_m$ -natural fiber bilinear transformation $C : TM \times_M VM \rightarrow VM$.*

Proof of Proposition 1. By the $\mathcal{V}\mathcal{B}_{m,n}$ -invariance, A is determined by the $\mathcal{M}f_m$ -natural fiber bilinear transformation

$$TM \times_M VM \ni (v, y) \rightarrow \langle A(v, e_1(\pi^T(v)) \otimes y), e_1^*(\pi^T(v)) \rangle \in VM ,$$

where e_1, \dots, e_n is the usual basis of sections of the trivial vector bundle $M \times \mathbf{R}^n$ and e_1^*, \dots, e_n^* is the dual basis, and $\pi^T : TM \rightarrow M$ is the tangent bundle projection.

2. PROOF OF THEOREM 1

We fix a basis in the vector space $F_0\mathbf{R}^{m,n}$.

Step 1. Consider

$$T\pi \circ B : (TFR^{m,n})_0 \cong \mathbf{R}^m \times F_0\mathbf{R}^{m,n} \times F_0\mathbf{R}^{m,n} \rightarrow T_0\mathbf{R}^m ,$$

where $\pi : FE \rightarrow M$ is the bundle projection. Using the invariance of B with respect to the fiber homotheties we deduce that $T\pi \circ B(a, u, v) = T\pi \circ B(a, tu, tv)$ for any $u, v \in F_0\mathbf{R}^{m,n}, a \in \mathbf{R}^m, t \neq 0$. Then $T\pi \circ B(a, u, v) = T\pi \circ B(a, 0, 0)$ for u, v, a as above. Then using the invariance of B with respect to $C \times \text{id}_{\mathbf{R}^n}$ for linear isomorphisms C of \mathbf{R}^n we deduce that $T\pi \circ B(a, 0, 0) = \lambda a$ for some real number λ . Then replacing B by $B - \lambda \text{Id}$ we have $T\pi \circ B(a, u, v) = 0$ for any a, u, v as above. Then B is of vertical type.

Step 2. Consider

$$\text{pr}_2 \circ B : (TFR^{m,n})_0 \cong \mathbf{R}^m \times F_0\mathbf{R}^{m,n} \times F_0\mathbf{R}^{m,n} \rightarrow F_0\mathbf{R}^{m,n} ,$$

where $(VFR^{m,n})_0 \cong F_0\mathbf{R}^{m,n} \times F_0\mathbf{R}^{m,n} \rightarrow F_0\mathbf{R}^{m,n}$ is the projection onto the second (essential) factor. Using the invariance of B with respect to the fiber homotheties we deduce that $\text{pr}_2 \circ B(a, tu, tv) = t \text{pr}_2 \circ B(a, u, v)$ for a, u, v as in Step 1. Then $\text{pr}_2 \circ B(a, u, v)$ is a system of linear combinations of the coefficients of u and v with coefficients being smooth maps in a because of the homogeneous function theorem. On the other hand, since B is an affinor, $\text{pr}_2 \circ B(a, u, v)$ is a

system of linear combinations of the coefficients of a and v with coefficients being smooth functions in u . Then

$$(*) \quad \text{pr}_2 \circ B(a, u, v) = G(a, u) + H(v)$$

for some bilinear map G and some linear map H .

Let $\Phi : \mathbf{R}^{m,n} \rightarrow \mathbf{R}^{m,n}$ be a $\mathcal{VB}_{m,n}$ -map such that $\Phi(x, v) = (x, e^{x^1}v)$, $(x, y) \in \mathbf{R}^{m,n}$. Then Φ sends $\frac{\partial}{\partial x^1}$ into $\frac{\partial}{\partial x^1} + L$, where L is the Liouville vector field on $\mathbf{R}^{m,n}$. Then using the invariance of B with respect to Φ we obtain

$$F\Phi(G(e_1, F\Phi^{-1}(v))) = G(e_1, v) + H(v),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbf{R}^m$. Since F is vertical, $F_0\Phi = \text{id}$. Hence $H(v) = 0$, and

$$\text{pr}_2 \circ B(a, u, v) = G(a, u).$$

Then by the $\mathcal{VB}_{m,n}$ -invariance of B we obtain the equivariant condition

$$F_0f(G(a, u)) = G(T_0f(a), F_0f(u))$$

for any a, u as above and any $\mathcal{VB}_{m,n}$ -map $f : \mathbf{R}^{m,n} \rightarrow \mathbf{R}^{m,n}$ preserving $0 \in \mathbf{R}^m$. Hence there is a $\mathcal{VB}_{m,n}$ -natural fiber bilinear transformation $A : TM \times_M FE \rightarrow FE$ corresponding to G . It is easy to see that $B = \text{Mod}(A)$. \square

3. APPLICATIONS

Let $T^{(p,q)} = \otimes^q T^* \otimes \otimes^p T : \mathcal{M}f_m \rightarrow \mathcal{VM}$ be the natural vector bundle of tensor fields of type (p, q) over m -manifolds. Let $F^{(p,r)} = F^{T^{(p,r)}} : \mathcal{VB}_m \rightarrow \mathcal{FM}$, $F^{(p,r)}E = E \otimes_M T^{(p,r)}M$, $F^{(p,q)}f = f \otimes_{\underline{f}} T^{(p,q)}\underline{f}$ be the corresponding vertical fiber product preserving gauge bundle functor (see Introduction).

Suppose that $C : TM \times_M T^{(p,r)}M \rightarrow T^{(p,q)}M$ is a fiber bilinear $\mathcal{M}f_m$ -natural transformation. Using the invariance of C with respect to base homotheties on $\mathbf{R}^{m,n}$ one can easily deduce that $C = 0$. Thus we have the following corollary

Corollary 1. *Any $\mathcal{VB}_{m,n}$ -natural affinator on $F^{(p,q)}$ as above is a constant multiple of the identity affinator.*

Similarly, any $\mathcal{M}f_m$ -natural fiber bilinear transformation $C : TM \times_M M \rightarrow M$, where M is treated as the zero vector bundle over M , is zero. Thus we have

Corollary 2. *Any $\mathcal{VB}_{m,n}$ -natural affinator on the vertical Weil bundle V^A is a constant multiple of the identity affinator.*

Let $T^{(r)} = (J^r(\cdot, \mathbf{R})_0)^* : \mathcal{M}f_m \rightarrow \mathcal{VB}$ be the linear r -tangent bundle functor. Let $F^{(r)} = F^{T^{(r)}} : \mathcal{VB}_m \rightarrow \mathcal{FM}$ be the corresponding vertical fiber product preserving gauge bundle functor.

Suppose that $C : TM \times_M T^{(r)}M \rightarrow T^{(r)}M$ is a $\mathcal{M}f_m$ -natural fiber bilinear transformation. By the rank theorem, C is determined by the contraction $\langle C, j_0^r x^1 \rangle : T_0^{(r)}\mathbf{R}^m \rightarrow \mathbf{R}$. Then using the invariance of C with respect to the base homotheties one can easily show that this contraction is zero. Then $C = 0$. Thus we have

Corollary 3. *Any $\mathcal{VB}_{m,n}$ -natural affinator on $F^{(r)}$ as above is a constant multiple of the identity one.*

Let $T^{r*} = J^r(\cdot, \mathbf{R})_0 : \mathcal{M}f_m \rightarrow \mathcal{VB}$ be the r -cotangent bundle functor. Let $F^{r*} = F^{T^{r*}} : \mathcal{VB}_m \rightarrow \mathcal{FM}$ be the corresponding vertical fiber product preserving gauge bundle functor.

Suppose that $C : TM \times_M T^{r*}M \rightarrow T^{r*}M$ is a $\mathcal{M}f_m$ -natural fiber bilinear transformation. By the rank theorem, C is determined by the evaluations $C(v, j_0^r x^1) \in T_0^{r*}\mathbf{R}^m$, where $v \in T_0\mathbf{R}^m$. Then using the invariance of C with respect to the base homotheties one can easily show that these evaluations are zero. Then $C = 0$. Thus we have

Corollary 4. *Any $\mathcal{VB}_{m,n}$ -natural affinator on F^{r*} as above is a constant multiple of the identity one.*

Let $E^{r*} = J^r(\cdot, \mathbf{R}) : \mathcal{M}f_m \rightarrow \mathcal{VB}$ be the extended r -cotangent bundle functor. As we know the corresponding vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m is equivalent to the vertical r -jet functor J_v^r (see Introduction).

Suppose that $C : TM \times_M E^{r*}M \rightarrow E^{r*}M$ is a $\mathcal{M}f_m$ -natural fiber bilinear transformation. By the rank theorem, C is determined by the evaluations $C(\frac{\partial}{\partial x^r}_0, j_0^r 1) \in E_0^{r*}\mathbf{R}^m$ and $C(\frac{\partial}{\partial x^r}_0, j_0^r x^1) \in E_0^{r*}\mathbf{R}^m$. Then using the invariance of C with respect to the base homotheties one can easily show that the second evaluation is a constant multiple of $j_0^r 1$ and the first one is zero. Then the vector space of all C in question is of dimension less or equal to 1. Thus we reobtain

Corollary 5 ([15]). *Any $\mathcal{VB}_{m,n}$ -natural affinator on J_v^r is a linear combination with real coefficients of the identity affinator and the affinator $\text{Mod}(A_v^r)$.*

Corollary 6. *Let F be a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . Any $\mathcal{VB}_{m,n}$ -natural 1-form ω on F is zero.*

Proof. Let L be the Liouville vector field on the vector bundle FE . Then $\omega \otimes L$ is a $\mathcal{VB}_{m,n}$ -natural affinator. Since it is not isomorphic, it is of the form $\omega \otimes L = \text{Mod}(A)$ for some bilinear $\mathcal{VB}_{m,n}$ -natural transformation $A : TM \times_M FE \rightarrow FE$. Then A is of the form $A(v, y) = \lambda(v)y$ for some uniquely (and then $\mathcal{M}f_m$ -natural) 1-form $\lambda : TM \rightarrow \mathbf{R}$ on M . But any such 1-form is zero. Then $A = 0$. Then $\omega = 0$.

Corollary 7. *Let F be a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . There is no $\mathcal{VB}_{m,n}$ -natural symplectic structure ω on F .*

Proof. Suppose that such ω exists. Then $\omega(L, \cdot)$ is a $\mathcal{VB}_{m,n}$ -natural 1-form on F . Then $\omega(L, \cdot) = 0$ because of Corollary 6. Then ω is degenerate. Contradiction.

Quite similarly one can prove

Corollary 8. *Let F be a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . Then there is no $\mathcal{VB}_{m,n}$ -natural non-degenerate Riemannian tensor field g on F .*

4. A DUAL VERSION OF THE MAIN RESULT

Let F be a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . Let F^* be the dual gauge bundle functor on $\mathcal{VB}_{m,n}$, $F^*E = (FE)^*$ and $F^*f = (Ff^{-1})^*$. Replacing in the proof of Theorem 1 F by F^* we obtain

Theorem 1'. *Let F be a vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m . Let F^* be the dual gauge bundle functor. Any $\mathcal{VB}_{m,n}$ -natural affinor B on F^* is of the form*

$$B = \lambda \text{Id} + \text{Mod}(A^*)$$

for some $\lambda \in \mathbf{R}$ and some $\mathcal{VB}_{m,n}$ -natural fiber bilinear transformation $A : TM \times_M FM \rightarrow FM$, where $A^* : TM \times_M F^*E \rightarrow F^*E$ is the $\mathcal{VB}_{m,n}$ -natural fiber bilinear transformation given by $A^*(v, \cdot) = (A(v, \cdot))^*$ for any $v \in TM$.

5. THE NOT NECESSARILY VERTICAL CASE

In our opinion, it is very probably that Theorem 1 holds for (not necessarily vertical) fiber product preserving gauge bundle functors on \mathcal{VB}_m . For example, in [15] we proved.

Fact 1 ([15]). *Any $\mathcal{VB}_{m,n}$ -natural affinor on the r -jet prolongation functor J^r , which is a not vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m , is a constant multiple of the identity affinor.*

The crucial property of J^r which we used to prove Fact 1 is that any $\mathcal{VB}_{m,n}$ -natural linear operator lifting linear vector fields from E to vector fields on J^rE is a constant multiple of the flow operator.

Replacing in [15] J^r be an arbitrary fiber product preserving gauge bundle functor F on \mathcal{VB}_m we can obtain

Proposition 2. *Let F be a (not necessarily vertical) fiber product preserving gauge bundle functor on \mathcal{VB}_m such that any $\mathcal{VB}_{m,n}$ -natural linear operator lifting linear vector fields from E into vector fields on FE is a constant multiple of the flow operator \mathcal{F} . Then any $\mathcal{VB}_{m,n}$ -natural affinor B on F is a constant multiple of the identity affinor.*

Proof. Clearly, $B \circ \mathcal{F}$ is a $\mathcal{VB}_{m,n}$ -natural linear operator lifting linear vector fields to F . By the assumption, there is $\lambda \in \mathbf{R}$ such that $B \circ \mathcal{F} = \lambda \mathcal{F}$. Next we use the same proof as the one of Theorem 1 up to the formula (*). Obviously, after Step 1, B satisfies $B(\mathcal{F}X) = 0$ for any linear vector field on $\mathbf{R}^{m,n}$. Putting in (*) $X = a \frac{\partial}{\partial x^1}$ (i.e. $(a, u, v) = (a, u, 0)$) we get $G(a, u) = 0$. Putting $X = L$, the Liouville vector field on $\mathbf{R}^{m,n}$ (i.e. $(a, u, v) = (0, v, v)$) we get $H(v) = 0$.

In [7], we proved that the assumption of Proposition 1 is satisfied for the vector r -tangent gauge bundle functor $T^{(r)\natural}$ on \mathcal{VB}_m defined as follows. Given a \mathcal{VB}_m -object $p : E \rightarrow M$, $T^{(r)\natural}E = (J_{\natural}^r(E, \mathbf{R})_0)^*$ is the vector bundle over M dual to $J_{\natural}^rE = \{j_x^r \gamma \mid \gamma : E \rightarrow \mathbf{R} \text{ is fiber linear, } \gamma_x = 0, x \in M\}$. For every \mathcal{VB}_m -map $f : E_1 \rightarrow E_2$ covering $\underline{f} : M_1 \rightarrow M_2$, $T^{(r)\natural}f : T^{(r)\natural}E_1 \rightarrow T^{(r)\natural}E_2$

is a vector bundle map covering \underline{f} such that $\langle T^{(r)\natural}f(\omega), j_{\underline{f}(x)}^r\xi \rangle = \langle \omega, j_x^r(\xi \circ f) \rangle$, $\omega \in T_x^{(r)\natural}E_1$, $j_{\underline{f}(x)}^r\xi \in J_{\natural}^r(E_2, \mathbf{R})_0$, $x \in M$. (The correspondence $T^{(r)\natural}$ is a not vertical fiber product preserving gauge bundle functor on \mathcal{VB}_m .) Thus we have

Fact 2. *Any $\mathcal{VB}_{m,n}$ -natural affinator on $T^{(r)\natural}$ is a constant multiple of the identity affinator.*

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