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COUPLED FIXED POINTS OF MIXED MONOTONE OPERATORS ON PROBABILISTIC BANACH SPACES

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Abstract. The existence of minimal and maximal fixed points for monotone operators defined on probabilistic Banach spaces is proved. We obtained sufficient conditions for the existence of coupled fixed point for mixed monotone condensing multivalued operators.

1. Introduction and Preliminaries

In his seminal paper Menger [11] introduced the notion of probabilistic metric space, which is a generalization of the metric space. The study of these spaces was extensively performed by Schweizer and Sklar [12, 13] and many other authors [4, 7, 14, 18]. The theory of probabilistic metric/normed spaces is of fundamental importance in probabilistic functional analysis. Recently a number of fixed point theorems and their applications in probabilistic metric spaces have been proved by several authors; Beg, Rehman and Shahzad [1], Bharucha-Reid [2], Cain and Kasriel [3], Hadzic [8], Stojakovic [15, 16] and others [5, 19]. In this paper, we introduce mixed monotone operators in probabilistic Banach spaces by definig a suitable ordering in these spaces and proved the existence of coupled minimal and maximal fixed points for these operators. The results obtained are probabilistic analogue of the results of [6, 9] and [17].

Let $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. A mapping $f : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing, left continuous with $\inf_{x\in\mathbb{R}} f(x) = 0$ and $\sup_{x\in\mathbb{R}} f(x) = 1$. We will denote by $\mathcal{L}$ the set of all distribution functions.

Definition 1.1 [4]**.** Let E be a vector space over the field $\mathbb{R}$. Let $\| ; \|$ be a mapping on E with values in $\mathcal{L}$. For each $p \in E$ the distribution function $\| ; \|(p)$ will be denoted by $\|p; \ \|$ and the value of $\| ; \|(p)$ at $t \in \mathbb{R}$ will be denoted by $\|p; t\|$. The function $\| ; \|$ is assumed to satisfy the following conditions:

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(PN1) $\|p;\ \| = H$ if and only if $p = 0 \in E$, where H is a distribution function defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0\,. \end{cases}$$

(PN2) For every $p \in E,\ t \in \mathbb{R}^+$ and $\lambda \in \mathbb{R} - \{0\}$

$$\|\lambda p; t\| \ = \ \|p; \frac{t}{|\lambda|}\|\,.$$

(PN3) If $\|p; t_1\| \ = \ 1,\ \|q; t_2\| \ = \ 1$, then

$$\|p + q; t_1 + t_2\| \ = \ 1$$

for all $p, q \in E$ and $t_1, t_2 \ > \ 0$.

The mapping $p \rightarrow \|p;\ \|$ is called the probabilistic norm on E and E together with this norm is called probabilistic normed space (or simply a PN-space). The PN-space $(E,\ \|\,;\|)$ becomes a PM-space under the probabilistic metric $\mathcal{F} : E \times E \rightarrow \mathcal{L}$ defined by $\mathcal{F}\ (p, q) = \|p - q;\ \|$.For detailed discussion on probabilistic normed spaces, we refer to [4, 5, 13, 18].

Let $(E, \|\,;\|)$ be a PN-space. Then a neighbourhood of an element $p \in E$, is defined by the set: $U_p(\epsilon, \lambda) = \ \{q \in E :\ \|p-q; \epsilon\| \ > \ 1-\lambda\}$,where $\epsilon \ > \ 0,\ \lambda \in (0, 1)$.

Definition 1.2. Let $(E, \|\,;\|)$ be a PN-space and A be a nonempty subset of E. The probabilistic diameter of A is a function D_A, defined on $\mathbb{R}^+$ by $D_A(t) = \sup_{s<t} \inf_{p,q \in A} \|p - q; s\|$. A subset A of a PN-space is said to be bounded if $\sup_{t>0} D_A(t) = 1$.

Definition 1.3 [13]**.** Let $\mu : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a triangle function. A probabilistic normed space under a triangle function μ is a triple $(E, \|\,;\|, \mu)$ satisfying PN1, PN2 and μ satisfies the following condition for any $t \in \mathbb{R}$:
(PN4) $\|p + q; t\| \geq \mu(\|p; t\|,\ \|q; t\|) = \ \mu(\|p; .\|,\ \|q; .\|)(t)$.

Definition 1.4. Let E be a probabilistic Banach space. A subset $P \subset E$ is a cone if and only if it satisfies the following conditions:

(i) P is closed and convex;

(ii) if $p \in P$, $tp \in P$ for every $t \geq 0$;

(iii) if both p and $-p$ are in P, then $p = 0$.

The order "$\leq$" is introduced by the cone P in E. That is, $p, q \in E, p \leq q$ if and only if $p - q \in P$. Thus E becomes a partially ordered probabilistic Banach space. Let E be a partially ordered probabilistic Banach space, $x, y \in E$ and $x \leq y$.A cone P is said to be normal if and only if there is some $m > 0$ such that $\|x; t\| \ \geq \ \|y; \frac{t}{m}\|,\ \ t \in \mathbb{R}$.

Let $u, v \in E$. The set of element $x \in E$ such that $u \leq x \leq v$ is called an ordered interval and is denoted by $[u, v]$. An ordered interval is closed and convex. It is bounded if the cone P is normal.

Definition 1.5. Let D be a subset of E. An operator $A : D \times D \to E$ is said to be mixed monotone if for each fixed $y \in D$, $A(x, y)$ is nondecreasing in x and for each fixed $x \in D$, $A(x, y)$ is nonincreasing in y.

2. The Results

Throughout this section the probabilistic Banach space $(E, \|\,;\|)$ will be denoted by E, which is partially ordered by a cone P in E.

Definition 2.1. Let D be a subset of E and $A : D \times D \to E$ be an operator:

(a) If $x, y \in D$ with $x \leq y$ can be found such that

$$x \;\leq\; A(x,y) \;\text{ and }\; A(y,x) \;\leq y\,,$$

then (x, y) is called a coupled lower and upper fixed point of A.

(b) If $x, y \in D$ with $x \leq y$ can be found such that

$$x \;=\; A(x,y) \;\text{ and }\; A(y,x) \;=\; y\,,$$

then (x, y) is called coupled fixed point of A. If a coupled fixed point $(x^\star, y^\star)$ can be found such that

$$x^\star \;\leq\; x \;\text{ and }\; y \;\leq\; y^\star$$

for every coupled fixed point (x, y) of A, then $(x^\star, y^\star)$ is called the minimal and maximal fixed point of A.

(c) A point $x^\star \in D$ is a fixed point of A if

$$A(x^\star, x^\star) \;=\; x^\star\,.$$

A sequence $\{x_n\}$ in E is said to be monotone increasing (decreasing) if and only if $x_{n-1} \;\leq\; x_n$ $(x_{n-1} \;\geq\; x_n)$ for all n.The operator A is called compact if for every bounded subset $D_1 \subset D$, $A(D_1 \times D_1)$ is relatively compact. We now state the following standard lemma whose proof is routine.

Lemma 2.2. *Let P be a cone in the probabilistic Banach space E. Let D be a subset of E and $A : D \times D \to E$ be mixed monotone operator. Assume that there is an ordered interval $[u, v] \subset D$ such that (u, v) is coupled lower and upper fixed point of A. Then:*

(i) *A maps $[u, v] \times [u, v]$ into $[u, v]$;*

(ii) *the sequences $\{x_n\}$, $\{y_n\}$ defined by setting $x_0 = u$, $y_0 = v$ and $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$ for $n \geq 1$ are monotone increasing and decreasing respectively.*

Theorem 2.3. *Let P be a cone in the probabilistic Banach space E. Let D be a subset of E and $A : D \times D \to E$ be continuous and mixed monotone operator. Assume that there is an ordered interval $[u, v] \subset D$ such that (u, v) is a coupled lower and upper fixed point of A. Then A has a coupled fixed point in $[u, v] \times [u, v]$ if P is normal and A is compact.*

Proof. Define sequences $\{x_n\}$ and $\{y_n\}$ by setting $x_0 = u$, $y_0 = v$ and $x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1})$ for $n \geq 1$. By Lemma 2.2, $\{x_n\}, \{y_n\}$ are monotone increasing and decreasing respectively. Since $[u, v]$ is bounded as P is normal, so the set $A([u, v], [u, v])$ is relatively compact. Thus it suffices to prove that a monotone sequence in a relatively compact set is convergent. By compactness $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit x (say). Since $\{x_{n_k}\}$ is monotone increasing $x_{n_r} - x_{n_k} \in P$ for $k \leq r$. By letting r tends to infinity, we have $x - x_{n_k} \in P$. That is, $x_{n_k} \leq x$ for all k. Now for $n \geq n_k$, $x_n \geq x_{n_k}$ implies $x - x_n \leq x - x_{n_k}$ and it follows as P is normal that there is an $m > 0$ such that

$$\|x - x_n; t\| \;\geq\; \|x - x_{n_k}; \frac{t}{m}\| \,.$$

Letting $k \to \infty$, we get $\|x - x_n; t\| \geq H(t)$ for $t \in \mathbb{R}$. Thus $\|x - x_n; t\| = H(t)$. It implies by [10, Theorem 11.1.7] that $x_n \to x$. Similarly the convergence of $\{y_n\}$ can be shown. Since $x = u$, $y = v$ and (u, v) is coupled lower and upper fixed point of A, therefore $x \leq y$. Now by the continuity of A, we have

$$x_n \;=\; A(x_{n-1}, y_{n-1})\,,$$

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} A(x_{n-1}, y_{n-1}) = A(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}) = A(x, y)$$

and

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} A(y_{n-1}, x_{n-1}) = A(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) = A(y, x)\,.$$

Thus (x, y) is a coupled fixed point of A. □

Theorem 2.4. *Let $u, v \in E$ with $u \leq v$ and $D = [u, v]$. Suppose that $A : D \times D \to E$ is mixed monotone operator and the following conditions holds:*

(i) *(u, v) is coupled lower and upper fixed point of A.*

(ii) *$A(D, D)$ is separable and sequentially compact in E.*

Then A has the coupled minimal and maximal fixed point in D.

Proof. Similar to the proof of [17, Theorem 1].

Theorem 2.5. *Let E be a probabilistic Banach space under the triangle function μ with $\mu = \inf$ and $0 < k < 1$. Let $u, v \in E$ with $u \leq v$ and $D = [u, v]$. Suppose that $A : D \times D \to E$ is an operator satisfying all the assumptions in Theorem 2.4, and also satisfies the following conditions:*

(iii) *For any fixed $x \in D$ and $t \in \mathbb{R}$*

$$\|A(u, x) - A(v, x); t\| \;>\; \|u - v; \frac{t}{k}\| \;\; \textit{for all} \;\; u, v \in D\,;$$

(iv) *for any fixed $y \in D$ and $t \in \mathbb{R}$*

$$\|A(y, u) - A(y, v); t\| \;>\; \|u - v; \frac{t}{k}\| \;\; \textit{for all} \;\; u, v \in D\,.$$

Then A has a unique fixed point $u^\star$ in D, and $x^\star = u^\star = y^\star$, where $(x^\star, y^\star)$ is a coupled minimal and maximal fixed point of A in D.

Proof. From Theorem 2.4, it follows that A has a coupled minimal and maximal fixed point $(x^\star, y^\star)$ in D. So it suffices to show that $x^\star = y^\star$. For this, let us assume the contrary, that is, $x^\star \neq y^\star$. By using condition (iii) and (iv) we get

$$\begin{aligned}
\|x^\star - y^\star; t\| &= \|A(x^\star, y^\star) - A(y^\star, x^\star); t\| \\
&= \|A(x^\star, y^\star) - A(x^\star, x^\star) + A(x^\star, x^\star) - A(y^\star, x^\star); t\| \\
&\geq \mu(\|A(x^\star, y^\star) - A(x^\star, x^\star); t\|, \|A(x^\star, x^\star) - A(y^\star, x^\star); t\|) \\
&> \mu(\|y^\star - x^\star; \frac{t}{k}\|, \|x^\star - y^\star; \frac{t}{k}\|) \\
&= \|x^\star - y^\star; \frac{t}{k}\| \\
&\geq \|x^\star - y^\star; t\|
\end{aligned}$$

which is a contradiction. So $x^\star = y^\star$ and $u^\star = x^\star = y^\star$ is a fixed point of A. The uniqueness of $u^\star$ in D follows from the fact that the set of coupled fixed points of A is a subset of $[x^\star, y^\star]$. □

Definition 2.6. Let D be a subset of a probabilistic Banach space E, which is partially ordered by a cone P of E, and $A : D \times D \to 2^E$ (the set of all subsets of E) a multivalued mapping. The operator A is said to be mixed monotone, if $A(x, y)$ is nondecreasing in x and nonincreasing in y, that is,

(i) for each $y \in D$ and any $x_1, x_2 \in D$, $x_1 \leq x_2$ $(x_1 \geq x_2)$, if $u_1 \in A(x_1, y)$ then there exists a $u_2 \in A(x_2, y)$ such that $u_1 \leq u_2$ $(u_1 \geq u_2)$;

(ii) for each $x \in D$ and any $y_1, y_2 \in D$, $y_1 \leq y_2$ $(y_1 \geq y_2)$, if $v_1 \in A(x, y_1)$ then there exists a $v_2 \in A(x, y_2)$ such that $v_1 \geq v_2$ $(v_1 \leq v_2)$.

A point $(x^\star, y^\star) \in D \times D$ is called coupled fixed point of A, if

$$x^\star \in A(x^\star, y^\star), \quad y^\star \in A(y^\star, x^\star).$$

Definition 2.7. An operator $A : D \times D \to 2^E$ is said to be probabilistically condensing if for any $D_1, D_2 \subset D$ either α_{D_1} or α_{D_2} is smaller than H, then

$$\alpha_{A(D_1, D_2)}(t) > \min\{\alpha_{D_1}(t), \alpha_{D_2}(t)\},$$

where α_{D_i} is probabilistic Kuratowski measure of noncompactness of $D_i (i = 1, 2)$. For more details regarding probabilistic Kuratowski measure of noncompactness we refer to Istratescu [10].

Remark 2.8. For continuity of multivalued operators in case of PN-spaces, we can easily modify the Definition 10.2.3 of [10].

Theorem 2.9. *Let E be a probabilistic Banach space. Let $u_0, v_0 \in E$ and $D = [u_0, v_0]$. Let $A : D \times D \to 2^D$ be a continuous mixed monotone multivalued operator with nonempty closed values. Suppose that A is probabilistically condensing operator and $[u_0, v_0]$ is bounded. Then A has a coupled fixed point $(x^\star, y^\star)$ in $D \times D$ and $x^\star = \lim_{n\to\infty} u_n$, $y^\star = \lim_{n\to\infty} v_n$, where $u_n \in A(u_{n-1}, v_{n-1})$ and $v_n \in A(v_{n-1}, u_{n-1})$, satisfy the following condition:*

$$u_0 \leq u_1 \leq \ \ldots \ \leq u_n \leq \ \ldots; \quad v_0 \geq v_1 \geq \ \ldots \ \geq v_n \geq \ \ldots,$$

and if $u_{n+1} = u_n,\ v_{n+1} = v_n$*, then* $u_{n+k} = u_n$ *and* $v_{n+k} = v_n$ *where* $k = 1, 2, \ldots$

Proof. If $u_0 \in A(u_0, v_0)$ and $v_0 \in A(v_0, u_0)$, take $x^\star = u_n = u_0$ and $y^\star = v_n = v_0$ $(n = 1, 2, \ldots)$, then the conclusion of the theorem is proved. Otherwise, taking $u_1 \in A(u_0, v_0) \subset [u_0, v_0]$ then $v_0 \geq u_1$ and $v_1 \leq v_0$. If $u_1 \in A(u_1, v_1),\ v_1 \in A(v_1, u_1)$, take $x^\star = u_n = u_1,\ y^\star = v_n = v_1$ $(n = 2, 3, \ldots)$, then the conclusion of theorem is proved; otherwise, by the mixed monotonicity of A, since $u_1 \in A(u_1, v_1),\ v_1 \in A(v_1, u_1)$ and $u_0 \leq u_1,\ v_1 \leq v_0$, there exist

$$u_2 \in A(u_1, v_1) \text{ such that } u_1 \leq u_2$$

and

$$v_2 \in A(v_1, u_1) \text{ such that } v_2 \leq v_1\,.$$

Choose $u_k \in A(u_{k-1}, v_{k-1})$, $v_k \in A(v_{k-1}, u_{k-1})$, such that $u_{k-1} \leq u_k$, $v_{k-1} \geq v_k$. If $u_k \in A(u_k, v_k)$, $v_k \in A(v_k, u_k)$, then take $x^\star = u_n = u_k$, $y^\star = v_n = v_k$ $(n = k+1, k+2, \ldots)$. Otherwise, by the mixed monotonicity of A, there exist $u_{k+1} \in A(u_k, v_k)$, $v_{k+1} \in A(v_k, u_k)$ such that

$$u_k \leq u_{k+1}\,, \quad v_k \geq v_{k+1}\,.$$

Repeating this process, either the conclusion of theorem is proved, or we can obtain a nondecreasing sequence and a nonincreasing sequence as follows:

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ \ldots; \quad v_0 \geq v_1 \geq \ldots \geq v_n \geq \ldots\,.$$

Letting $D_1 = \{u_n\}$, $D_2 = \{v_n\}$, $(n = 0, 1, 2, \ldots)$, then $D_1 \subset A(D_1, D_2) \cup \{u_0\}$, $D_2 \subset A(D_2, D_1) \cup \{v_0\}$. By the properties of Kuratowski probabilistic measure α_{D_1} of noncompactness (for detailed properties of Kuratowski probabilistic measure α_{D_1} of noncompactness see [10, Theorem 11.3.2])

$$\begin{aligned} \alpha_{D_1}(t) &\geq \min\{\alpha_{A(D_1,D_2)}(t), \alpha_{\{u_0\}}(t)\} \\ &= \min\{\alpha_{A(D_1,D_2)}(t), H(t)\} \\ &= \alpha_{A(D_1,D_2)}(t)\,. \end{aligned}$$

Similarly $\alpha_{D_2}(t) \geq \alpha_{A(D_1,D_2)}(t)$. If either α_{D_1} or α_{D_2} is smaller than H, then by the condensing condition of A,

$$\alpha_{A(D_1,D_2)}(t) > \min\{\alpha_{D_1}(t), \alpha_{D_2}(t)\}$$

and

$$\alpha_{A(D_2,D_1)}(t) > \min\{\alpha_{D_2}(t), \alpha_{D_1}(t)\}\,.$$

Therefore

$$\alpha_{D_1}(t) \ > \min\{\alpha_{D_1}(t), \alpha_{D_2}(t)\}, \alpha_{D_2}(t)\} > \min\{\alpha_{D_2}(t), \alpha_{D_1}(t)\}\,.$$

Hence $\min\{\alpha_{D_1}(t), \alpha_{D_2}(t)\} > \min\{\alpha_{D_1}(t), \alpha_{D_2}(t)\}$, a contradiction. This shows that we must have $\alpha_{D_1}(t) = \alpha_{D_2}(t) = H(t)$ for each $t \in \mathbb{R}$. Istratescu [10, Theorem 11.3.5] further implies that set associated to both the sequences $\{u_n\}$ and $\{v_n\}$ are precompact. Therefore there exist convergent subsequences $\{u_{n_k}\}$, $\{v_{n_k}\}$ of $\{u_n\}$ and $\{v_n\}$ respectively such that

$$\lim_{k\to\infty} u_{n_k} = x^\star \in [u_0, v_0]\,, \quad \lim_{k\to\infty} v_{n_k} = y^\star \in [u_0, v_0]\,.$$

Let $\lim_{n\to\infty} u_n \neq x^\star$. Then there exists $\epsilon_0 > 0$ and $\lambda_0 \in (0,1)$ and a subsequence $\{u'_{n_i}\}$ of $\{u_n\}$ such that

$$\|u'_{n_i} - x^\star; \epsilon_0\| \leq 1 - \lambda_0 .$$

By the precompactness of $\{u_n\}$, there is a subsequence $\{u'_{n_{i_j}}\}$ of $\{u'_{n_i}\}$ such that $\lim_{j\to\infty} u'_{n_{i_j}} = x'$. Hence for any given k it follows from the nondecreasingness of $\{u_n\}$ that when j is large enough

$$u_{n_k} \leq u'_{n_{i_j}} .$$

First letting $j \to \infty$ and then letting $k \to \infty$ we have $x^\star \leq x'$. Similarly we can also prove that $x' \leq x^\star$. Hence $x' = x^\star$. It follows that when j is large enough

$$\|u'_{n_{i_j}} - x^\star; \epsilon_0\| \ > \ 1 - \lambda_0 .$$

This contradicts $\|u'_{n_{i_j}} - x^\star; \epsilon_0\| \leq 1 - \lambda_0$ (for all i). Hence $\lim_{n\to\infty} u_n = x^\star$. Similarly it can be proved that $\lim_{n\to\infty} v_n = y^\star$. Since $u_n \in A(u_{n-1}, v_{n-1})$ and $v_n \in A(v_{n-1}, u_{n-1})$, it follows by the continuity of A that $x^\star \in A(x^\star, y^\star)$ and $y^\star \in A(y^\star, x^\star)$. Thus $(x^\star, y^\star)$ is a coupled fixed point of A. □

Corollary 2.10. *Let all the conditions of Theorem 2.8 and the following condition be satisfied: for any $x_1 \leq x_2,\ y_1 \geq y_2,\ (x_1, y_1) \neq (x_2, y_2)$ and any $u \in A(x_1, y_1)$ and any $v \in A(x_2, y_2),\ u \leq v$. Then*

$$u_n \leq x^\star \leq y^\star \leq v_n ,$$

and the coupled fixed point $(x^\star, y^\star)$ is minimal and maximal in the sense that for any coupled fixed point $(\bar{x}, \bar{y}) \in D \times D$ of A,

$$x^\star \leq \bar{x} \leq y^\star, \quad x^\star \leq \bar{y} \leq y^\star .$$

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References

[1] Beg, I., Rehman, S. and Shahzad, N., *Fixed points of generalized contraction mappings on probabilistic metric spaces*, Pakistan J. Statist. **8**(2) (1992), 35–52.

[2] Bharucha-Reid, A. T., *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. **82** (1976), 641–657.

[3] Cain, G. L., Jr. and Kasriel, R. H., *Fixed and periodic points of local contraction mappings on probabilistic metric spaces*, Math. Systems Theory **9** (1970), 289–297.

[4] Chang, S. S., *Basic theory and applications of probabilistic metric spaces (I)*, App. Math. Mech. **9**(2) (1988), 123–133.

[5] Chang, S. S., Cho, Y. J. and Kang, S. M., *Probabilistic Metric Spaces and Nonlinear Operator Theory*, Sichuan University Press, Chendgu, China, 1994.

[6] Chang, S. S. and Ma, Y. H., *Coupled fixed points for mixed monotone condensing operators and existence theorem of the solutions for a class of functional equation arising in dynamic programming*, J. Math. Anal. Appl. **160** (1991), 468–479.

[7] Egbert, R. J., *Products and quotients of probabilistic metric spaces*, Pacific J. Math. **24** (1968), 437–455.

[8] Hadzic, O., *Some theorems on the fixed points in probabilistic metric and random normed spaces*, Boll. Un. Mat. Ital. B(6), 1-B (1982), 381–391.

[9] Hutson, V. and Pym, J. S., *Applications of Functional Analysis and Operator Theory*, Academic Press, London, 1980.

[10] Istratescu, V. I., *Fixed Point Theory*, D. Reidel Publishing Company, Dordrecht, 1988.

[11] Menger, K., *Statistical metric*, Proc. Natl. Acad. Sci. USA, **28** (1942), 535–537.

[12] Schweizer, B. and Sklar, A., *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313–334.

[13] Schweizer, B. and Sklar, A., *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics **5** 1983.

[14] Sherwood, H., *Complete probabilistic metric spaces*, Z. Wahrsch. Verw. Gebiete **20** (1971), 117–128.

[15] Stojakovic, M., *Fixed point theorems in probabilistic metric spaces*, Kobe J. Math. **2**(1) (1985), 1–9.

[16] Stojakovic, M., *Common fixed point theorems in complete metric and probabilistic metric spaces*, Bull. Austral. Math. Soc. **36** (1987), 73–88.

[17] Sun, Y., *A fixed point theorem for monotone operators with applications*, J. Math. Anal. Appl. **156** (1991), 240–252.

[18] Serstnev, A. N., *On the notion of random normed spaces*, Dokl. Akad. Nauk **149** (1963), 280–283.

[19] Tan, N. X., *Generalized probabilistic metric spaces and fixed point theorems*, Math. Nachr. **129** (1986), 205–218.

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