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Almost Q-rings

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ALMOST Q-RINGS

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ABSTRACT. In this paper we establish some new characterizations for Q-rings and Noetherian Q-rings.

1. INTRODUCTION

Throughout this paper R is assumed to be a commutative ring with identity. $L(R)$ denotes the lattice of all ideals of R. R is said to be a Q-ring [4], if every ideal is a finite product of primary ideals. It is well known that if R is a Q-ring, then R_M is a Q-ring for every maximal ideal M of R [4, Lemma 4]. But in general the converse need not be true. For example, if R is an almost Dedekind domain which is not a Dedekind domain, then R_M is a Q-ring, for every maximal ideal M of R, but R is not a Q-ring. We call a ring R an almost Q-ring if R_M is a Q-ring, for every maximal ideal M of R. The goal of this paper is to characterize those almost Q-rings which are also Q-rings. We prove that R is an almost Q-ring if and only if every non-maximal prime ideal is locally principal (see Theorem 1). Using this result, we characterize Q-rings in terms of almost Q-rings (see Theorem 2). Finally, we establish some equivalent conditions for Noetherian Q-rings (see Theorem 3).

For any $A, B \in L(R)$, we denote $A \setminus B = \{x \in A \mid x \notin B\}$. We use $\subset$ for proper set containment. For any $x \in R$, the principal ideal generated by x is denoted by (x). For any ideal $I \in L(R)$, we denote $\theta(I) = \sum\{(I_1 : I) \mid I_1 \subseteq I$ and I_1 is a finitely generated ideal$\}$. Recall that an ideal I of R is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. If I is a multiplication ideal, then I is locally principal [1, Theorem 1 and Page 761]. An ideal M of R is called a *quasi-principal ideal* [9, Exercise 10, Page 147] (or a principal element of $L(R)$ [11]) if it satisfies the following identities (i) $(A \cap (B : M))M = AM \cap B$ and (ii) $(A + BM) : M = (A : M) + B$, for all A, $B \in L(R)$. Obviously, every quasi-principal ideal is a multiplication ideal. It should be mentioned that every quasi-principal ideal is finitely generated and also a finite product of quasi-principal ideals of R is again a quasi-principal ideal [9, Exercise 10, Page 147]. In fact, an ideal I of R is quasi-principal if and only if it is finitely generated and locally

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principal (see [6, Theorem 4]) or [11, Theorem 2]). A B_w-prime of I is a prime ideal P such that P is minimal over $(I : x)$ for some $x \in R$. R is said to be a *Laskerian ring* [8], if every ideal is a finite intersection of primary ideals. It is well known that R is a Q-ring if and only if R is a Laskerian ring in which every non-maximal prime ideal is quasi-principal [4, Theorem 13]. R is a π-ring if every principal ideal is a finite product of prime ideals. We say that R has Noetherian spectrum, if R satisfies the ascending chain condition for radical ideals [12]. It is well known that R has Noetherian spectrum if and only if every prime ideal is the radical of a finitely generated ideal [12, Corollary 2.4]. Also it is well known that if R has Noetherian spectrum, then every ideal has only finitely many minimal primes.

For general background and terminology, the reader is referred to [9].

We shall begin with the following definition.

Definition 1. A quasi-local ring R with maximal ideal M is said to satisfy the condition $(*)$ if for each non-maximal prime ideal P with $P = PM$, there exists $t \in M$ such that $P + (t)$ is finitely generated.

Note that valuation rings (i.e., any two ideals are comparable), quasi-local rings in which the maximal ideals are principal and one dimensional quasi-local domains are examples of quasi-local rings satisfying the condition $(*)$.

Lemma 1. *Let R be a quasi-local Q-ring with maximal ideal M. Then R satisfies the condition $(*)$.*

Proof. The proof of the lemma follows from [4, Lemma 5]. □

Lemma 2. *Let R be a quasi-local ring with maximal ideal M satisfying the condition $(*)$. Suppose every principal ideal is a finite product of primary ideals. If P is a non-maximal prime ideal with $P = PM$, then $P = (0)$.*

Proof. Suppose P is a non-maximal prime ideal with $P = PM$. By hypothesis, there exists $a \in M$ such that $P + (a)$ is finitely generated. If $a \in P$, then $P = P + (a)$ is finitely generated, so by Nakayama's lemma, $P = 0$. Suppose $a \notin P$. Since $P+(a)$ is finitely generated, it follows that $P+(a) = P_1+(a)$ for some finitely generated ideal $P_1 \subseteq P$. Since $P = PM$, we have $(P + (a))M = PM + (a)M = P_1M + (a)M$, so $P + (a)M = P_1M + (a)M$ and hence $P + (a) = P_1M + (a)$. Again since $P_1 \subseteq P + (a) = (a) + P_1M$ and P_1 is finitely generated, by Nakayama's lemma, it follows that $P_1 \subset (a)$. Therefore $P \subset (a)$. Let $x \in P$. By hypothesis $(x) = QA$ for some primary ideal $Q \subseteq P$ and $A \in L(R)$. Since $Q \subset (a)$, it follows that $Q = (a)Q$. Therefore $(x) = QA = Q(a)A = (x)(a)$ and hence by Nakayama's lemma, $(x) = (0)$. This shows that $P = (0)$. □

Lemma 3. *Let R be a quasi-local ring with maximal ideal M. Suppose every ideal generated by two elements is a finite product of primary ideals. If P is a non-maximal prime ideal with $P \neq PM$, then P is principal.*

Proof. Let P be a non-maximal prime ideal with $P \neq PM$. Choose any element $a \in P$ such that $a \notin PM$. Let $t \in M$ be any element such that $t \notin P$. Suppose

$x \in P$. Then by hypothesis, $(a) + (xt)$ is a finite product of primary ideals. Since $a \notin PM$, it follows that $(a) + (xt)$ is primary. Again since $(xt) \subseteq (a) + (xt)$ and $t \notin \sqrt{(a) + (xt)} \subseteq P$, it follows that $x \in (a) + (xt)$, so by Nakayama's lemma $(x) \subseteq (a)$. Therefore $P = (a)$. □

Lemma 4. *Let R be a quasi-local ring with maximal ideal M satisfying the condition $(*)$. Suppose every ideal generated by two elements is a finite product of primary ideals. Then the non-maximal prime ideals are principal. Hence* $\dim R \leq 2$.

Proof. By Lemma 2 and Lemma 3, every non-maximal prime ideal is principal. Again as shown in the last paragraph of the proof of Lemma 5 of [4], $\dim R \leq 2$. This completes the proof of the lemma. □

Lemma 5. *Suppose I is an ideal of R such that every prime minimal over I is finitely generated. Then I contains a finite product of prime ideals minimal over I. Further I has only finitely many minimal primes.*

Proof. Suppose I does not contain a finite product of prime ideals minimal over I. Let $\Im = \{J \in L(R) \mid I \subseteq J$ and J does not contain a finite product of prime ideals minimal over $I\}$. By Zorn's lemma, $\Im$ has a maximal element, say P. It can be easily shown that P is a prime ideal. Again note that P contains a prime ideal P_0 which is minimal over I, a contradiction. Therefore I contains a finite product of prime ideals minimal over I. Consequently, I has only finitely many minimal primes. □

Lemma 6. *Suppose R is a quasi-local ring. Then the following statements are equivalent:*

(i) *R is a Q-ring.*
(ii) *R satisfies the condition $(*)$ and every ideal generated by two elements is a finite product of primary ideals.*
(iii) *Every non-maximal prime ideal is principal.*

Proof. (i)⇒(ii) follows from Lemma 1.

(ii)⇒(iii) follows from Lemma 4.

(iii)⇒(i). Suppose (iii) holds. Then every ideal I is either M-primary (M is a maximal ideal of R) or by Lemma 5, I has only finitely many minimal primes. Again by the last paragraph of the proof of [4, Lemma 5], R is Laskerian. Now the result follows from [4, Theorem 10]. □

Lemma 7. *Let R be an almost Q-ring. Suppose every principal ideal is a finite product of primary ideals. Then every non-maximal prime ideal of R is a multiplication ideal.*

Proof. Using Lemma 6 and by imitating the proof of [4, Lemma 7], we can get the result. □

Lemma 8. *Let* $\dim R \leq 2$ *and let every ideal generated by two elements has only finitely many minimal primes. Then R has Noetherian spectrum.*

Proof. First we show that every minimal prime ideal is the radical of a finitely generated ideal. By hypothesis, R has only finitely many minimal primes. Let $P_1, P_2, \ldots, P_n$ be the distinct minimal prime ideals. If $n = 1$, then P_1 is the radical of the zero ideal. Suppose $n > 1$. Then $P_1 \not\subseteq \bigcup_{i=2}^{n} P_i$. Choose any $x \in P_1$ such that $x \notin \bigcup_{i=2}^{n} P_i$. Let $Q_1, Q_2, \ldots, Q_m$ be the distinct primes minimal over (x). Then $P_1 = Q_j$ for some j, say $P_1 = Q_1$. If $m = 1$, then P_1 is the radical of (x). Suppose $m > 1$. Then $P_1 \not\subseteq \bigcup_{i=2}^{m} Q_i$. Choose any $y \in P_1$ such that $y \notin \bigcup_{i=2}^{m} Q_i$. By hypothesis, $(x) + (y)$ has only finitely many minimal primes. Let $Q'_1, Q'_2, \ldots, Q'_k$ be the distinct primes minimal over $(x) + (y)$. Note that $P_1 = Q'_j$ for some j, say $P_1 = Q'_1$. If $k = 1$, then P_1 is the radical of $(x) + (y)$. Suppose $k > 1$. Observe that any Q'_j different from P_1 contains Q_i properly, for some $i \neq 1$, and each Q_i different from P_1, is non-minimal. So each Q'_j is maximal, for $j = 2, 3, \ldots, k$. Choose any element $z \in P_1$ such that $z \notin \bigcup_{i=2}^{k} Q'_i$. Now it can be easily shown that P_1 is the radical of $(x) + (y) + (z)$. Thus we have shown that every minimal prime ideal is the radical of a finitely generated ideal.

Next we show that every non-minimal prime ideal is the radical of a finitely generated ideal. Let P be a non-minimal prime ideal. Then $P \not\subseteq \bigcup_{i=1}^{n} P_i$. Choose any $x \in P$ such that $x \notin \bigcup_{i=1}^{n} P_i$. Let $Q_1, Q_2, \ldots, Q_m$ be the distinct primes minimal over (x). Then $P \supseteq Q_j$ for some j, say $P \supseteq Q_1$. If $m = 1$ and $P = Q_1$, then P is the radical of (x) and so we are through. Suppose $m \geq 1$ and $Q_1 \subset P$. Then $P \not\subseteq \bigcup_{i=1}^{m} Q_i$. Choose any $y \in P$ such that $y \notin \bigcup_{i=1}^{m} Q_i$. Then $(x) + (y)$ has only finitely many minimal primes and every prime minimal over $(x)+(y)$ is a maximal ideal. Therefore there exists a finitely generated ideal I such that P is the radical of I. Finally assume that $m > 1$ and $P = Q_1$. Then $P \not\subseteq \bigcup_{i=2}^{m} Q_i$. Choose any $y \in P$ such that $y \notin \bigcup_{i=2}^{m} Q_i$. Let $Q'_1, Q'_2, \ldots, Q'_k$ be the distinct primes minimal over $(x) + (y)$. Note that $P_1 \supseteq Q'_j$ for some j, say $P_1 \supseteq Q'_1$. Since $x \in Q'_1$ and $Q_1 = P \supseteq Q'_1$, it follows that $P = Q_1 = Q'_1$. If $k = 1$, then P_1 is the radical of $(x) + (y)$. Suppose $k > 1$. Then $P \not\subseteq \bigcup_{i=2}^{k} Q'_i$ and each Q'_i different from P, is maximal. Choose any element $z \in P$ such that $z \notin \bigcup_{i=2}^{k} Q'_i$. Then P is the radical of $(x) + (y) + (z)$. Thus every prime ideal is the radical of a finitely generated ideal and hence R has Noetherian spectrum. □

For any $I \in L(R)$ and for any prime ideal P minimal over I, we denote $P_I = \cap\{Q \in L(R) \mid Q \text{ is a } P\text{-primary ideal containing } I\}$. It can be easily seen that P_I is the smallest P-primary ideal containing I. For any $x \in R$, and for any prime ideal P minimal over (x), we denote $P_x = \cap\{Q \in L(R) \mid Q \text{ is a } P\text{-primary ideal containing } (x)\}$.

For any $x \in R$, we denote $(x)^* = \cap\{P_x \mid P \text{ is a prime ideal minimal over } (x)\}$.

Lemma 9. *Let P be a prime minimal over an ideal I of R and let P_1 be a prime properly containing P. Then the following statements hold:*

(i) *If P is a multiplication ideal, then $P \subset ((I + PP_I) : P_I)$.*
(ii) *If P_1 is a B_w-prime of I, then $(P_I)_M \neq I_M$ (in R_M) for all maximal ideals M containing P_1.*

Proof. (i) Consider the ideal $((I + PP_I) : P_I)$. Note that $P \subseteq ((I + PP_I) : P_I)$. Suppose $P = ((I + PP_I) : P_I)$. We claim that $I + PP_I$ is P-primary. Let $yz \in I + PP_I$ and $z \notin P$. Then $yz \in P_I$, so $y \in P_I$. Since P is a multiplication ideal, by [3, Lemma 1] and [2, Corollary], P_I is a multiplication ideal. As P_I is a multiplication ideal, it follows that $(y) = P_I C$ for some ideal C of R. If $C \subseteq P$, then we are through. Suppose $C \nsubseteq P$. Then $(yz) = (z)P_I C \subseteq I + PP_I$, so $zC \subseteq ((I + PP_I) : P_I) = P$, a contradiction. Therefore $I + PP_I$ is P-primary and hence $P_I = I + PP_I$. Consequently, $1 \in ((I + PP_I) : P_I) = P$, a contradiction. Therefore $P \subset ((I + PP_I) : P_I)$.

(ii) Suppose P_1 is a B_w-prime of I. Then P_1 is minimal over $(I : r)$ for some $r \in R$. Since$(I : r)r \subseteq I \subseteq P_I$, $(I : r) \nsubseteq P$ and P_I is P-primary, it follows that $r \in P_I$. If $(P_I)_M = I_M$ for some maximal ideal M containing P_1, then $rs \in I$ for some $s \notin M$. So $s \in (I : r) \subseteq M$, a contradiction. Therefore the result is true. □

Lemma 10. *Let every non-maximal prime ideal of R be a multiplication ideal. Suppose P is a non-maximal minimal prime and minimal over an ideal I of R. Then the following statements hold:*

(i) *Any B_w-prime of I which contains P properly, is a rank one maximal ideal and minimal over $((I + PP_I) : P_I)$.*
(ii) *If the maximal ideals of R are finitely generated, then the ideal $((I + PP_I) : P_I)$* has only finitely many minimal primes.

Proof. (i) Let P_1 be any B_w-prime of I which contains P properly. If M is a maximal ideal containing P_1, then by Lemma 9(ii), R_M is not a domain. Note that by Lemma 6, R is an almost Q-ring. As R is an almost Q-ring, by [4, Corollary 6], it follows that rank $M = 1$, so P_1 is a rank one maximal ideal. If $((I + PP_I) : P_I) \nsubseteq P_1$, then $(P_I)_{P_1} \subseteq I_{P_1} + (PP_I)_{P_1}$. As P is a multiplication ideal, it follows that P_I is a multiplication ideal, so P_I is locally principal, and hence by Nakayama's lemma, it follows that $(P_I)_{P_1} = I_{P_1}$. But this contradicts the statement of Lemma 9(ii). Therefore $((I + PP_I) : P_I) \subseteq P_1$ and hence by Lemma 9(i), P_1 is minimal over $((I + PP_I) : P_I)$.

(ii) Note that by hypothesis, R is an almost Q-ring and so dim $R \leq 2$. By Lemma 9(i), every prime minimal over $((I + PP_I) : P_I)$ is either a non-minimal maximal ideal or a rank one non-maximal prime. As every non-maximal prime is a multiplication ideal, by [2, Theorem 3], the rank one non-maximal primes are quasi-principal. By hypothesis, the minimal primes over $((I + PP_I) : P_I)$ are finitely generated and so by Lemma 5, the ideal $((I + PP_I) : P_I)$ has only finitely many minimal primes. □

Lemma 11. *Suppose every ideal (of R) generated by two elements has only finitely many minimal primes and the non-maximal prime ideals are multiplication ideals. Then the non-maximal prime ideals are quasi-principal.*

Proof. Let P be a non-maximal prime ideal. As $\dim R \leq 2$, it follows that P is either minimal or a rank one prime. If P is non-minimal, then P is quasi-principal [2, Theorem 3]. Suppose P is minimal. By Lemma 8, $P = \sqrt{I}$ for some finitely generated ideal I of R. Note that every B_w-prime of I contains P, and by Lemma 8, the ideal $((I + PP_I) : P_I)$ has only finitely many minimal primes. Therefore by Lemma 10(i), I has only finitely many B_w-primes. Again note that by Lemma 10(i), for every finitely generated ideal I_0 with $I \subseteq I_0 \subseteq P$, I_0 has only finitely many B_w-primes. As $\dim R \leq 2$, by [7, Theorem 1.3], P is finitely generated and hence quasi-principal. □

Lemma 12. *Suppose every non-maximal prime ideal of R is a multiplication ideal, the maximal ideals of R are finitely generated and every principal ideal has only finitely many minimal primes. Then every principal ideal is a finite intersection of primary ideals.*

Proof. Note that by hypothesis, R is an almost Q-ring, so by Lemma 4, $\dim R \leq 2$. Let $x \in R$. Then by hypothesis, $(x)^*$ is a finite intersection of primary ideals. Suppose (x) is not contained in any minimal prime. We show that $(x) = (x)^*$. Let M be a maximal ideal. If $x \notin M$, then $(x)_M = (x)^*{}_M$. Suppose $x \in M$. If M is minimal over (x), then $(x)_M = (x)^*{}_M$. Suppose M is not minimal over (x). Then rank $M = 2$, so by [4, Corollary 6], R_M is a π-domain. Therefore $(x)_M = (x)^*{}_M$ (see the proof of [10, Theorem 1.2] or [5, Theorem 3]). This shows that $(x)_M = (x)^*{}_M$ for all maximal ideals containing x and hence $(x) = (x)^*$.

Now assume that $P_1, P_2, \ldots, P_m$ be the primes minimal over (x). Let $P_1, P_2, \ldots, P_t$ be the non-maximal minimal primes and let $P_{t+1}, P_{t+2}, \ldots, P_m$ be the primes which are either maximal or rank one non-maximal primes. By Lemma 10(ii), the ideals $((x) + P_i(P_i)_x : (P_i)_x)$ for $i = 1, 2, \ldots, t$ have only finitely many minimal primes, say $M_1, M_2, \ldots, M_n$. Again by the proof of Lemma 10(ii), these are either non-minimal maximal ideals or rank one non-maximal prime ideals. Without loss of generality, assume that $M_1, M_2, \ldots, M_k$ are the rank one maximal prime ideals and $M_{k+1}, M_{k+2}, \ldots, M_n$ are either rank two maximal ideals or rank one non-maximal prime ideals. Let M be any maximal ideal different from $M_1, M_2, \ldots, M_k$. We claim that $(x)_M = (x)^*{}_M$. Obviously, if $x \notin M$, then $(x)_M = (x)^*{}_M$. Suppose $x \in M$. If either M is minimal over (x) or rank $M = 2$, then $(x)_M = (x)^*{}_M$. Suppose M is not minimal over (x) and rank $M = 1$. Then M is different from $M_1, M_2, ..., M_n$, so $((x) + P_i(P_i)_x : (P_i)_x) \nsubseteq M$ for $i = 1, 2, \ldots, t$ and hence $((P_i)_x)_M = (x)_M$ for $i = 1, 2, \ldots, t$. Consequently, $(x)_M = (x)^*{}_M$. If $(x)_{M_i} = (x)^*{}_{M_i}$ for $i = 1, 2, \ldots, k$, then $(x)_M = (x)^*{}_M$ for all maximal ideals, so $(x) = (x)^*$. Suppose $(x)_{M_i} \neq (x)^*{}_{M_i}$ for $i = 1, 2, \ldots, l$ $(1 \leq l \leq k)$. As R_{M_i} is a Laskerian ring, it follows that there exist M_i-primary Q_i such that $(x)_{M_i} = ((x)^*)_{M_i} \cap (Q_i)_{M_i}$ for $i = 1, 2, \ldots, l$. Then $(x)_M = ((x)^* \cap Q_1 \cap Q_2 \cap \cdots \cap Q_l)_M$ for all maximal ideals M of R. Therefore $(x) = (x)^* \cap Q_1 \cap Q_2 \cap \cdots \cap Q_l$ and hence (x) is a finite intersection of primary ideals. This completes the proof of the lemma. □

Lemma 13. *Suppose R is a quasi-local ring in which the maximal ideal M is finitely generated. If every ideal generated by two elements is a finite product of primary ideals, then R is a Noetherian Q-ring.*

Proof. If M is minimal, then we are through. Suppose M is non-minimal. By Lemma 6, it is enough if we show that R satisfies the condition $(*)$. Let P be a non-maximal prime ideal with $P = PM$. Let $\Psi = \{P_\alpha \mid P \subseteq P_\alpha, P_\alpha \text{ is prime and } P_\alpha = P_\alpha M\}$. Clearly $\Psi \neq \emptyset$ and by Zorn's lemma, Ψ has a maximal element, say P_0. Note that $P_0 \neq M$. If $P_0 \subset P_1 \subset M$ for some prime ideal P_1, then $P_1 \neq P_1 M$, so by Lemma 3, P_1 is principal and hence P is contained in a principal ideal. Now assume that M covers P_0. Choose any $x \in M$ such that $x \notin P_0$. Then $P_0 + (x)$ is M-primary. As M is finitely generated, it follows that $M^k \subseteq P_0 + (x)$ for some positive integer k. Again since $P_0 = P_0 M$, it follows that $P_0 \subseteq M^n$ for all positive integers n. Therefore $M^k \subseteq P_0 + (x) \subseteq (x) + M^{k+1} = (x) + M^k M$ and hence by Nakayama's lemma $P_0 \subset M^k \subseteq (x)$. This shows that P is properly contained in (x) and hence R satisfies the condition $(*)$. □

Lemma 14. *Suppose every finitely generated ideal of R is a finite product of primary ideals. Suppose I is an ideal of R such that I is locally finitely generated and every prime minimal over I is a maximal ideal. Then I is finitely generated.*

Proof. We claim that $\theta(I) = R$. Suppose $\theta(I) \neq R$. Then $\theta(I) \subseteq M$ for some maximal ideal M of R. Since I is locally finitely generated, it follows that $I_M = (I_1)_M$ for some finitely generated ideal I_1 contained in I. By hypothesis, there exist primary ideals $Q_1, Q_2, \ldots, Q_n$ such that $I_1 = Q_1 Q_2 \ldots Q_n$. Without loss of generality, assume that $Q_i \subseteq M$ for $i = 1, 2, \ldots, k$ and $Q_j \nsubseteq M$ for $j = k+1, k+2, \ldots, n$. Then $I_M = (I_1)_M = (Q_1)_M (Q_2)_M \ldots (Q_k)_M$. Since $I_M \subseteq (Q_i)_M$, it follows that $I \subseteq Q_i$ for $i = 1, 2, \ldots, k$. Since M is minimal over I, it follows that each Q_i is M-primary and hence $Q_1 Q_2 \ldots Q_k$ is M-primary. Therefore $I \subseteq Q_1 Q_2 \ldots Q_k$. Choose elements $x_j \in Q_j$ such that $x_j \notin M$ for $j = k+1, k+2, \ldots, n$. Let $z = x_{k+1} x_{k+2} \ldots x_n$. Since $I \subseteq Q_1 Q_2 \ldots Q_k$ and $z \in Q_{k+1} Q_{k+2} \ldots Q_n$, it follows that $Iz \subseteq Q_1 Q_2 \ldots Q_n = I_1$, so $z \in (I_1 : I) \subseteq \theta(I) \subseteq M$, which is a contradiction. Therefore $\theta(I) = R$ and hence $R = \sum_{i=1}^{n} (I_i : I)$, where I_i's are finitely generated ideals contained in I. So $I = \sum_{i=1}^{n} I_i$. This shows that I is a finitely generated ideal. □

Theorem 1. *R is an almost Q-ring if and only if every non-maximal prime ideal is locally principal.*

Proof. The result follows from Lemma 6. □

Corollary 1. *Suppose every principal ideal is a finite product of primary ideals. Then R is an almost Q-ring if and only if every non-maximal prime ideal is a multiplication ideal.*

Proof. The proof of the corollary follows from Theorem 1 and Lemma 7. □

Corollary 2. *Suppose every principal ideal is a finite intersection of primary ideals. Then R is an almost Q-ring if and only if every non-maximal prime ideal is quasi-principal.*

Proof. The proof of the corollary follows from Theorem 1 and [4, Theorem 12]. □

Theorem 2. *The following statements on R are equivalent:*

(i) *R is a Q-ring.*
(ii) *R is an almost Q-ring in which every ideal generated by two elements is a finite intersection of primary ideals.*
(iii) *R is an almost Q-ring in which every ideal generated by two elements is a finite product of primary ideals.*
(iv) *Every ideal generated by two elements is a finite product of primary ideals and for every maximal ideal M of R, R_M satisfies the condition $(*)$.*
(v) *Every non-maximal prime ideal is a multiplication ideal and every ideal generated by two elements has only finitely many minimal primes.*

Proof. (i)⇒(ii) and (i)⇒(iii) follow from [4, Lemma 4 and Theorem 10].
(ii)⇒(v) follows from Corollary 2.
(iii)⇒(iv) follows from Lemma 1.
(iv)⇒(v) follows from Lemma 6 and Corollary 1.
(v)⇒(i). Suppose (v) holds. By Lemma 4 and Lemma 6, $\dim R \leq 2$. By Lemma 8, R has Noetherian spectrum. Also by Lemma 11, every non-maximal prime ideal is quasi-principal. Therefore by [4, Lemma 1], every primary ideal whose radical is non-maximal is a power of its radical and hence quasi-principal. Consequently, every primary ideal whose radical is non-maximal is finitely generated. Again by [8, Corollary 2.3], R is Laskerian and hence by [4, Theorem 13], R is a Q-ring. □

The following theorem gives some new equivalent conditions for Noetherian Q-rings.

Theorem 3. *The following statements on R are equivalent:*

(i) *R is a Noetherian Q-ring.*
(ii) *The maximal ideals are locally finitely generated and every ideal generated by two elements is a finite product of primary ideals.*
(iii) *R is an almost Q-ring in which the maximal ideals are finitely generated and every principal ideal is a finite product of primary ideals.*

Proof. (i)⇒(ii) is obvious.
(ii)⇒(iii). Suppose (ii) holds. By Lemma 13, R is locally Noetherian and an almost Q-ring. By Theorem 2, R is a Q-ring and so by Lemma 14, the maximal ideals are finitely generated. Therefore (iii) holds.
(iii)⇒(i). Suppose (iii) holds. By Corollary 1, Corollary 2 and Lemma 12, R is a Noetherian ring and hence by Theorem 2, R is a Noetherian Q-ring. This completes the proof of the theorem. □

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