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GAP PROPERTIES OF HARMONIC MAPS AND SUBMANIFOLDS

QUN CHEN* AND ZHEN-RONG ZHOU†

Abstract. In this article, we obtain a gap property of energy densities of harmonic maps from a closed Riemannian manifold to a Grassmannian and then, use it to Gaussian maps of some submanifolds to get a gap property of the second fundamental forms.

1. Introduction. Main Theorems

Let $f : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds, $e(f) = \frac{1}{2}|df|^2$ be the energy density of f. f is called a harmonic map if it is a critical point of the energy functional

$$E(f) = \int_M e(f) dv_M \,. \tag{1}$$

It is known that (see [7]) if the Ricci curvature $\mathrm{Ric}^M \geq A > 0$ and the Riemannian sectional curvature $\mathrm{Riem}^N \leq B, B > 0$, and if f is harmonic, then $e(f) = 0$ or $e(f) = \frac{mA}{2(m-1)B}$ whenever $e(f) \leq \frac{mA}{2(m-1)B}$.

Let N be a Grassmannian, M a general closed Riemannian manifold, f a harmonic map from M to N. In this paper, we find some non-negative numbers A, B ($A < B$) such that if $A \leq e(f) \leq B$, then $e(f)$ equals to A or B.

We denote the Laplace-Beltrami operator on (M^m, g) by Δ_M. Then $-\Delta_M$ has a discrete spectrum:

$$\mathrm{spec}(\Delta_M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty\}. \tag{2}$$

Let

$$A(p,k) = \frac{p}{2(2p-1)}\left(\lambda_k + \lambda_{k+1} - \sqrt{\lambda_k^2 + \lambda_{k+1}^2 + \frac{4-6p}{p}\lambda_k\lambda_{k+1}}\right) \tag{3}$$

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and

$$B(p,k)=\frac{p}{2(2p-1)}\left(\lambda_k+\lambda_{k+1}+\sqrt{\lambda_k^2+\lambda_{k+1}^2+\frac{4-6p}{p}\lambda_k\lambda_{k+1}}\right). \tag{4}$$

Then $A(p,0)=0$, $B(p,0)=\frac{p}{2p-1}\lambda_1$; $A(1,k)=\lambda_k$, $B(1,k)=\lambda_{k+1}$. Let $G_{m,p}$ be the Grassmannian consisting of linear oriented m-subspaces of the Euclidean $m+p$-space. One can embedding it into the Euclidean space of m-wedge vectors. We denote the image of $G_{m,p}$ under this embedding still by $G_{m,p}$. We obtain

Theorem A. *Let $f: M^q\to G_{m,p}$ be harmonic. If $A(p,k)\le 2e(f)\le B(p,k)$ for some k, then $2e(f)=A(p,k)$ or $2e(f)=B(p,k)$. Especially, we have*

(1) *Let $f: M\to S^m(1)$ be harmonic. If $\lambda_k\le 2e(f)\le\lambda_{k+1}$ for some $k\ge 0$, then $2e(f)=\lambda_k$ or λ_{k+1}.*

(2) *Let $f: M\to G_{m,p}$ be harmonic. If $2e(f)\le\frac{p}{2p-1}\lambda_1$, then $2e(f)=\frac{p}{2p-1}\lambda_1$ or 0.*

As a corollary, we have

Theorem B. *Let M^m be a closed submanifold of E^{m+p} with parallel mean curvature, σ the square length of the second fundamental form. If $A(p,k)\le\sigma\le B(p,k)$ for some $k\ge 0$, then $\sigma=A(p,k)$ or $\sigma=B(p,k)$.*

Especially, we have

(1) *if $p=1$ and $\lambda_k\le\sigma\le\lambda_{k+1}$, then $\sigma=\lambda_k$ or λ_{k+1};*

(2) *if $p\ge 2$ and $\sigma\le\frac{p}{2p-1}\lambda_1$, then $\sigma=0$ or $\frac{p}{2p-1}\lambda_1$.*

S. S. Chern et al proved that if the square length σ of the second fundamental form of a minimal submanifold of spheres satisfies $\sigma\le\frac{mp}{2p-1}$, then $\sigma=0$ or $\frac{mp}{2p-1}$. Our Theorem B shows that the similar gap phenomenon exists for submanifolds of the Euclidean space with parallel mean curvature. Our method is very different from theirs.

2. Preliminaries

Let M^m and N^n be two Riemannian manifolds, $f: M\to N$ be a smooth map. On M, we choose a local orthonormal field of frame around $x\in M$: $e=\{e_i, i=1,\dots,m\}$. The dual is denoted by $\omega=\{\omega_i\}$. The corresponding fields around $f(x)$ are $e^*=\{e^*_\alpha,\alpha=1,\dots,n\}$ and $\omega^*=\{\omega^*_\alpha\}$. We use the convention of summation. The ranges of indices in this section are:

$$i,j,\dots=1,2,\dots,m\,;\qquad \alpha,\beta,\dots=1,2,\dots,n\,. \tag{5}$$

Then the Riemann metrics of M and N can be written respectively as

$$ds^2_M=\sum\omega_i^2\,;\qquad ds^2_N=\sum\omega^{*2}_\alpha\,. \tag{6}$$

Let

$$f^*\omega^*_\alpha=\sum a_{\alpha i}\omega_i\,. \tag{7}$$

then

$$f^*ds^2_N=\sum a_{\alpha i}a_{\alpha j}\omega_i\omega_j\,. \tag{8}$$

Hence, the energy density of f is:

$$e(f) = \frac{1}{2}\mathrm{tr} f^* ds_N^2 = \frac{1}{2}\sum(a_{\alpha i})^2 \,. \tag{9}$$

The structure equations of M are:

$$\mathrm{d}\omega_i = \sum \omega_j \wedge \omega_{ji}, \omega_{ij} + \omega_{ji} = 0 \,, \tag{10}$$

$$\mathrm{d}\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \Omega_{ij} = -\frac{1}{2}\sum R_{ijkl}\omega_k \wedge \omega_l \,, \tag{11}$$

where R_{ijkl} is the Riemannian curvature tensor of M. Take exterior differentiation in (7) and use the structure equations of M and N. we have

$$\sum Da_{\alpha i} \wedge \omega_i = 0 \tag{12}$$

where

$$Da_{\alpha i} := \mathrm{d}a_{\alpha i} + \sum a_{\alpha j}\omega_{ji} + \sum a_{\beta i}\omega^*_{\beta\alpha} \circ f =: \sum a_{\alpha ij}\omega_j \,. \tag{13}$$

By Cartan's Lemma, we have

$$a_{\alpha ij} = a_{\alpha ji} \,. \tag{14}$$

Define

$$b(f) = \sum a_{\alpha ij}\omega_i \otimes \omega_j \otimes e^*_\alpha \circ f \in \Gamma(T^*M \otimes T^*M \otimes f^{-1}TN) \,. \tag{15}$$

We call $b(f)$ the second fundamental form of f, $\tau(f) := \mathrm{tr}\, b(f) = \sum a_{\alpha ii} e^*_\alpha \circ f$ the tension field of f. Then $\tau(f) = 0$ if and only if f is harmonic. If $b(f) = 0$, we say that f is totally geodesic. Apparently,

$$\tau(f) = 0 \Longleftrightarrow \sum a_{\alpha ii} = 0 \,; \quad b(f) = 0 \Longleftrightarrow a_{\alpha ij} = 0 \,. \tag{16}$$

Let P be the set of all orthonormal frame of the $m+p$-dimensional Euclidean space E^{m+p} with the positive orientation. On P, we introduce an equivalent relation $\sim$: $e = (e_1, \dots, e_{m+p}) \sim \overline{e} = (\overline{e}_1, \dots, \overline{e}_{m+p})$ if and only if $(\overline{e}_1, \dots, \overline{e}_m) = (e_1, \dots, e_m) \cdot g$, if and only if $(\overline{e}_{m+1}, \dots, \overline{e}_{m+p}) = (e_{m+1}, \dots, e_{m+p}) \cdot h$ where $g \in SO(m)$ and $h \in SO(p)$. We denote $P/\sim$ by $G_{m,p}$. It can be identified with $\frac{SO(m+p)}{SO(m)\times SO(p)}$, also with the space consisting of oriented m-linear subspace of E^{m+p}. We call it a Grassmannian.

Let $V = \wedge^m E^{m+p}$ be the space of m-degree wedge product of E^{m+p}. There is a natural inner product in V:

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_m}, e_{j_1} \wedge \cdots \wedge e_{j_m} \rangle = \delta^{i_1 \dots i_m}_{j_1 \dots j_m} \,, \tag{17}$$

with respect to which, V forms a $K = C^m_{m+p}$-dimensional Euclidean space, where $(e_1, \dots, e_{m+p}) \in P$ and $i_k, j_k \in \{1, \dots, m+p\}$, $k = 1, \dots, m$.

We define a map $i : G_{m,p} \to V$ by:

$$X \mapsto e_1 \wedge \cdots \wedge e_m \tag{18}$$

for any $X = [e_1, \dots, e_{m+p}] \in G_{m,p}$, the equivalent class of $(e_1, \dots, e_{m+p}) \in P$ with respect to the relation $\sim$. Then i is an embedding (see [1]) from $G_{m,p}$ to V (precisely to S^{K-1}). We denote $i(G_{m,p})$ still by $G_{m,p}$.

In the rest of this section, our indice ranges are:

$$\begin{aligned} & i,j,k,l = 1,\dots,m\,; \quad a,b,c,d = m+1,\dots,m+p\,; \\ & A,B,C,D = 1,\dots,m+p\,. \end{aligned} \tag{19}$$

The motion equation of point x in E^{m+p} is:

$$\mathrm{d}x = \sum \omega_A e_A\,, \tag{20}$$

and the motion equation of the frame $\{e_A\}$ is:

$$\mathrm{d}e_A = \sum \omega_{AB} e_B\,. \tag{21}$$

Then the structure equations of E^{m+p} are:

$$\mathrm{d}\omega_A = \sum \omega_B \wedge \omega_{BA}, \omega_{AB} + \omega_{BA} = 0\,, \tag{22}$$

$$\mathrm{d}\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB}\,. \tag{23}$$

For any $X \in G_{m,p}$, we can set $X = e_1 \wedge \cdots \wedge e_m$. We have

$$\begin{aligned} \mathrm{d}X &= \mathrm{d}(e_1 \wedge \cdots \wedge e_m) \\ &= \sum_i e_1 \wedge \cdots \wedge e_{i-1} \wedge \mathrm{d}e_i \wedge e_{i+1} \wedge \cdots \wedge e_m \\ &= \sum_i e_1 \wedge \cdots \wedge e_{i-1} \wedge (\sum_j \omega_{ij} e_j + \sum_a \omega_{ia} e_a) \wedge e_{i+1} \wedge \cdots \wedge e_m \\ &= \sum \omega_{ia} E_{ia} \end{aligned} \tag{24}$$

where $E_{ia} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_a \wedge e_{i+1} \wedge \cdots \wedge e_m$. Hence, $\{E_{ia}\}$ forms a base of $T_X G_{m,p}$. Let $ds_G^2 = \sum(\omega_{ia})^2$. Then it is a Riemannian metric making $\{E_{ia}\}$ orthonormal.

Let M be an m-dimensional submanifold of E^{m+p}. Identify the oriented tangent space at any point of M with an oriented m-dimensional linear subspace of E^{m+p} in the natural way. Suppose that $(e_1,\dots,e_m)$ is a frame of the tangent space with the positive orientation. Then, $\omega_a = 0$. Therefore, $\omega_{ia} = \sum h^a_{ij}\omega_j$, $h^a_{ij} = h^a_{ji}$. We call (h^a_{ij}) the Weingarten matrix of M in E^{m+p}. We define the Gaussian map $g : M \to G_{m,p}$ of M by

$$g(x) = e_1 \wedge \cdots \wedge e_m\,. \tag{25}$$

Then, by (24) we have, the tangent and the cotangent map g_* and g^* of g at x are

$$g_* e_i = dg(e_i) = \sum \omega_{ja}(e_i) E_{ja} = \sum h^a_{ji} E_{ja}\,, \tag{26}$$

$$g^*\omega_{ia} = \sum h^a_{ij}\omega_j\,. \tag{27}$$

By (7), (9) and (27) we know that the energy density of g is

$$e(g) = \frac{1}{2}\sum (h^a_{ij})^2 = \frac{1}{2}\sigma\,, \tag{28}$$

where σ is the square length of the second fundamental form of M in E^{m+p}.

Hence we have

Lemma 2.1 *Let M^m be a submanifold of E^{m+p}, g the Gussian map of M^m, σ the square length of the second fundamental form of the submanifold. Then we have*

$$\sigma = 2e(g)\,. \tag{29}$$

□

Suppose that M^q is any q-dimensional closed manifold. Consider the following composition:

$$M \xrightarrow{f} G_{m,p} \xrightarrow{\iota} V\,, \tag{30}$$

where ι is the the inclusion of $G_{m,n}$ in V (noting that we have embedded $G_{m,n}$ into V). Let $F = \iota \circ f$. In the following, we calculate the Laplacian of F.

For any $x \in M$, set $f(x) = e_1 \wedge \cdots \wedge e_m \in G_{m,p}$, where $(e_1, \ldots e_{m+p}) \in P$. Then $F(x) \in V$. The ranges of indices in this section are the same as the above section. But $u \in \{1, \ldots, q\}$. Let $\{\epsilon_u, u = 1, \ldots, q\}$ be a local orthonormal field of frame around x, whose dual is $\{\theta_u\}$, and let

$$f^*\omega_{ia} = \sum a^a_{iu}\theta_u\,. \tag{31}$$

Then we have

Lemma 2.2

$$-\,\Delta_M F = \tau(f) + 2e(f)F + G\,, \tag{32}$$

where

$$G = \begin{cases} 2\sum_{i<j,a<b}\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})E_{ia,jb}\circ f\,, & m,p \ge 2\,;\\ 0\,, & \text{otherwise.}\end{cases} \tag{33}$$

Here $E_{ia,jb} = E_{jb,ia} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_a \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_b \wedge e_{j+1} \wedge \cdots \wedge e_m$. It is a normal vector of $G_{m,p}$ in V.

Proof. Notice that $\{E_{ia}\}$ is an orthonormal base, whose dual is $\{\omega_{ia}\}$. By the structure equation (23) we have

$$\begin{aligned} \mathrm{d}\omega_{ia} &= \sum \omega_{ij}\wedge\omega_{ja} + \sum \omega_{ib}\wedge\omega_{ba}\\ &= \sum \omega_{jb}\wedge(-\omega_{ij}\delta_{ba} + \omega_{ba}\delta_{ij})\\ &\equiv \omega_{jb}\wedge\omega^*_{jb,ia}\circ f \end{aligned} \tag{34}$$

where $\omega^*_{jb,ia}\circ f = -\omega_{ij}\delta_{ba} + \omega_{ba}\delta_{ij}$ are the connection forms of $G_{m,p}$.

The tension field of f is

$$\tau(f) = \sum a^a_{iuu}E_{ia}\circ f \tag{35}$$

where (see (13))

$$\sum a^a_{iuv}\theta_v = \mathrm{d}a^a_{iu} - \sum a^a_{iv}\theta_{uv} + \sum a^b_{ju}f^*\omega^*_{jb,ia}\,. \tag{36}$$

Let $f_* = f_u\theta_u$. Then by (31) we have $f_u = \sum a^a_{iu}E_{ia}\circ f$.

Therefore

$$\sum f_{uv}\theta_v = \mathrm{d}f_u - \sum f_v\theta_{uv} = \sum \mathrm{d}a^a_{iu}\cdot E_{ia}\circ f + \sum a^a_{iu}\mathrm{d}(E_{ia}\circ f) - \sum a^a_{iv}E_{ia}\circ f\theta_{uv}\,. \tag{37}$$

It is not difficult to check that if $m, p \geq 2$, we have

$$\mathrm{d}(E_{ia}\circ f) = -f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{jb}E_{jb,ia}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f\,,$$

and that if $m = 1$ or $p = 1$, we have

$$\mathrm{d}(E_{ia}\circ f) = -f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f\,.$$

When $m, p \geq 2$,

$$\begin{aligned}\sum f_{uv}\theta_v &= \sum(a^a_{iuv}\theta_v + a^a_{iv}\theta_{uv} - a^b_{ju}f^*\omega^*_{jb,ia})E_{ia}\circ f\\ &\quad + \sum a^a_{iu}(-f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{jb}E_{jb,ia}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f)\\ &\quad - \sum a^a_{iv}E_{ia}\circ f\theta_{uv}\\ &= \sum(a^a_{iuv}\theta_v + a^a_{iv}\theta_{uv} - a^b_{ju}(-f^*\omega_{ij}\delta_{ba} + f^*\omega_{ba}\delta_{ij}))E_{ia}\circ f\\ &\quad + \sum a^a_{iu}(-f^*\omega_{ji}E_{ja}\circ f + f^*\omega_{jb}E_{jb,ia}\circ f + f^*\omega_{ai}F + f^*\omega_{ab}E_{ib}\circ f)\\ &\quad - \sum a^a_{iv}E_{ia}\circ f\theta_{uv}\\ &= \sum_{i,a,v} a^a_{iuv}E_{ia}\theta_v + \sum_{i\neq j, a\neq b} a^a_{iu}a^b_{jv}E_{ia,jb}\theta_v - \sum_{i,a,v} a^a_{iu}a^a_{iv}F\theta_v\,.\end{aligned} \tag{38}$$

Because $\Delta F = \Delta f = \sum f_{uu}$, we have

$$\Delta_M F = \tau(f) - 2e(f)F + 2\sum_{i<j,a<b}\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})E_{ia,jb}\circ f\,. \tag{39}$$

Similarly, When $m = 1$ or $p = 1$, we have

$$\Delta_M F = \tau(f) - 2e(f)F\,. \tag{40}$$

The lemma follows. □

The following theorem is well known:

Lemma 2.3 (Ruh-Vilms' Theorem) *Suppose that M is a submanifold of the Euclidean space. Then M has a parallel mean cavature if and only if its Gaussian map is harmonic.*

For the proofs, see [6] and [3]. Here we give another one.

Proof. Let $g_* = \sum A_{(ja)i}\omega_i \otimes E_{ja}\circ g \in \Gamma(T^*M\otimes g^{-1}(TG_{m,p}))$. Then by (26), we have $A_{(ka)i} = h^a_{ki}$. The latter is in $\Gamma(T^*M\otimes T^*M\otimes NM)$ where NM is the normal bundle of M. We denote the covariant derivative of h^a_{ki} in $\Gamma(T^*M\otimes g^{-1}(TG_{m,p}))$

by $h^a_{ki;j}$, and that in $\Gamma(T^*M\otimes T^*M\otimes NM)$ by $h^a_{ki|j}$. Then

$$
\begin{aligned}
\sum h^a_{ki;j}\omega_j &= \mathrm{d}h^a_{ki}+\sum h^a_{kj}\omega_{ji}+\sum h^b_{li}\omega^*_{(lb)(ka)}\circ g\\
&= \mathrm{d}h^a_{ki}+\sum h^a_{kj}\omega_{ji}+\sum h^b_{li}(-\omega_{kl}\delta_{ba}+\omega_{ba}\delta_{kl})\\
&= \mathrm{d}h^a_{ki}+\sum h^a_{kj}\omega_{ji}-\sum h^a_{li}\omega_{kl}+\sum h^b_{ki}\omega_{ba}\\
&= \sum h^a_{ki|j}\omega_j\,.
\end{aligned}\tag{41}
$$

Hence $\tau(g)_{(ka)}=h^a_{ki;i}=h^a_{ki|i}=h^a_{ik|i}=h^a_{ii|k}$. The lemma follows. □

Let A be a $m\times n$ matrix, A' its transport. Define $N(A)=\mathrm{tr}(AA')$. Then, we have

Lemma 2.4 *$N(AB'-BA')\le 2N(A)N(B)$ for $m\times n$ matrices A and B*

This inequality is proved by G. R. Wu and W. H. Chen in [9]. For completeness, we prove it in the following.

Proof. $N(A)$ is invariant under orthogonal transformations. Put $C=AB'-BA'$. It is anti-symmetric. By the theory of linear algebra, $\exists U\in O(m)$ such that

$$
UCU'=\tilde C=\mathrm{diag}\left(\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix},\dots,\begin{pmatrix}0&\lambda_p\\-\lambda_p&0\end{pmatrix},0\right)\tag{42}
$$

where $2p=\mathrm{rank}C$, $\lambda_1,\dots,\lambda_p$ are non-zero real numbers, the last 0 is a zero matrix of $(m-2p)\times(m-2p)$. Let $\tilde A=UA=(\xi^\alpha_i)$ and $\tilde B=UB=(\eta^\alpha_i)$. Then we have

$$
\tilde C_{2r-1,2r}=\sum_\alpha(\xi^\alpha_{2r-1}\eta^\alpha_{2r}-\xi^\alpha_{2r}\eta^\alpha_{2r-1})=\lambda_r\,,\quad 1\le r\le p\,.\tag{43}
$$

Hence we have

$$
\begin{aligned}
N(C)=N(\tilde C)&=2\sum_{r=1}^p\left(\sum_\alpha(\xi^\alpha_{2r-1}\eta^\alpha_{2r}-\xi^\alpha_{2r}\eta^\alpha_{2r-1})\right)^2\\
&=2\sum_{r=1}^p(X_r\cdot Y_r)^2
\end{aligned}\tag{44}
$$

where $X_r=(\xi^1_{2r-1},\dots,\xi^n_{2r-1},\xi^1_{2r},\dots,\xi^n_{2r})$, $Y_r=(\eta^1_{2r},\dots,\eta^n_{2r},-\eta^1_{2r-1},\dots,-\eta^n_{2r-1})$, $X_r\cdot Y_r$ stands for the euclidean inner product. By Schwarz inequality we have

$$
\begin{aligned}
N(C)&=2\sum_{r=1}^p(X_r\cdot Y_r)^2\le 2\sum_{r=1}^p|X_r|^2|Y_r|^2\\
&\le 2\sqrt{\sum_{r=1}^p|X_r|^4}\sqrt{\sum_{r=1}^p|Y_r|^4}\le 2\sum_{r=1}^p|X_r|^2\sum_{r=1}^p|Y_r|^2\\
&\le 2N(\tilde A)N(\tilde B)=2N(A)N(B)
\end{aligned}\tag{45}
$$

as desired. □

3. Proofs of Theorems A and B

Proof of Theorem A

Expand F as $F = F_0 + \sum_{s\geq 1} F_s$, where F_0 is a constant vector called the mass center of F or f, $F_s,\ s \geq 0$ are eigenfunctions of Δ_M with respect to the eigenvalues λ_s, i.e.

$$\Delta_M F_s = -\lambda_s F_s\,. \tag{46}$$

If $F_0 = 0$, we say that F or f is mass-symmetric. If $\exists u_i \geq 1, i = 1, \ldots, k$, such that $F = F_0 + \sum_{i=1}^{k} F_{u_i}$, then F or f is called of k-type and $\{u_1, \ldots, u_k\}$ is by definition the order of F or f. For example, if f is a minimal isometric immersion of M^q into S^{q+p}, then $F = i \circ f$ is mass symmetric, of 1-type and its order is $\{k\}$ for some $k \geq 1$ by Takahashi theorem([8]):

$$\Delta_M F = HF - qF \tag{47}$$

where H is the mean curvature of f.

Denote

$$\Psi_k = -\int_M \langle \Delta_M F, F\rangle dv_M - \lambda_k \int_M \langle F, F\rangle dv_M\,, \tag{48}$$

$$\Theta_k = \int_M \langle \Delta_M F, \Delta_M F\rangle dv_M + \lambda_k \int_M \langle \Delta_M F, F\rangle dv_M\,. \tag{49}$$

Then

$$\begin{aligned}\Psi_k &= \int_M \langle \sum \lambda_s F_s, \sum F_s\rangle dv_M - \lambda_k \int_M \langle \sum F_s, \sum F_s\rangle dv_M \\ &= \sum \lambda_s \int_M \langle F_s, F_s\rangle dv_M - \sum \lambda_k \int_M \langle F_s, F_s\rangle dv_M \\ &= \sum \lambda_s a_s - \sum \lambda_k a_s\end{aligned} \tag{50}$$

where $a_s = \int_M \langle F_s, F_s\rangle dv_M$. Similarly

$$\Theta_k = \sum \lambda_s^2 a_s - \lambda_k \sum \lambda_s a_s\,. \tag{51}$$

Accordingly

$$\begin{gathered}\Theta_k - \lambda_{k+1}\Psi_k = \lambda_k \lambda_{k+1} a_0 + \sum_{s\geq 1} (\lambda_s - \lambda_k)(\lambda_s - \lambda_{k+1}) a_s \geq 0\,, \\ \forall k \geq 0\,,\end{gathered} \tag{52}$$

and the equality holds if and only if F is

(a) of 1-type and its order is $\{1\}$ when $k = 0$;

(b) of 2-type and its order is $\{k, k + 1\}$ when $k \geq 1$.

On the other hand, by (32), and noting that $E_{ia,jb}$ is normal to $G_{m,p}$ at $f(x)$, and also normal to $F(x)$ (as a vector in V), we have:

$$\int_M \langle F, F\rangle dv_M = V_M \text{ the volume of } M^q\,; \tag{53}$$

$$\int_M \langle \Delta_M F, F\rangle dv_M = -2E(f)\,, \quad \text{by Lemma 2.2 and noting that } \tau(f)(x) \perp F(x)\,; \tag{54}$$

$$\int_M \langle \Delta_M F, \Delta_M F\rangle dv_M = \int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M |df|^4 dv_M + \int_M |G|^2 dv_M\,. \tag{55}$$

Hence,

$$\Psi_k = 2E(f) - \lambda_k V_M\,; \tag{56}$$

$$\Theta_k = \int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M |df|^4 dv_M + \int_M |G|^2 dv_M - 2\lambda_k E(f)\,. \tag{57}$$

From (52), (56) and (57) we get:

$$\int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M |G|^2 dv_M + \int_M (|df|^2 - \lambda_k)(|df|^2 - \lambda_{k+1}) dv_M \geq 0\,. \tag{58}$$

So, when p is 1, we have

$$\int_M \langle \tau(f), \tau(f)\rangle dv_M + \int_M (|df|^2 - \lambda_k)(|df|^2 - \lambda_{k+1}) dv_M \geq 0\,, \tag{59}$$

whence, if $\tau = 0$, we have

$$\int_M (|df|^2 - \lambda_k)(|df|^2 - \lambda_{k+1}) dv_M \geq 0\,,$$

i.e.

$$\int_M (2e(f) - \lambda_k)(2e(f) - \lambda_{k+1}) dv_M \geq 0\,. \tag{60}$$

When $m, p \geq 2$, we put $A_a = (a^a_{iu})$ be $m \times q$ matrices. From Lemma 2.4, we have

$$
\begin{aligned}
|G|^2 &= 2\sum_{i<j,a<b}\Big(\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})\Big)^2 = \sum_{a<b}\sum_{i,j}\Big(\sum_u (a^a_{iu}a^b_{ju} - a^b_{iu}a^a_{ju})\Big)^2 \\
&= \sum_{a<b} N(A_aA'_b - A_bA'_a) \leq 2\sum_{a<b} N(A_a)N(A_b) \\
&= \Big(\Big(\sum_a N(A_a)\Big)^2 - \sum_a (N(A_a))^2\Big) \leq \frac{p-1}{p}\Big(\sum_a N(A_a)\Big)^2 \\
&= \frac{(p-1)}{p}|df|^4 \,. \qquad (61)
\end{aligned}
$$

Insert it into (58), we have

$$
\begin{aligned}
&\int_M \langle \tau(f), \tau(f)\rangle dv_M \\
&\quad + \int_M \Big(\frac{2p-1}{p}|df|^4 - (\lambda_k + \lambda_{k+1})|df|^2 + \lambda_k\lambda_{k+1}\Big) dv_M \geq 0 \,, \qquad (62)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\int_M \langle \tau(f), \tau(f)\rangle dv_M \\
&\quad + \frac{2p-1}{p}\int_M (|df|^2 - A(p,k))(|df|^2 - B(p,k)) dv_M \geq 0 \,. \qquad (63)
\end{aligned}
$$

If f is harmonic, then $\tau(f) = 0$. Therefore (63) becomes

$$
\int_M (|df|^2 - A(p,k))(|df|^2 - B(p,k)) dv_M \geq 0 \,, \qquad (64)
$$

i.e.

$$
\int_M (2e(f) - A(p,k))(2e(f) - B(p,k)) dv_M \geq 0 \,. \qquad (65)
$$

This inequality is also valid for $p = 1$ by (60). Hence if $A(p,k) \leq 2e(f) \leq B(p,k)$ for some $p \geq 1$ and some $k \geq 0$, then the integrand in (65) is non-positive, hence vanishing. So $2e(f) = A(p,k)$ or $2e(f) = B(p,k)$. Theorem A follows. □

Proof of Theorem B

By Theorem A, Ruh-Vilms' Theorem (Lemma 2.3) and Lemma 2.1, Theorem B follows. □

Remark 3.1. The order of the map in Theorem A must be $\{1\}$ when $k = 0$ or $\{k, k+1\}$ when $k \geq 1$.

Remark 3.2. When $p = 1$, $G_{m,p} = S^m$. From (60) we conclude that

(i) If f is mass symmetric and of order $\{k, k+1\}$, and $2e(f) \leq \lambda_k$ or $2e(f) \geq \lambda_{k+1}$ for some $k \geq 1$, then f is harmonic, and $2e(f) = \lambda_k$ or $2e(f) = \lambda_{k+1}$.

(ii) If f is of order $\{1\}$ and $2e(f) \geq \lambda_1$, then f is harmonic and $2e(f) = \lambda_1$.

References

[1] Chen, W. H., *Geometry of Grassmannian manifolds as submanifolds* (in Chinese), Acta Math. Sinica **31**(1) (1998), 46–53.

[2] Chen, X. P., *Harmonic maps and Gaussian maps* (in Chinese), Chin. Ann. Math. **4A**(4) (1983), 449–456.

[3] Chern, S. S., Goldberg, S. I., *On the volume decreasing property of a class of real harmonic mappings*, Amer. J. Math. **97**(1) (1975), 133–147.

[4] Chern, S. S., doCarmo, M., Kobayashi, S., *Minimal submanifolds of a sphere with second fundamental form of constant length*, Funct. Anal. Rel. Fields (1970), 59–75.

[5] Eells, J., Lemaire, L., *Selected topics on harmonic maps*, Expository Lectures from the CBMS Regional Conf. held at Tulane Univ., Dec. 15–19, 1980.

[6] Ruh, E. A. Vilms, J., *The tension field of the Gauss map*, Trans. Amer. Math. Soc. **149** (1970), 569–573.

[7] Sealey, H. C. J., *Harmonic maps of small energy*, Bull. London Math. Soc. **13** (1981), 405–408.

[8] Takahashi, T., *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan. **18** (1966), 380–385.

[9] Wu, G. R., Chen, W. H., *An inequality on matrix and its geometrical application* (in Chinese), Acta Math. Sinica **31**(3) (1988), 348–355.

[10] Yano, K., Kon, M., *Structures on Manifolds*, Series in Pure Math. **3** (1984), World Scientific.

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