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Archivum Mathematicum, Vol. 40 (2004), No. 4, 383--393

Persistent URL: `http://dml.cz/dmlcz/107922`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 383 – 393

ON θ-CLOSED SETS AND SOME FORMS OF CONTINUITY

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Abstract. In this paper, we further the study of θ-compactness a generalization of quasi-H-closed sets and its applications to some forms of continuity using θ-open and δ-open sets. Among other results, it is shown a weakly θ-retract of a Hausdorff space X is a δ-closed subset of X.

1. Introduction

The concepts of δ-closure, θ-closure, δ-interior and θ-interior operators were first introduced by Veličko. These operators have since been studied intensively by many authors. Although θ-interior and θ-closure operators are not idempotents, the collection of all δ-open sets in a topological space (X, Γ) forms a topology Γ_s on X, called the semiregularization topology of Γ, weaker than Γ and the class of all regular open sets in Γ forms an open basis for Γ_s. Similarly, the collection of all θ-open sets in a topological space (X, Γ) forms a topology Γ_θ on X, weaker than Γ.

So far, numerous applications of such operators have been found in studying different types of continuous like maps, separation of axioms, and above all, to many important types of compactness [6], [7], [8], [10], [11], [12], [15], [16], [17]. A subset A of a space X is called a closure compact subset of X if every open cover of A has a finite subcollection whose closures cover A. A closure compact Hausdorff space is called H-closed, first defined by Alexandroff and Urysohn.

In the present paper, we further the study of θ-compactness a generalization of quasi-H-closed sets and its applications to some forms of continuity using θ-open and δ-open sets. Among other results, it is shown a weakly θ-retract of a Hausdorff space X is a δ-closed subset of X.

We also give a new generalization of closure continuity (*c.c.*) using θ-open sets. In Section 2, we give several characterizations to weak θ-continuity (*w.θ.c.*), and we study the relations between these functions and their graphs. Among other results, it is shown that a function $f : X \to Y$ is *w.θ.c.* iff its graph mapping

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2000 *Mathematics Subject Classification*: 54C08, 54D05, 54D30.
Key words and phrases: almost closure continuity, θ-open, θ-closed, quasi-H-closed, S-Hausdorff spaces, n-compactness.
Received November 25, 2002.

$g = (x, f(x))$ is *w.θ.c.* Theorem 3.1 proves that an *w.θ.c.* retract of a Hausdorff space is δ-closed which is a stronger result of Theorem 5 in [8] and Theorem 3.1 in [16].

For a set A in a space X, let us denote by $\mathrm{Int}(A)$ and $\overline{A}$ for the interior and the closure of A in X, respectively. Following Veličko, a point x of a space X is called a θ-adherent point of a subset A of X iff $\overline{U} \cap A \neq \emptyset$, for every open set U containing x. The set of all θ-adherent points of A is called the θ-closure of A, denoted by $\mathrm{cls}_\theta A$. A subset A of a space X is called θ-closed iff $A = \mathrm{cls}_\theta A$. The complement of a θ-closed set is called θ-open. Similarly, the θ-interior of a set A in X, written $\mathrm{Int}_\theta A$, consists of those points x of A such that for some open set U containing x, $\overline{U} \subseteq A$. A set A is θ-open iff $A = \mathrm{Int}_\theta A$, or equivalently, $X \backslash A$ is θ-closed. A point x of a space X is called a δ-adherent point of a subset A of X iff $\mathrm{Int}(\overline{U}) \cap A \neq \emptyset$, for every open set U containing x. The set of all δ-adherent points of A is called the δ-closure of A, denoted by $\mathrm{cls}_\delta A$. A subset A of a space X is called δ-closed iff $A = \mathrm{cls}_\delta A$. The complement of a δ-closed set is called δ-open. Similarly, the δ-interior of a set A in X, written $\mathrm{Int}_\delta A$, consists of those points x of A such that for some regularly open set U containing x, $U \subseteq A$. A set A is *δ-open* iff $A = \mathrm{Int}_\delta A$, or equivalently, $X \backslash A$ is *δ-closed*.

A function $f : X \to Y$ is *almost continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq \mathrm{Int}(\overline{V})$. If this condition is satisfied at each $x \in X$, then f is said to be *almost continuous* (briefly, *a.c.*). A function $f : X \to Y$ is said to be *almost strongly θ-continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subset \mathrm{Int}(\overline{V})$. If this condition is satisfied at each $x \in X$, then f is said to be *almost strongly θ-continuous* (briefly, *a.s.c.*). A function $f : X \to Y$ is *closure continuous or θ-continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq \overline{V}$. If this condition is satisfied at each $x \in X$, then f is said to be *closure continuous* (θ-continuous) (briefly, *θ.c.*). A function $f : X \to Y$ is *weakly continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq \overline{V}$. If this condition is satisfied at each $x \in X$, then f is said to be weakly continuous (briefly, *w.c.*). A function $f : X \to Y$ is *δ-continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\mathrm{Int}(\overline{U})) \subset \mathrm{Int}(\overline{V})$. If this condition is satisfied at each $x \in X$, then f is said to be *δ-continuous* (briefly, *δ.c.*). A function $f : X \to Y$ is *super continuous* (*in the sense of Munshi and Basan*) at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\mathrm{Int}(\overline{U})) \subset V$. If this condition is satisfied at each $x \in X$, then f is said to be *super continuous* (briefly, *s.c.*). A function $f : X \to Y$ is *weak θ-continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\mathrm{Int}(\overline{U})) \subset \overline{V}$. If this condition is satisfied at each $x \in X$, then $f$is said to be *weak θ-continuous* (briefly, *w.θ.c.*). A function $f : X \to Y$ is *strongly θ-continuous* at $x \in X$ if given any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\overline{U}) \subseteq V$. If this condition is satisfied at each $x \in X$, then f is said to be *strongly θ-continuous* (briefly, *s.θ.c.*). A space X is called *Urysohn* if for every $x, y \in X$, there exists an open set U containing x and an open set V containing y such that $\overline{U} \cap \overline{V} = \emptyset$. A space X is called *semi-regular* if for every $x \in X$ and for every open set U of x there exists an open set W such that $x \in W \subset \operatorname{Int}(\overline{W}) \subset U$. A space X is called *almost regular* if for every regularly closed set F and for every $x \notin F$ there exists disjoint open sets U, V such that $x \in U, F \subset V$. Equivalently, a space X is almost-regular if for every $x \in X$ and for every regularly open set U containing x there exists a regularly open set W containing x such that $x \in W \subset \overline{W} \subset U$.

2. Basic Results

One could easily prove the following, *s.θ.c.* $\Rightarrow$ *s.c.* $\Rightarrow$ *continuity* $\Rightarrow$ *a.c.* $\Rightarrow$ *θ.c.* $\Rightarrow$ *w.θ.c.* $\Rightarrow$ *w.c.*, *s.θ.c.* $\Rightarrow$ *s.c.* $\Rightarrow$ *δ.c.* $\Rightarrow$ *a.c.*, and *s.θ.c.* $\Rightarrow$ *a.s.c.* $\Rightarrow$ *a.c.*, but neither continuity implies *δ.c.* nor *δ.c.* implies continuity, neither *a.s.c.* implies continuity nor *a.s.c.* implies continuity, and neither *a.s.c.* implies *δ.c.* nor *s.c.* implies *a.s.c.*

Example 2.1. Let $X = \{1,2,3\}$, with $\Im_X = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$, $Y = \{1,2,3\}$ with $\Im_Y = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$. Let $f : X \to Y$ be the identity map then f is continuous but f is not *δ.c.* nor *a.s.c.* since $\operatorname{Int}(\overline{\{1\}}) = \{1\}$ in Y, but $\operatorname{Int}(\overline{\{1\}}) = \{1,3\}$ in X and $\overline{\{1\}} = \{1,3\}$ in X, but $\operatorname{Int}(\overline{\{1\}}) = \{1\}$ in Y.

Example 2.2. Let $f : (R_U) \to (R_C)$ be the identity map, where R_U, R_C the usual and the cocountable topologies, respectively. Then f is *a.s.c.* and *δ.c.* but neither *s.c.* nor continuous.

Example 2.3. Let $X = R$ with the topology $\Im$ generated by a basis with members of the form (a,b) and $(a,b)\backslash K$, where $K = \{\frac{1}{n} : n \in Z^+\}$. Let $f : (X, \Im) \to (X, \Im)$, be the identity map. Then f is continuous but not strongly θ-continuous.

Example 2.4. Let $X = \{1,2,3,4\}$ with $\Im_X = \{\emptyset, \{1,2,3\}, \{3\}, \{3,4\}, \{1,2,3,4\}\}$, $Y = \{1,2,3\}$ with $\Im_Y = \{\emptyset, \{2\}, \{4\}, \{2,4\}, \{1,2\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}\}$. Let $f : X \to Y$ be the identity map. Then f is weakly continuous but f is not *w.θ.c.* since $\overline{\{2\}} = \{1,2,3\}$ in Y, but $\operatorname{Int}(\overline{\{1,2,3\}}) = \{1,2,3,4\}$ in X.

Example 2.5. Let $X = R$ with the cocountable topology and let $Y = \{1,2,3\}$ with $\Im_Y = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$. Let $f : X \to Y$ be defined as $f(x) = 1$, if x is rational, and $f(x) = 2$, if x is irrational. Then f is *w.θ.c.* but f is not even weakly continuous and thus not *θ.c.*

The proofs of the following follow straightforward from the definitions.

Theorem 2.6. *Let $f : X \to Y$ be an a.c. function and let X be an almost regular space then f is δ.c.*

Theorem 2.7. *Let $f : X \to Y$ be δ.c. and let Y be an almost regular space then f is s.c.*

Theorem 2.8. *Let $f : X \to Y$ be an a.c. function and let X be a regular space then f is a.s.c.*

Theorem 2.9. *Let $f : X \to Y$ be an open w.θ.c. function. Then f is δ.c.*

Theorem 2.10. *Let $f : X \to Y$ be an a.s.c. function and let Y be a semi-regular space then f is s.θ.c.*

Theorem 2.11. *Let $f : X \to Y$ be an θ.c. function and let Y be an almost-regular space then f is a.s.c.*

Corollary 2.12. *An open map f is w.θ.c. iff f is δ.c.*

Corollary 2.13. *An open a.c. map is δ.c.*

Corollary 2.14. *An open continuous map is δ.c.*

Theorem 2.15. *Let $f : X \to Y$ be an δ.c. function and let Y be an almost regular space then f is s.c.*

Theorem 2.16. *Let $f : X \to Y$ be an a.s.c. function and let Y be an almost regular space then f is s.c.*

Theorem 2.17. *Let $f : X \to Y$ be a continuous function and let X be an almost regular space then f is s.c.*

Corollary 2.18. *Let $f : X \to Y$ be a continuous function and let X be an almost regular space then f is δ.c.*

Theorem 2.19. *Let $f : X \to Y$ be an w.θ.c. function and let X be a regular space then f is θ.c.*

Theorem 2.20. *Let $f : X \to Y$ be an w.c. function and let X be an almost regular space then f is w.θ.c.*

Theorem 2.21. *Let $f : X \to Y$ be an w.θ.c. function and let Y be a regular space then f is s.c.*

Theorem 2.22. *Let $f : X \to Y$ be a function and let X, Y be regular spaces. Then the following are equivalent: f is w.c., f is θ.c., f is a.c., f is continuous, f is a.s.c., f is s.θ.c., f is δ.c., f is w.θ.c., f is s.c.*

The following is a useful characterization of weak θ-continuity.

Theorem 2.23. *Let $f : X \to Y$. Then the following are equivalent:*

(a) *$f(\operatorname{cls}_\delta A) \subset \operatorname{cls}_\theta f(A)$ for every $A \subset X$;*
(b) *the inverse image of θ-closed is δ-closed;*
(c) *the inverse image of the closure of every open set is δ-open;*
(d) *for every $x \in X$, and for every V an open subset of Y containing $f(x)$ there exists a regularly open set U_x such that $f(U_x) \subset \overline{V}$;*
(e) *f is weakly θ-continuous.*

In [7] it is shown that a function f is almost continuous iff its graph mapping g, where $g(x) = (x, f(x))$ is almost continuous. In [10] this result was extended to weak continuity and in [19] this was extended it to faint and quasi-θ-continuity. Similarly, it is shown in [13] that a function f is weakly θ-continuous iff its graph mapping g, where $g(x) = (x, f(x))$ is weakly θ-continuous.

Similar to δ-continuity [12] and following a similar argument as in [7, Theorems, 6, 7], we get the following results.

Theorem 2.24. *Let $f : X \to \prod_{\alpha\in I}\prod X_\alpha$ be given. Then f is w.θ.c. iff the composition with each projection π_αw.θ.c.*

Theorem 2.25. *Define $\prod_{\alpha\in I} f_\alpha : \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$. Then $\prod_{\alpha\in I} f_\alpha$ is w.θ.c. iff each $f_\alpha : X_\alpha \to Y_\alpha$ is w.θ.c.*

3. Hausdorffness and Weak Forms of Compactness

Definition 3.1. A space X is said to be S-Hausdorff if for every $x \neq y \in X$, there exists θ-open sets U_x, V_y such that $U_x \cap V_y = \emptyset$.

Lemma 3.2. *A space X is Hausdorff if for every $x \neq y \in X$, there exist regularly--open sets U_x, V_y such that $U_x \cap V_y = \emptyset$.*

It is clear that every S-Hausdorff space is Urysohn, but a Urysohn space need not be S-Hausdorff [19].

By a weak θ-retraction we mean a weak θ-continuous function $f : X \to A$ where $A \subset X$ and $f|_A$ is the identity function on A. In this case, A is said to be a weak θ-retraction of X.

The next theorem is an improvement of Theorem 5 in [8] and Theorem 3.1 in [21].

Theorem 3.3. *Let $A \subset X$ and let $f : X \to A$ be a weak θ-retraction of X onto A. If X is Hausdorff, then A is an δ-closed subset of X.*

Proof. Suppose not, then there exists a point $x \in \operatorname{cls}_\delta A\backslash A$. Since f is a weak θ-retraction we have $f(x) \neq x$. Since X is Hausdorff, there exist regularly open sets U and V of x and $f(x)$ respectively, such that $U \cap V = \emptyset$. Now let W be any open set in X containing x. Then $U \cap \operatorname{Int}(\overline{W})$ is a regularly open set containing x and hence $\operatorname{Int}(\overline{U}) \cap \operatorname{Int}(\overline{W}) \cap A \neq \emptyset$, since $x \in \operatorname{cls}_\delta A$. Therefore, there exists a point $y \in \operatorname{Int}(\overline{U}) \cap \operatorname{Int}(\overline{W}) \cap A$. Since $y \in A$, $f(y) = y \in \operatorname{Int}(\overline{U})$ and hence $f(y) \notin \overline{V}$. This shows that $f(\operatorname{Int}(\overline{W}))$ is not contained in $\overline{V}$. This contradicts the hypothesis that f is a weak θ-continuous. Thus A is δ-closed as claimed. □

Corollary 3.4 (Theorem 3.1 [12]). *Let $A \subset X$ and let $f : X \to A$ be a closure retraction of X onto A. If X is Hausdorff, then A is an δ-closed subset of X.*

Theorem 3.5. *Let $f : X \to Y$ be an w.θ.c. and injective function. If Y is Urysohn, then X is Hausdorff.*

Proof. For any distinct points $x_1 \neq x_2 \in X$, we have $f(x_1) \neq f(x_2)$ since f is injective. Since Y is Urysohn, there exist open sets V_1, V_2 containing $f(x_1)$

and $f(x_2)$, respectively, such that $\overline{V}_1 \cap \overline{V}_2 = \emptyset$. But since f is $w.\theta.c.$ there exist open sets U_1, U_2 containing x_1, x_2, respectively, such that $f(\mathrm{Int}\,(\overline{U_1})) \subset \overline{V}_1$, and $f(\mathrm{Int}\,(\overline{U_2})) \subset \overline{V}_2$. Thus $\mathrm{Int}\,(\overline{U}_1) \cap \mathrm{Int}\,(\overline{U_2}) = \emptyset$, proving that X is Hausdorff. □

Corollary 3.6. *Let $A \subset X$ and let $f : X \to A$ be a $w.\theta.c.$ and bijective map. If A is Urysohn, then A is an $\delta-$closed subset of X.*

Theorem 3.7. *Let f, g be $w.\theta.c.$ from a space X into a Urysohn space Y. Then the set $A = \{x \in X : f(x) = g(x)\}$ is an δ-closed set.*

Proof. We will show that $X\backslash A$ is δ-open. Let $x \in A^c$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets $W_{f(x)}$ and $V_{g(x)}$ such that $\overline{W} \cap \overline{V} = \emptyset$. By $w.\theta.c.$ of f and g there exist regularly open sets U_1, U_2 of x such that $f(U_1) \subset \overline{W}$ and $g(U_2) \subset \overline{V}$. Clearly $U = U_1 \cap U_2 \subset X\backslash A$. Thus $X\backslash A$ is δ-open and hence A is δ-closed. □

The above theorem leads to a generalization of a well-known principle of the extension of identities.

Recall that a subset A of a space X is called θ-dense if $\mathrm{cls}_\theta\, A = X$.

Corollary 3.8. *Let f, g be $w.\theta.c.$ from a space X into a Urysohn space Y. If f and g agree on a θ-dense subset of X then $f = g$ every where.*

Theorem 3.9. *Let $f : X \to Y$ be $w.\theta.c.$ map and let $A \subset X$. Then $f : A \to Y$ is $w.\theta.c.$*

Proof. Straightforward. □

Remark 3.10. If a function $f : X \to Y$ is $w.\theta.c.$ map. Then $f : X \to f(X)$ need not be $w.\theta.c.$

Example 3.11. Let $X = R$ with the usual topology, $Y = R$ with the cocountable topology and let $f : X \to Y$ be defined as $f(\text{rationals}) = 1$, $f(\text{irrationals}) = 0$ then f is $\delta.c, s.a.c.$, but $f : X \to f(X)$ is not even $w.c.$

In [2] it is shown that the image of compact is closure compact under weakly continuous functions. Similarly, the image of closure compact is closure compact under closure continuous functions and the image of closure compact is compact under strongly θ-continuous functions. The following results are similar to that applied to almost closure continuity.

Definition 3.12. A subset A of a space X is called closure compact or quasi-H-closed if every open cover of A has a finite subcollection whose closures cover A.

Clearly every compact set is closure compact but not conversely as it is shown in the next example.

Example 3.13. Let X be any uncountable space with the cocountable topology then every subset of X is closure compact but the only compact subsets of X are the finite ones.

Definition 3.14. A subset A of a space X is said to be theta-compact (briefly, θ-compact) if every cover of θ-open sets has a finite subcover.

Definition 3.15. A subset A of a space X is called nearly compact (briefly, n--compact) if every open cover has a finite subcollection whose interior of the closures cover the space.

Lemma 3.16. *A subset A of a space X is n-compact iff every cover of δ-open sets has a finite subcover.*

Theorem 3.17. *Let $f : X \to Y$ be w.θ.c. and let K be an n-compact subset of X. Then $f(K)$ is a closure compact subset of Y.*

Proof. Let $\mathcal{V}$ be an open cover of $f(K)$. For each $k \in K$, $f(k) \in V_k$ for some $V_k \in \mathcal{V}$. By $w.\theta.c.$ of f, $f^{-1}(\overline{V}_k)$ is regularly open. The collection $\{f^{-1}(\overline{V}_k) : k \in K\}$ is a regular open cover of K and so since K is compact there is a finite subcollection $\{f^{-1}(\overline{V}_k) : k \in \mathcal{V}_0\}$ where $\mathcal{V}_0$ is a finite subset of K and $\{f^{-1}(\overline{V}_k) : k \in \mathcal{V}_0\}$ covers K. Clearly $\{\overline{V}_k : k \in V_0\}$ covers $f(K)$ and thus $f(K)$ is a closure compact subset of Y. $\square$

It is well-known that every closed subset of a compact space is compact. The next theorem approximates this result for n.compactness.

Theorem 3.18. *An n-compact subset of a Hausdorff space is δ-closed.*

Proof. Let A be a nearly compact subset of a Hausdorff space X. We will show that $X \backslash A$ is δ-open. Let $x \in X \backslash A$ then for each $a \in A$ there exist δ-open sets $U_{x,a}$ and V_a such that $U_{x,a} \cap V_a = \emptyset$. The collection $\{V_a : a \in A\}$ is a δ-open cover of A. Therefore, there exists a finite subcollection $V_1, \ldots, V_n$ that covers A. Let $U = U_1 \cap \cdots \cap U_n$, then $U \cap A = \emptyset$. Thus $X \backslash A$ is δ-open, proving that A is δ-closed. $\square$

Theorem 3.19. *Every δ-closed subset of an n-compact space is n-compact.*

Proof. Let X be an n-compact and let A be an δ-closed subset of X. Let $\mathcal{C}$ be an δ-open cover of A, then $\mathcal{C}$ plus $X \backslash A$ is an δ-open cover of X. Since X is n-compact, this collection has a finite subcollection that covers X. But then $\mathcal{C}$ has a finite subcollection that covers A as we need. $\square$

Theorem 3.20. *An n-compact subset of an S-Hausdorff space is θ-closed.*

Proof. Let A be a nearly compact subset of an S-Hausdorff space X. We will show that $X \backslash A$ is θ-open. Let $x \in X \backslash A$ then for each $a \in A$ there exist θ-open sets $U_{x,a}$ and V_a such that $U_{x,a} \cap V_a = \emptyset$. The collection $\{V_a : a \in A\}$ is a θ-open cover of A. Therefore, there exists a finite subcollection $V_1, \ldots, V_n$ that covers A. Let $U = U_1 \cap \cdots \cap U_n$, then $U \cap A = \emptyset$. Thus $X \backslash A$ is θ-open, proving that A is θ-closed. $\square$

Theorem 3.21. *Let $f : X \to Y$ be a surjective w.θ.c. and let X be connected. Then Y is connected.*

Proof. Suppose Y is disconnected. Then there exist disjoint open sets V, W such that $Y = V \cup W$. By weak θ-continuity of f, $f^{-1}(\overline{V}) = f^{-1}(V)$ and $f^{-1}(\overline{W}) = f^{-1}(W)$ are open in X. But $X = f^{-1}(W) \cup f^{-1}(V)$ and $f^{-1}(W) \cap f^{-1}(V) = \emptyset$. Thus X is disconnected, a contradiction. Therefore, Y is connected. $\square$

4. Some extensions of open and closed maps.

In this section we give some characterizations of regularly closed and regularly open maps.

Definition 4.1. A function f is said to be θ-open if the image of every open set is θ-open. Similarly, a function f is said to be θ-closed if the image of every closed set is θ-closed.

Definition 4.2. A function f is said to be almost open if the image of every δ--open set is open. Similarly, a function f is said to be almost closed if the image of every δ-closed set is closed.

Definition 4.3. A function f is said to be almost θ-open if the image of every δ-open set is θ-open. Similarly, a function f is said to be almost θ-closed if the image of every δ-closed set is θ-closed.

Definition 4.4. A function f is said to be δ-open if the image of every δ-open set is δ-open. Similarly, a function f is said to be δ-closed if the image of every δ-closed set is δ-closed.

Definition 4.5. A function f is called regularly open if the image of every regularly open set is open. Similarly, a function f is called regularly closed if the image of every regularly closed set is closed.

Definition 4.6. A function f is said to be θ-regularly open if the image of every regularly open set is θ-open. Similarly, a function f is said to be θ-regularly closed if the image of every regularly closed set is θ-closed.

The next theorems are improvements of [18, Theorem 2.13].

Theorem 4.7. *Let $f : X \to Y$ be weakly θ-continuous 1-1, onto. If X is n-compact, Y Urysohn then f is almost θ-open.*

Proof. Let U be an δ-open subset of X, and thus $X \backslash U$ is an δ-closed subset of X. Hence, by Theorem 4.3 $X \backslash U$ is n-compact. Since f is weakly θ-continuous, $f(X \backslash U)$ is closure compact. Therefore, $f(X \backslash U) = Y \backslash f(U)$ is θ-closed, and thus $f(U)$ is almost θ-open. □

Theorem 4.8. *Let $f : X \to Y$ be weakly θ-continuous. If X is n-compact, Y Urysohn then f is almost θ-closed.*

Proof. Let U be an δ-closed subset of X. Then U is n-compact. Since f is weakly θ-continuous, $f(U)$ is closure compact. Therefore, $f(U)$ is θ-closed. Thus, f is almost θ-closed. □

Following similar arguments as in the above two theorems we get the following.

Theorem 4.9. *Let $f : X \to Y$ be w.θ .c. 1-1, onto. If X is compact, Y Urysohn then f is θ-open.*

Theorem 4.10. *Let $f : X \to Y$ be w.θ.c. If X is compact, Y Urysohn then f is θ-closed.*

Theorem 4.11. *Let $f : X \to Y$ be w.θ.c. 1-1, onto. If X is n-compact, Y Hausdorff then f is almost θ- open.*

Theorem 4.12. *Let $f : X \to Y$ be w.θ.c. If X is n-compact, Y Hausdorff then f is almost closed.*

Theorem 4.13. *Let $f : X \to Y$ be θ-continuous 1-1, onto. If X is closure compact, Y Hausdorff then f is δ-open.*

Theorem 4.14. *Let $f : X \to Y$ be θ-continuous. If X is closure compact, Y Hausdorff then f is regularly closed.*

Theorem 4.15. *Let $f : X \to Y$ be θ-continuous 1-1, onto. If X is closure compact, Y Urysohn then f is θ-regularly open.*

Theorem 4.16. *Let $f : X \to Y$ be θ-continuous. If X is closure compact, Y Urysohn then f is θ-regularly closed.*

Theorem 4.17. *Let $f : X \to Y$ be a continuous 1-1, onto. If X is compact, Y Hausdorff then f is θ-open.*

Theorem 4.18. *Let $f : X \to Y$ be continuous. If X is compact, Y then f is θ-closed.*

Theorem 4.19. *Let $f : X \to Y$ be δ.c. 1-1, onto. If X is n-compact, Y S--Hausdorff then f is δ-open.*

Theorem 4.20. *Let $f : X \to Y$ be δ.c. If X is n-compact, Y Hausdorff then f is δ-closed.*

Theorem 4.21. *Let $f : X \to Y$ be s.c. 1-1, onto. If X is n-compact, Y Hausdorff then f is almost θ-open.*

Theorem 4.22. *Let $f : X \to Y$ be s.c. If $X$is n-compact, Y Hausdorff then f is almost θ-closed.*

Theorem 4.23. *Let $f : X \to Y$ be a.s.c. 1-1, onto. If X is n-compact, Y Hausdorff then f is θ-open.*

Theorem 4.24. *Let $f : X \to Y$ be a.s.c. 1-1, onto. If X is n-compact, Y Hausdorff then f is θ-closed.*

Theorem 4.25. *Let $f : X \to Y$ be a.s.c. If X is n-compact, Y Hausdorff then f is θ-closed.*

Remark 4.26. A function $f : (X, \Gamma) \to (Y, \Im)$ is *a.s.c.* iff $f : (X, \Gamma_\theta) \to (Y, \Im_\delta)$ is continuous and $f : (X, \Gamma) \to (Y, \Im)$ is *a.c.* iff $f : (X, \Gamma) \to (Y, \Im_\delta)$ is continuous.

Remark 4.27. A subset $K \subset (X, \Gamma)$ is an n-compact subset iff $K \subset (X, \Gamma_\delta)$ is compact and $K \subset (X, \Gamma)$ is a θ-compact subset iff $K \subset (X, \Gamma_\theta)$ is compact.

One might prove directly most of the results in Sections 3, 4 using Remarks 4.1 and 4.2. Also from these Remarks and the facts that $f : (X, \Gamma) \to (Y, \Im)$ is strongly θ-continuous iff $f : (X, \Gamma_\theta) \to (Y, \Im)$ is continuous, we get Theorems 14, 15 in [9].

Acknowledgment. Part of this paper was written during the stay of the author at Ohio University under a Fulbright scholarship. The author wishes to thank the members of the Department of Mathematics and the Center of Ring Theory for the warm hospitality and the Fulbright for the financial support.

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