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Xian Zhang; Zhong Peng Yang; Chong-Guang Cao

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MATRIX INEQUALITIES INVOLVING THE KHATRI-RAO PRODUCT

XIAN ZHANG, ZHONG-PENG YANG AND CHONG-GUANG CAO

Abstract. We extend three inequalities involving the Hadamard product in three ways. First, the results are extended to any partitioned blocks Hermitian matrices. Second, the Hadamard product is replaced by the Khatri-Rao product. Third, the necessary and sufficient conditions under which equalities occur are presented. Thereby, we generalize two inequalities involving the Khatri–Rao product.

1. Introduction

Let $\mathbf{C}^{m\times n}$ denote the set of $m \times n$ complex matrices. H_m (H_m^+) be the set of (positive definite) Hermitian matrices of order m. For a positive integer n, let $\langle n\rangle = \{1, \cdots, n\}$. If $\alpha \subset \langle n\rangle$, then $|\alpha|$ equals the cardinality of α. Let $A \in \mathbf{C}^{m\times n}$, for nonempty index sets $\alpha \subset \langle m\rangle$ and $\beta \subset \langle n\rangle$, we denote by $A(\alpha, \beta)$ the submatrix of A lying in the rows indicated by α and the columns indicated by β. If $\alpha \subset \langle m\rangle \cap \langle n\rangle$, then the submatrix $A(\alpha,\ \alpha)$ is abbreviated by $A(\alpha)$. Let $\alpha \subset \langle m\rangle \cap \langle n\rangle$, $\alpha_1 = \langle m\rangle - \alpha$ and $\alpha_2 = \langle n\rangle - \alpha$. Then

$$A/\alpha = A(\alpha_1, \alpha_2) - A(\alpha_1, \alpha)\,[A(\alpha)]^{-1}\,A(\alpha, \alpha_2)$$

is called the Schur complement with respect to $A(\alpha)$.

For two matrices M and N in H_m, $M \geq N$ $(N \leq M)$ means $M - N$ is positive semidefinite. Let $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{p\times q}$ be partitioned as the following:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1t} \\ \cdots & \cdots & \cdots \\ A_{s1} & \cdots & A_{st} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & \cdots & B_{1t} \\ \cdots & \cdots & \cdots \\ B_{s1} & \cdots & B_{st} \end{pmatrix},$$

where $A_{ij} \in \mathbf{C}^{m_i\times n_j}$ and $B_{ij} \in \mathbf{C}^{p_i\times q_j}$ for any $i \in \langle s\rangle$ and $j \in \langle t\rangle$, the Khatri-Rao and Tracy-Singh products of A and B are defined as

$$A \odot B = (A_{ij} \odot B)_{ij} = \left((A_{ij} \otimes B_{kl})_{kl}\right)_{ij} \in \mathbf{C}^{mp\times nq}$$

and

$$A * B = (A_{ij} \otimes B_{ij})_{ij} \in \mathbf{C}^{\left(\sum_{i=1}^{s} m_i p_i\right)\times\left(\sum_{j=1}^{t} n_j q_j\right)},$$

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respectively. Obviously, $A * B$ and $A \odot B$ become $A \circ B$ and $A \otimes B$ if $A_{ij} \in \mathbf{C}$ and $B_{ij} \in \mathbf{C}$ for any i and j (The definitions of the four matrix products also see [6]). We denote by I_n the $n \times n$ identity matrix, and by I when the order is clear. For a matrix $A \in \mathbf{C}^{m \times n}$, let A^T (A^H) denotes the (conjugate) transpose of A.

For any A and B in H_m^+, the following classical result (1) is frequently cited. For example, [3, Theorem 7.7.9 (a)], [5, (2.6)], [2, (10)] and [8, (8)].

$$(A \circ B)^{-1} \leq A^{-1} \circ B^{-1} \tag{1}$$

Wang and Zhang in [2, Theorem 1] and Zhan in [8, Theorem 2] extend (1) to

$$(C \circ D)^H (A \circ B)^{-1} (C \circ D) \leq (C^H A^{-1} C) \circ (D^H B^{-1} D)\,, \tag{2}$$

for all $C, D \in \mathbf{C}^{m \times n}$. If $A = B = I$, then (2) becomes the Amemiya inequality [1]

$$(C^H \circ D^H)(C \circ D) \leq (C^H C) \circ (D^H D)\,, \tag{3}$$

for all $C, D \in \mathbf{C}^{m \times n}$. Liu in [6, Theorem 4 and Theorem 8 (18)] generalize (1) and (3) to

$$(M * N)^{-1} \leq M^{-1} * N^{-1} \tag{4}$$

and

$$(C^H * D^H)(C * D) \leq (C^H C) * (D^H D)\,, \tag{5}$$

for any 2×2 partitioned matrices $M \in H_m^+$, $N \in H_p^+$, $C \in \mathbf{C}^{m \times n}$ and $D \in \mathbf{C}^{p \times q}$.

However, the equality conditions of (1) – (5) have not be found up to now. In addition, the following Example 1.1 shows that A and B in H_m^+ is not the necessary condition of (1) holds.

Example 1.1. Let $A = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$. Obviously, $A \notin H_2^+$ and

$$A^{-1} \circ B^{-1} - (A \circ B)^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} -12 & -6 \\ -6 & -6 \end{pmatrix} = \begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix} \geq O\,,$$

i.e., A don't satisfy the hypothesis of (1), but $(A \circ B)^{-1} \leq A^{-1} \circ B^{-1}$.

Recently, Liu [7, Lemma 2] gave up the condition of A and B in H_m^+, and generalized (2) to

$$(C^H \circ D^H)(A \circ B)^+ (C \circ D) \leq (C^H A^+ C) \circ (D^H B^+ D)\,,$$

where A and B are semidefinite Hermitian matrices of order m, C and D in $\mathbf{C}^{m \times n}$ with $AA^+ C = C$ and $BB^+ D = D$.

Under the enlightenment of [7], [6] and Example 1.1, in this paper we extend the inequalities (1), (2) and (3) in three ways. First, the results are extended to any partitioned blocks Hermitian matrices. Second, the Hadamard product is replaced by the Khatri-Rao product. Third, the necessary and sufficient conditions under which equalities occur are presented. Thereby, we generalize (4) and (5). Our methods are different to those in [7] and [2].

To arrive the purpose of the paper, we first give an inequality of Schur complement in the Section 2, and then show relations between the Khatri-Rao product and the Tracy-Singh product for any partitioned matrices in Section 3. In section 4, Several inequalities involving the Khatri-Rao product are proved.

2. An inequality of Schur complement

The following theorem is the basic tool of our discussion.

Theorem 2.1. *Let $\alpha \subset \langle m \rangle$, $\alpha' = \langle m \rangle - \alpha$, $\beta \in \langle n \rangle$ and $\beta' = \langle n \rangle - \alpha$. If $A \in H_m$ is nonsingular and $A(\alpha) \in H^+_{|\alpha|}$, then*

$$(C^H AC)(\beta') \geq [C(\alpha', \beta')]^H \left(A^{-1}(\alpha')\right)^{-1} C(\alpha', \beta'), \quad \forall C \in \mathbf{C}^{m\times n} \tag{6}$$

and the equality holds if and only if

$$A(\alpha)C(\alpha, \beta') + A(\alpha,\ \alpha')C(\alpha', \beta') = O\,. \tag{7}$$

Proof. Obviously, there exist two permutation matrices P and R such that

$$PAP^T = \begin{pmatrix} A(\alpha) & A(\alpha, \alpha') \\ [A(\alpha, \alpha')]^H & A(\alpha') \end{pmatrix},$$

$$PCR = \begin{pmatrix} C(\alpha, \beta) & C(\alpha, \beta') \\ C(\alpha', \beta) & C(\alpha',\ \beta') \end{pmatrix}$$

and

$$R^T(C^H AC)R = \begin{pmatrix} (C^H AC)(\beta) & (C^H AC)(\beta,\ \beta') \\ \left[(C^H AC)(\beta,\ \beta')\right]^H & (C^H AC)(\beta') \end{pmatrix}. \tag{8}$$

Let $Q = \begin{pmatrix} I & -[A(\alpha)]^{-1} A(\alpha, \alpha') \\ O & I \end{pmatrix}$. Then

$$Q^H PAP^T Q = \begin{pmatrix} A(\alpha) & O \\ O & A/\alpha \end{pmatrix} \tag{9}$$

and

$$Q^{-1}PCR = \begin{pmatrix} \star & X \\ C(\alpha', \beta) & C(\alpha', \beta') \end{pmatrix}, \tag{10}$$

where $X = C(\alpha, \beta') + [A(\alpha)]^{-1} A(\alpha, \alpha')C(\alpha', \beta')$ and $\star \in \mathbf{C}^{|\alpha|\times|\beta|}$. Applying (8)–(10) and $R^T(C^H AC)R = (Q^{-1}PCR)^H(Q^H PAP^T Q)(Q^{-1}PCR)$, we have

$$\begin{aligned} (C^H AC)(\beta') &= \begin{pmatrix} X^H & [C(\alpha', \beta')]^H \end{pmatrix} \begin{pmatrix} A(\alpha) & O \\ O & A/\alpha \end{pmatrix} \begin{pmatrix} X \\ C(\alpha', \beta') \end{pmatrix} \\ &= X^H A(\alpha)X + [C(\alpha', \beta')]^H (A/\alpha)C(\alpha', \beta')\,, \end{aligned}$$

and hence

$$(C^H AC)(\beta') \geq [C(\alpha', \beta')]^H (A/\alpha)\, C(\alpha', \beta')\,, \quad \forall C \in \mathbf{C}^{m\times n} \tag{11}$$

and the equality holds if and only if $X^H A(\alpha)X = O$. Since $A \in H_m$ is nonsingular, it follows from [2, (4)] that $A^{-1}(\alpha') = (A/\alpha)^{-1}$, and hence (6) holds and the equality holds if and only if $X^H A(\alpha)X = O$ by (11). Noting $A(\alpha) \in H^+_{|\alpha|}$, we obtain that the equality holds in (6) if and only if $X = O$ (i.e., $A(\alpha)C(\alpha,\beta') + A(\alpha,\alpha')C(\alpha',\beta') = O$). □

3. Relations with the Khatri-Rao product and the Tracy-Singh product

It follows from [6] that the order of $A * B$ is relative to the partitioned block method of A and B. Hence we first discuss the relations with $A * B$ and $A \odot B$ for any partitioned block matrices A and B.

Theorem 3.1. *There exist real matrices Z_1 of order $\left(\sum_{i=1}^{s} m_i p_i\right) \times mp$ and Z_2 of order $nq \times \sum_{j=1}^{t} n_j q_j$ such that*

(i) $Z_1 Z_1^T = I$ *and* $Z_2^T Z_2 = I$;

(ii) $F * G = Z_1(F \odot G)Z_2$ *for any* $F \in \mathbf{C}^{m\times n}$ *and* $G \in \mathbf{C}^{p\times q}$ *which are partitioned as in the following:*

$$F = \begin{pmatrix} F_{11} & \cdots & F_{1t} \\ \cdots & \cdots & \cdots \\ F_{s1} & \cdots & F_{st} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & \cdots & G_{1t} \\ \cdots & \cdots & \cdots \\ G_{s1} & \cdots & G_{st} \end{pmatrix}, \tag{12}$$

where $F_{ij} \in \mathbf{C}^{m_i\times n_j}$ *and* $G_{ij} \in \mathbf{C}^{p_i\times q_j}$ *for any* $i \in \langle s\rangle$ *and* $j \in \langle t\rangle$;

(iii) $M * N = Z_1(M \odot N)Z_1^T$ *for any* $M \in \mathbf{C}^{m\times m}$ *and* $N \in \mathbf{C}^{p\times p}$ *which are partitioned as in the following:*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1s} \\ \cdots & \cdots & \cdots \\ M_{s1} & \cdots & M_{ss} \end{pmatrix}, \quad N = \begin{pmatrix} N_{11} & \cdots & N_{1s} \\ \cdots & \cdots & \cdots \\ N_{s1} & \cdots & N_{ss} \end{pmatrix}, \tag{13}$$

where $M_{ii} \in \mathbf{C}^{m_i\times m_i}$ *and* $N_{ii} \in \mathbf{C}^{p_i\times p_i}$ *for any* $i \in \langle s\rangle$;

(iv) $U * V = Z_2^T(U \odot V)Z_2$ *for any* $U \in \mathbf{C}^{n\times n}$ *and* $V \in \mathbf{C}^{q\times q}$ *which are partitioned as in the following:*

$$U = \begin{pmatrix} U_{11} & \cdots & U_{1t} \\ \cdots & \cdots & \cdots \\ U_{t1} & \cdots & U_{tt} \end{pmatrix}, \quad V = \begin{pmatrix} V_{11} & \cdots & V_{1t} \\ \cdots & \cdots & \cdots \\ V_{t1} & \cdots & V_{tt} \end{pmatrix}, \tag{14}$$

where $U_{jj} \in \mathbf{C}^{n_j\times n_j}$ *and* $V_{jj} \in \mathbf{C}^{q_j\times q_j}$ *for any* $j \in \langle t\rangle$.

Proof. Let

$$Z_i^{(1)} = \begin{pmatrix} O_{i1} & \cdots & O_{i\ i-1} & I_{m_i p_i} & O_{i\ i+1} & \cdots & O_{is} \end{pmatrix}, \qquad i = 1, 2, \cdots, s$$

and

$$Z_j^{(2)} = \begin{pmatrix} O_{j1} & \cdots & O_{j\ j-1} & I_{n_j q_j} & O_{j\ j+1} & \cdots & O_{jt} \end{pmatrix}', \qquad j = 1, 2, \cdots, t,$$

where O_{ik} is the $m_ip_i \times m_ip_k$ zero matrix for any $k \neq i$ and O_{jl} is the $n_jq_j \times n_jq_l$ zero matrix for any $l \neq j$. Then $Z_i^{(1)} \left(Z_i^{(1)}\right)^T = I$, $\left(Z_j^{(2)}\right)^T Z_j^{(2)} = I$ and

$$Z_i^{(1)}(F_{ij} \odot G)Z_j^{(2)} = Z_i^{(1)} \begin{pmatrix} F_{ij} \odot G_{11} & \cdots & F_{ij} \odot G_{1t} \\ \cdots & \cdots & \cdots \\ F_{ij} \odot G_{s1} & \cdots & F_{ij} \odot G_{st} \end{pmatrix} Z_j^{(2)} = F_{ij} \otimes G_{ij}$$

for any $i \in \langle s\rangle$ and $j \in \langle t\rangle$. Similarly, $Z_i^{(1)}(M_{ij} \odot N)\left(Z_j^{(1)}\right)^T = M_{ij} \otimes N_{ij}$ for any $i, j \in \langle s\rangle$ and $\left(Z_i^{(2)}\right)^T (U_{ij} \odot V)Z_j^{(2)} = U_{ij} \otimes V_{ij}$ for any $i, j \in \langle t\rangle$.

Letting $Z_1 = \begin{pmatrix} Z_1^{(1)} & & \\ & \ddots & \\ & & Z_s^{(1)} \end{pmatrix}$ and $Z_2 = \begin{pmatrix} Z_1^{(2)} & & \\ & \ddots & \\ & & Z_t^{(2)} \end{pmatrix}$,
the theorem follows by a direct computation. □

If $s = t = 2$, $M \in H_m$ and $N \in H_p$ in Theorem 3.1, then (ii) becomes [6, (12)] and (iii) becomes [6, (13)].

Corollary 3.2. *There exist $\gamma \subset \langle mp\rangle$ and $\delta \subset \langle nq\rangle$ such that*
(i) $F * G = (F \odot G)(\gamma, \delta)$ *if F and G are as in* (12)*;*
(ii) $M * N = (M \odot N)(\gamma)$ *if M and N are as in* (13)*;*
(iii) $U * V = (U \odot V)(\delta)$ *if U and V are as in* (14)*.*

Proof. The corollary follows from the structure of Z_1 and Z_2 in Theorem 3.1. □

Obviously, Corollary 3.2 generalizes the equation $A \circ B = (A \otimes B)(\alpha)$ in [4].

4. Inequalities involving the Khatri-Rao product

Lemma 4.1. *If A and B are compatibly partitioned, then the below equalities hold.*
(i) $(A \odot B)(C \odot D) = (AC) \odot (BD)$*;*
(ii) $(A \odot B)^+ = A^+ \odot B^+$*, where A^+ is the Moore-Penrose inverse of A;*
(iii) $(A \odot B)^H = A^H \odot B^H$*.*

Proof. (i) and (ii) see [6, Theorem 1 (a)(b)]. Now we prove (iii).

$$\begin{aligned}(A \odot B)^H &= \left((A_{ij} \odot B)_{ij}\right)^H = \left(((A_{ij} \otimes B_{kl})_{kl})_{ij}\right)^H \\ &= \left(((A_{ij} \otimes B_{kl})_{kl})^H\right)_{ji} = \left(\left((A_{ij} \otimes B_{kl})^H\right)_{lk}\right)_{ji} \\ &= \left(\left(A_{ij}^H \otimes B_{kl}^H\right)_{lk}\right)_{ji} = \left(A_{ij}^H \odot B^H\right)_{ji} \\ &= A^H \odot B^H .\end{aligned}$$
□

Theorem 4.2. *Suppose two nonsingular matrices $M \in H_m$ and $N \in H_p$ are partitioned as in* (13)*. Let γ and δ be as in Corollary* 3.2*. If $\left(M^{-1} \odot N^{-1}\right)(\gamma') \in H^+_{|\gamma'|}$,* *then*

$$(F^H * G^H)(M * N)^{-1}(F * G) \leq (F^H M^{-1} F) * (G^H N^{-1} G) \tag{15}$$

for any F and G as in (12) *and the equality holds if and only if*

$$(M^{-1} \odot N^{-1})(\gamma')(F \odot G)(\gamma', \delta) + (M^{-1} \odot N^{-1})(\gamma', \gamma)(F * G) = O\,, \tag{16}$$

where $\gamma' = \langle mp\rangle - \gamma$.

Proof. It follows from (ii) and (iii) of Lemma 4.1 that $(M \odot N)^{-1} = M^{-1} \odot N^{-1} \in H_{mp}$. Replacing α' by γ, β' by δ, A by $(M \odot N)^{-1}$ and C by $F \odot G$ in Theorem 2.1, respectively, we have

$$\begin{aligned}&\left[(F \odot G)^H (M \odot N)^{-1}\ (F \odot G)\right](\delta)\\ &\qquad \geq \left[(F \odot G)(\gamma, \delta)\right]^H \left[(M \odot N)(\gamma)\right]^{-1} (F \odot G)(\gamma, \delta)\end{aligned} \tag{17}$$

and the equality holds if and only if

$$\begin{aligned}&(M^{-1} \odot N^{-1})(\gamma')(F \odot G)(\gamma', \delta)\\ &\qquad + (M^{-1} \odot N^{-1})(\gamma', \gamma)(F \odot G)(\gamma, \delta) = O\,.\end{aligned} \tag{18}$$

Again applying Corollary 3.2 and Lemma 4.1, we obtain

$$(M \odot N)(\gamma) = M * N\,, \qquad (F \odot G)(\gamma,\ \delta) = F * G \tag{19}$$

and

$$\begin{aligned}&\left[(F \odot G)^H (M \odot N)^{-1} (F \odot G)\right](\delta)\\ &\quad = \left[(F^H \odot G^H)(M^{-1} \odot N^{-1})(F \odot G)\right](\delta)\\ &\quad = \left[(F^H M^{-1} F) \odot (G^H N^{-1} G)\right](\delta)\\ &\quad = (F^H M^{-1} F) * (G^H N^{-1} G)\,.\end{aligned} \tag{20}$$

Combining (17) - (20), the theorem follows. □

Corollary 4.3. *Assume the hypothesis of Theorem* 4.2. *Then*

(i) $(M * N)^{-1} \leq M^{-1} * N^{-1}$ *and the equality holds if and only if* $P^T (M \odot N) P = (M \odot N)(\gamma') \oplus (M * N)$ *for some permutation matrix P.*

(ii) $(F^H * G^H)(F * G) \leq (F^H F) * (G^H G)$ *for any $F \in \mathbf{C}^{m\times n}$ and $G \in \mathbf{C}^{p\times q}$ and the equality holds if and only if $(F \odot G)(\gamma', \delta) = O$.*

Proof. (i) follows by choosing $F = I_m$ and $G = I_p$ and (ii) follows by choosing $M = I_m$ and $N = I_p$ in Theorem 4.2. □

Remark 4.4. If $s = t = 2$, then (ii) of Corollary 4.3 becomes (5); If $A \in H_m^+$ and $B \in H_p^+$, then A and B automatically satisfy the hypothesis of Corollary 4.3. So, (i) of Corollary 4.3 generalizes (4), and the following example show that this is not a simple generalization.

Example 4.5. Let

$$A = \left(\begin{array}{cc|c} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{5}{3} & -\frac{1}{3} \\ \hline -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|cc} \frac{2}{5} & 0 & \frac{1}{5} \\ \hline 0 & -\frac{1}{2} & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{array}\right).$$

Then A and B are not satisfy the hypothesis of (4). But

$$(A^{-1} \odot B^{-1})(\gamma') = \begin{pmatrix} 4 & 0 & -2 & 0 & 0 \\ 0 & 4 & 0 & -2 & -1 \\ -2 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 4 \end{pmatrix} > O,$$

where $\gamma = \{1, 2, 8, 9\}$. Hence $A^{-1} * B^{-1} \geq (A * B)^{-1}$ from (i) of Corollary 4.3.

Corollary 4.6. *Suppose A and B in H_m are nonsingular, $\gamma = \{1, m+2, 2m+3, \cdots, m^2\}$, $\delta = \{1, n+2, 2n+3, \cdots, n^2\}$ and $\gamma' = \langle m^2 \rangle - \gamma$. If $(A^{-1} \otimes B^{-1})(\gamma') \in H^+_{|\gamma'|}$, then*

(i) *$(C \circ D)^H (A \circ B)^{-1} (C \circ D) \leq (C^H A^{-1} C) \circ (D^H B^{-1} D)$ for any C and D in $\mathbf{C}^{p \times q}$ and the equality holds if and only if*

$$(A^{-1} \otimes B^{-1})(\gamma')(C \otimes D)(\gamma', \delta) + (A^{-1} \otimes B^{-1})(\gamma', \gamma)(C \circ D) = O;$$

(ii) *$(A \circ B)^{-1} \leq A^{-1} \circ B^{-1}$ and the equality holds if and only if $P^T (A \otimes B) P = (A \otimes B)(\gamma') \oplus (A \circ B)$ for some permutation matrix P.*

(iii) *$(C^H \circ D^H)(C \circ D) \leq (C^H C) \circ (D^H D)$ for any C and D in $\mathbf{C}^{m \times n}$ and the equality holds if and only if $(C \otimes D)(\gamma', \delta) = O$.*

Proof. The corollary follows by replacing the Khatri-Rao and Tracy-Singh products with the Hadamard and Kronecker products in Theorem 4.2 and Corollary 4.3, respectively. □

Obviously, the corollary generalizes (1), (2) and (3).

Remark 4.7. In Example 1.1, it follows from $(A^{-1} \otimes B^{-1})(2, 3) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in H^+_2$ that A and B satisfy the hypothesis of Corollary 4.6, and hence (1) holds. The following Examples 4.8 and 4.9 show that the needed conditions are reasonable in Theorem 4.2 and Corollary 4.6.

Example 4.8. Let $A = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$ and $B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$. Then $(A^{-1} \otimes B^{-1})(2, 3) = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} \notin H^+_2$ and $(A \circ B)^{-1} - A^{-1} \circ B^{-1} = \begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix} > O$, i.e., $(A \circ B)^{-1} \geq A^{-1} \circ B^{-1}$.

Example 4.9. Let $A = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$. Then $(A^{-1} \otimes B^{-1})(2,3) = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \notin H_2^+$ and $D = (A \circ B)^{-1} - A^{-1} \circ B^{-1} = \begin{pmatrix} 17 & 13 \\ 13 & 8.5 \end{pmatrix}$. Obviously, $\det D = -24.5$. Hence $(A \circ B)^{-1}$ is not to be compared with $A^{-1} \circ B^{-1}$.

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Xian Zhang
Department of Mathematics, Heilongjiang University
Harbin, 150080, P. R. C.
and
School of Mechanical and Manufacturing Engineering
The Queen's University of Belfast
Ashby Building, Belfast, BT9 5AH
Northern Ireland, United Kingdom
E-mail: X.Zhang@qub.ac.uk

Zhong-Peng Yang
Department of Mathematics, Putian College
Putian, Fujian, 351100, P. R. C.

Chong-Guang Cao
Department of Mathematics, Heilongjiang University
Harbin, 150080, P. R. C.