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# BOUNDED SOLUTIONS AND ASYMPTOTIC STABILITY OF NONLINEAR DIFFERENCE EQUATIONS IN THE COMPLEX PLANE

EUGENIA N. PETROPOULOU* AND PANAYIOTIS D. SIAFARIKAS

Abstract. An existence and uniqueness theorem for solutions in the Banach space $l_1$ of a nonlinear difference equation is given. The constructive character of the proof of the theorem predicts local asymptotic stability and gives information about the size of the region of attraction near equilibrium points.

## 1. Introduction

We consider the $m$-th order non-linear difference equation of the form

$$f(n+m)+\sum_{p=1}^{m}(\alpha_p+\beta_p(n))f(n+m-p)=\sum_{i=1}^{N}c_i(n)f(n+q_{i1})f(n+q_{i2})$$

$$+\sum_{j=1}^{M}d_j(n)f(n+q_{j3})f(n+q_{j4})f(n+q_{j5}) \tag{1.1}$$

where $m, N, M$ are positive integers, $q_{i1}$, $q_{i2}$, $i=1,\dots,N$, $q_{j3}$, $q_{j4}$, $q_{j5}$, $j=1,\dots,M$ are non-negative integers, $\alpha_p$, $p=1,\dots,m$ in general complex numbers, with the initial conditions

$$f(p)=u_p, p=1,\dots,m \tag{1.2}$$

Under suitable assumptions on the sequences $\beta_p(n)$,$p=1,\dots,m$, $c_i(n)$, $i=1,\dots,N$, $d_j(n)$, $j=1,\dots,M$ and the roots of the polynomial $r^m+\alpha_1 r^{m-1}+\dots+\alpha_m=0$, we prove that there exists a unique solution of (1.1), (1.2) in the Banach space $l_1$ of all bounded complex sequences $f(n)$ which satisfy the condition $\sum_{n=1}^{\infty}|f(n)|<+\infty$. For the motivation of seeking solutions of nonlinear difference

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equations in $l_1$ see [3]. More general nonlinearities concerning the right-hand side of equation (1.1) are considered in a forthcoming paper.

The method used is a functional analytic method based on a method which is developed by E. K. Ifantis in [3]. Using this method, equation (1.1) is reduced equivalently to an operator equation of the form

$$f = Q(f) + \xi_1 = \phi(f)\,, \tag{1.3}$$

where $\xi_1$ is a fixed element in an abstract Banach space $H_1$ depending on the initial conditions $f(p) = u_p, p = 1, \ldots, m$ and Q a non-linear operator which is defined on $H_1$. Under suitable assumptions on the sequences $c_i(n)$, $i = 1, \ldots, N$, $d_j(n)$, $j = 1, \ldots, M$, it is shown that the non-linear operator $\phi : B(0, R_0) \to B(0, R_0)$, where $B(0, R_0)$ is an open ball centered at the origin of $H_1$, is a holomorphic map in $B(0, R_0)$, i.e. its Fréchet derivative exists at every point in the open ball $B(0, R_0)$ in $H_1$ and $\phi(B(0, R_0) \subset B(0, R_0)$. For holomorphic maps the following result of Earle and Hamilton [2] holds:

*If $f : X \to X$ is holomorphic and f(X) lies strictly inside X, then $f$ has a unique fixed point in $X$, where $X$ is a bounded, connected and open subset of a Banach space $E$.*

By saying that a subset $X'$ of $X$ lies strictly inside $X$ we mean that there exists a $\epsilon_1 > 0$ such that $\|x' - y\| > \epsilon_1$ for all $x' \in X'$ and $y \in E - X$.

Using the above it is proved that the equation (1.3) has a unique fixed point in the space $H_1$. This means equivalently that the initial value problem (1.1), (1.2) has a unique solution in the space $l_1$.

The above result establishes local asymptotic stability and gives information about the size of the region of attraction near equilibrium points. Moreover, we can see in particular cases how the size of the region of attraction depends on the parameters of equation (1.1) and the initial conditions (1.2). Also, it is of some interest to note that we can actually obtain in some cases an explicit upper bound for the solution $f(n)$ of equation (1.1) which satisfies the initial conditions (1.2).

In some cases we can also find the radius of convergence of the power series $f(z) = \sum_{n=1}^{\infty} f(n) z^{n-1}$. This power series corresponding to $f(n)$ is called generating function and it may be a formal solution of a differential or integral equation.

In Section 2, equation (1.1) together with the initial conditions (1.2) is reduced to an operator equation of the form (1.3). In Section 3 a theorem is proved for the existence and uniqueness of solutions in $l_1$ of the initial value problem (1.1), (1.2). The proof of the theorem is based on two lemmas. In the first lemma we give the abstract forms of the right-hand side (nonlinear) part of the difference equation (1.1) defined on $H_1$, and in the second lemma we prove that the nonlinear operators which are the abstract forms presented in the first lemma are Fréchet differentiable in $H_1$. Moreover we find the Fréchet derivative for each one of them. These two lemmas together with the results of Section 2 and the Earle-Hamilton fixed point theorem give Theorem 3.1. Finally in Section 4 we apply the theorem for some nonlinear difference equations which can be deduced from equation (1.1).

## 2. The abstract form of nonlinear difference equations

In the following $H$ is used to denote an abstract separable Hilbert space with the orthonormal basis $e_n, n = 1, 2, 3, \ldots$. We use the symbols $(\cdot, \cdot)$ and $\|\cdot\|$ to denote scalar product and norm in $H$ respectively. By $H_1$ we mean the Banach space consisting of those elements $f$ in $H$ which satisfy the condition $\sum_{n=1}^{\infty} |(f, e_n)| < +\infty$. The norm in $H_1$ is denoted by $\|f\|_1 = \sum_{n=1}^{\infty} |(f, e_n)|$. By $f(n)$ we mean an element of the Banach space $l_1$ and by $f = \sum_{n=1}^{\infty} f(n)e_n$ we mean that element in $H_1$ generated by $f(n) \in l_1$. The norm in $l_1$ is denoted by $\|f(n)\|_{l_1} = \sum_{n=1}^{\infty} |f(n)|$. Finally by $V$ we mean the shift operator on $H$

$$V : Ve_n = e_{n+1}\,, \quad n = 1, 2, \ldots$$

and by $V^*$ its adjoint

$$V^* : V^*e_n = e_{n-1}\,, \quad n = 2, 3, \ldots, V^*e_1 = 0\,.$$

One can easily prove that the function

$$\phi : H_1 \to l_1$$

which is defined as follows:

$$\phi(f) = (f, e_n) = f(n)$$

is an isomorphism from $H_1$ onto $l_1$. We call $f$ the abstract form of $f(n)$.

In general, if $G$ is a mapping in $l_1$ and $N$ is a mapping in $H_1$, we call $N(f)$ the abstract form of $G(f(n))$ if

$$G(f(n)) = (N(f), e_n) \tag{2.1}$$

It follows easily that $V^*f$ is the abstract form of $f(n+1)$, since

$$f(n+1) = (f, e_{n+1}) = (V^*f, e_n)\,,$$

$Vf$ is the abstract form of $f(n-1)$, since

$$f(n-1) = (f, e_{n-1}) = (Vf, e_n)$$

and $B_pf$, $p = 1, 2, \ldots, m$ are the abstract forms of $\beta_p(n)f(n)$, where $B_p$ are the diagonal operators

$$B_pe_n = \beta_p(n)e_n\,, \quad n = 1, 2, \ldots, m \tag{2.2}$$

since $\beta_p(n)f(n) = \beta_p(n)(f, e_n) = (f, \beta_p^*(n)e_n) = (B_pf, e_n)$.

It follows readily from the above that the abstract form of the left-hand side (linear) part of (1.1) is

$$V^{*m}f+\sum_{p=1}^{m}(\alpha_p+B_p)V^{*m-p}f$$

because

$$\begin{aligned} &f(n+m)+\sum_{p=1}^{m}(\alpha_p+\beta_p(n))f(n+m-p)\\ &=(V^{*m}f+\sum_{p=1}^{m}(\alpha_p+B_p)V^{*m-p}f,e_n)\,,\quad n=1,2,\dots \end{aligned} \tag{2.3}$$

From (2.1), taking into account (2.3), we obtain the abstract form of equation (1.1):

$$V^{*m}f+\sum_{p=1}^{m}(\alpha_p+B_p)V^{*m-p}f=\sum_{i=1}^{N}N_i(f)+\sum_{j=1}^{M}K_j(f)\,,$$

where the second part of this equation is the abstract form of the right-hand side (nonlinear) part of (1.1) and the nonlinear operators $N_i(f), 1\le i\le N$ and $K_j(f)$, $1\le j\le M$ are defined in $H_1$. Since $V^*V=I$ the above equation can be written as follows:

$$\begin{aligned} &V^*(I-r_1V)(I-r_2V)\dots(I-r_mV)f+\sum_{p=1}^{m}B_pV^{*m-p}f\\ &=\sum_{i=1}^{N}N_i(f)+\sum_{j=1}^{M}K_j(f) \end{aligned} \tag{2.4}$$

where $r_p$, $p=1,2,\dots,m$ are the roots of the equation:

$$r^m+\alpha_1r^{m-1}+\dots+\alpha_m=0 \tag{2.5}$$

and $I$ is the identity operator, or

$$V^{*m}\Gamma f=\sum_{i=1}^{N}N_i(f)+\sum_{j=1}^{M}K_j(f) \tag{2.6}$$

where

$$\Gamma=(I-r_1V)(I-r_2V)\dots(I-r_mV)+V^m\sum_{p=1}^{m}B_pV^{*m-p}f\,. \tag{2.7}$$

The operator $\Gamma$ leaves invariant the space $H_1$, i.e. for every $x\in H_1$, $\Gamma x\in H_1$ and therefore equation (2.4) can be considered as an equation in $H_1$ whenever $\Gamma$ is defined from $H_1$ into $H_1$ [3]. This means that $f$ is a solution of (2.4) or (2.6) in $H_1$ if and only if $\{f(n)\}$ is a solution of (1.1) in $l_1$. Also the operator $\Gamma$ has a bounded inverse on $H_1$ provided that $|r_p|<1$, $p=1,2,\dots,m$ [3].

Taking into account the above properties of the operator $\Gamma$ it can be proved similarly as in [3] the following theorem:

**Theorem 2.1.** *Assume that the roots $r_p$, $p = 1, 2, \ldots, m$ of the equation $r^m + \alpha_1 r^{m-1} + \cdots + \alpha_m = 0$ satisfy the conditions $|r_p| < 1$, $p = 1, 2, \ldots, m$. Then equation (1.1) in the space $l_1$ together with the initial conditions*

$$f(1) = u_1, f(2) = u_2, \ldots, f(m) = u_m \tag{2.8}$$

*is equivalent to the following operator equation in $H_1$:*

$$f = \Gamma^{-1}u + \Gamma^{-1}V^m \left( \sum_{i=1}^{N} N_i(f) + \sum_{j=1}^{M} K_j(f) \right) , \tag{2.9}$$

*where*

$$\begin{aligned} u =&u_1 e_1 + (\alpha_1 u_1 + u_2)e_2 + \ldots \\ &+ (\alpha_{m-1}u_1 + \alpha_{m-2}u_2 + \cdots + u_m)e_m \, . \end{aligned} \tag{2.10}$$

**Proof.** Equation (2.6) can be written as follows:

$$V^{*m}(\Gamma f - \sum_{i=1}^{N} V^m N_i(f) - \sum_{j=1}^{M} V^m K_j(f)) = 0 \tag{2.11}$$

The null space of $V^{*m}$ is spanned by the elements $e_1, e_2, \ldots, e_m$. Thus we obtain from (2.11):

$$\Gamma f - \sum_{i=1}^{N} V^m N_i(f) - \sum_{j=1}^{M} V^m K_j(f)) = c_1 e_1 + c_2 e_2 + \cdots + c_m e_m \, .$$

From (2.4) and (2.9) we find easily that:

$$c_1 = u_1, c_2 = \alpha_1 u_1 + u_2, \ldots, c_m = u_m + \cdots + u_2 \alpha_{m-2} + u_1 \alpha_{m-1}$$

and the theorem follows from the properties of the operator $\Gamma$. □

## 3. Existence and uniqueness theorem

In this Section we shall prove a theorem which predicts a unique solution of the nonlinear difference equation (1.1) in $l_1$, which satisfies the initial conditions (1.2). The proof of the theorem is based on two lemmas. In the first lemma we give the abstract forms of the right-hand side (nonlinear) part of the difference equation (1.1) defined on $H_1$, and on the second lemma we prove that the nonlinear operators which are the abstract forms presented in the first lemma are Fréchet differentiable in $H_1$. Moreover we find the Fréchet derivative for each one of them. Combining these two lemmas together with Theorem 2.1 and the fixed point theorem of Earle and Hamilton [2] we obtain the following:

**Theorem 3.1.** *Assume that the complex sequences $\beta_p(n), c_i(n)$ and $d_j(n)$, $p = 1,\dots,m$, $i = 1,\dots,N$, $j = 1,\dots,M$ satisfy:*

$$\lim_{n\to\infty} \beta_p(n) = 0\,, \quad \sup_n |c_i(n)| \le \gamma_i, \quad \sup_n |d_j(n)| \le \delta_j\,, \tag{3.1}$$

*and the roots of the equation $r^m + \alpha_1 r^{m-1} + \cdots + \alpha_m = 0$ satisfy the conditions $|r_p| < 1$, $p = 1, 2, \dots, m$. Then there exist positive numbers $R_0$ and $P_0$ such that for*

$$|u| = |u_1| + |\alpha_1 u_1 + u_2| + \cdots + |\alpha_{m-1}u_1 + \alpha_{m-2}u_2 + \cdots + u_m| < P_0 \tag{3.2}$$

*the equation*

$$\begin{aligned} f(n+m) &+ \sum_{p=1}^{m} (\alpha_p + \beta_p(n)) f(n+m-p) \\ &= \sum_{i=1}^{N} c_i(n) f(n+q_{i1}) f(n+q_{i2}) + \sum_{j=1}^{M} d_j(n) f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5}) \end{aligned} \tag{3.3}$$

*together with the initial conditions*

$$f(1) = u_1, f(2) = u_2, \dots, f(m) = u_m \tag{3.4}$$

*where $\alpha_p$, $u_p$ are in general complex numbers, $m, N, M$ are positive integers, $q_{i1}$, $q_{i2}$, $i = 1,\dots,N$, $q_{j3}$, $q_{j4}$, $q_{j5}$, $j = 1,\dots,M$ are non-negative integers, has a unique solution $f(n)$ in $l_1$. Moreover*

$$\sum_{n=1}^{\infty} |f(n)| < R_0\,. \tag{3.5}$$

**Lemma 1.** (i) *Consider the diagonal operators $C_i : H \to H$ such that:*

$$C_i : C_i e_n = c_i(n) e_n\,, \quad n = 1, 2, \dots, \quad i = 1, .., N \tag{3.6}$$

*and the nonlinear operators $N_i'$, which are defined on $H_1$ as follows:*

$$N_i'(f) = (f, e_{n+q_{i1}})(f, e_{n+q_{i2}}) e_n = f(n+q_{i1}) f(n+q_{i2}) e_n, \tag{3.7}$$

*$1 \le i \le N$. Then the nonlinear operators:*

$$N_i : N_i(f) = c_i(n) N_i'(f)\,, \quad i = 1, \dots, N \tag{3.8}$$

*are defined in $H_1$ and are the abstract forms of the operators:*

$$G_i : G_i(f(n+q_{i1}), f(n+q_{i2})) = c_i(n) f(n+q_{i1}) f(n+q_{i2})\,, \quad i = 1, \dots, N \tag{3.9}$$

*in $l_1$.*

(ii) *Consider the diagonal operators $D_j : H \to H$, such that:*

$$D_j : D_j e_n = d_j(n) e_n \,, \quad n = 1, 2, \dots \,, \; j = 1, .., M \tag{3.10}$$

*and the nonlinear operators $K'_j$, which are defined on $H_1$ as follows:*

$$\begin{aligned} K'_j(f) &= (f, e_{n+q_{j3}})(f, e_{n+q_{j4}})(f, e_{n+q_{j5}}) e_n \\ &= f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5}) e_n \,, \end{aligned} \tag{3.11}$$

$1 \le j \le M$. *Then the nonlinear operators:*

$$K_j : K_j(f) = d_j(n) K_j(f) \,, \quad j = 1, \dots, M \tag{3.12}$$

*are defined in $H_1$ and are the abstract forms of the operators:*

$$\begin{aligned} F_j &: F_j(f(n+q_{j3}), f(n+q_{j4}), f(n+q_{j5})) \\ &= d_j(n) f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5}) \,, \quad j = 1, \dots, M \end{aligned} \tag{3.13}$$

*in $l_1$.*

**Proof.** (i) From (3.7) we have:

$$\begin{aligned} \|N'_i(f)\|_1 &= \sum_{r=1}^{\infty} |(N'_i(f), e_r)| = \sum_{r=1}^{\infty} |(f, e_{n+q_{i1}})(f, e_{n+q_{i2}})(e_n, e_r)| \\ &\Rightarrow \|N'_{ik}(f)\|_1 = |(f, e_{n+q_{i1}})| \cdot |(f, e_{n+q_{i2}})| \le \|f\|_1^2 \end{aligned} \tag{3.14}$$

Since $\|C_i\|_1 = \sup_n |c_i(n)| \le \gamma_i$ we obtain from (3.8) and (3.14):

$$\|N_i(f)\|_1 \le \gamma_i \|f\|_1^2 < \infty \,, \tag{3.15}$$

since $f \in H_1$. Thus the nonlinear operators (3.8) are defined in $H_1$. Moreover we see that

$$\begin{aligned} &(N_i(f), e_n) = (c_i(n) f(n+q_{i1}) f(n+q_{i2}) e_n, e_n) \\ &\qquad \Rightarrow (N_i(f), e_n) = c_i(n) f(n+q_{i1}) f(n+q_{i2}) = G_i(f(n+q_{i1}), f(n+q_{i2})) \,, \end{aligned}$$

which means that $N_i$ are the abstract forms of $G_i$ in $l_1$.

(ii) Similarly from (3.11) we have:

$$\|K'_j(f)\|_1 \le \|f\|_1^3 \tag{3.16}$$

Since $\|D_j\|_1 = \sup_n |d_j(n)| \le \delta_j$ we obtain from (3.12) and (3.16):

$$\|K_j(f)\|_1 \le \delta_{tk} \|f\|_1^3 < \infty \,. \tag{3.17}$$

Thus the nonlinear operators (3.12) are defined in $H_1$. Moreover we see that

$$\begin{aligned} &(K_j(f), e_n) = (d_j(n) f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5}) e_n, e_n) \\ &\qquad \Rightarrow (K_j(f), e_n) = F_j(f(n+q_{j3}), f(n+q_{j4}), f(n+q_{j5})) \,, \end{aligned}$$

which means that $K_j$ are the abstract forms of $F_j$. □

**Lemma 2.** (i) *The nonlinear operators (3.8) are Fréchet differentiable in $H_1$ and their Fréchet derivatives at the point $f_0 \in H_1$ are*

$$A_i(f_0)f = c_i(n)[(f, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) + (f_0, e_{n+q_{i1}})(f, e_{n+q_{i2}})]e_n\,. \tag{3.18}$$

(ii) *The nonlinear operators (3.12) are Fréchet differentiable in $H_1$ and their Fréchet derivatives at the point $f_0 \in H_1$ are*

$$\begin{aligned} A_j(f_0)f = d_j(n)[&(f, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) \\ &+ (f_0, e_{n+q_{j3}})(f, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) \\ &+ (f_0, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f, e_{n+q_{j5}})]e_n\,. \end{aligned} \tag{3.19}$$

**Proof.** (i) We shall first prove that the linear operators:

$$A_i'(f_0)f = [(f, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) + (f_0, e_{n+q_{i1}})(f, e_{n+q_{i2}})]e_n$$

are the Fréchet derivatives of the nonlinear operators (3.7) at the point $f_0 \in H_1$.

Indeed, $A_i'$ are bounded operators for $f_0 \in H_1$, since:

$$\|A_i'(f_0)f\|_1 = \sum_{r=1}^{\infty} |(A_i'(f_0)f, e_r)|$$

$$= \sum_{r=1}^{\infty} |([(f, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) + (f_0, e_{n+q_{i1}})(f, e_{n+q_{i2}})]e_n, e_r)|$$

$$\Rightarrow \|A_i'(f_0)f\|_1 \le |(f, e_{n+q_{i1}})| \cdot |(f_0, e_{n+q_{i2}})| + |(f_0, e_{n+q_{i1}})| \cdot |(f, e_{n+q_{i2}})|$$

$$\Rightarrow \|A_i'(f_0)f\|_1 \le 2\|f_0\|_1\|f\|_1$$

and $\|f_0\|_1 < \infty$, since $f_0 \in H_1$. Also for $f_0 \in H_1$ and $h \in H_1$, we have:

$$\begin{aligned} \|N_i'(f_0 + h) - N_i'(f_0) - A_i'(f_0)h\|_1 &= \sum_{r=1}^{\infty} |(N_i'(f_0 + h) - N_i'(f_0) - A_i'(f_0)h, e_r)| \\ &= |(f_0 + h, e_{n+q_{i1}})(f_0 + h, e_{n+q_{i2}}) - (f_0, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) \\ &\quad -(h, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) - (f_0, e_{n+q_{i1}})(h, e_{n+q_{i2}})| \end{aligned}$$

$$\begin{aligned} = |[f_0(n + q_{i1}) + h(n + q_{i1})][f_0(n + q_{i2}) + h(n + q_{i2})] - f_0(n + q_{i1})f_0(n + q_{i2}) \\ -h(n + q_{i1})f_0(n + q_{i2}) - f_0(n + q_{i1})h(n + q_{i2})| \end{aligned}$$

$$\Rightarrow \|N_i'(f_0 + h) - N_i'(f_0) - A_i'(f_0)h\|_1 = |h(n + q_{i1})| \cdot |h(n + q_{i2})|$$

$$\Rightarrow \|N_i'(f_0 + h) - N_i'(f_0) - A_i'(f_0)h\|_1 = |(h, e_{n+q_{i1}})| \cdot |(h, e_{n+q_{i2}})| \le \|h\|_1^2$$

$$\Rightarrow \frac{\|N_i'(f_0 + h) - N_i'(f_0) - A_i'(f_0)h\|_1}{\|h\|_1} \le \|h\|_1 \to 0$$

for $\|h\|_1 \to 0$. Thus the linear operators $A_i'(f_0)f$ are the Fréchet derivatives of the nonlinear operators (3.7) at the point $f_0 \in H_1$. Now we can prove that the linear operators (3.18) are the Fréchet derivatives of the nonlinear operators (3.8) at the point $f_0 \in H_1$. Indeed, the linear operators (3.18) are bounded, since:

$$\|A_i(f_0)f\|_1 \leq |c_i(n)| \cdot \|A_i'(f_0)f\|_1 \leq 2\gamma_i\|f_0\|_1\|f\|_1 \,.$$

Also

$$\|N_i(f_0+h) - N_i(f_0) - A_i(f_0)h\|_1 \leq \gamma_i\|N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h\|_1$$
$$\Rightarrow \frac{\|N_i(f_0+h) - N_i(f_0) - A_i(f_0)h\|_1}{\|h\|_1} \leq \gamma_i\|h\|_1 \to 0$$

for $\|h\|_1 \to 0$. Thus the linear operators (3.18) are the Fréchet derivatives of the nonlinear operators (3.8) at the point $f_0 \in H_1$.

(ii) Similarly, we shall first prove that the linear operators:

$$A_j'(f_0)f = [(f, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f_0, e_{n+q_{j5}})$$
$$+(f_0, e_{n+q_{j3}})(f, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) + (f_0, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f, e_{n+q_{j5}})]e_n$$

are the Fréchet derivatives of the nonlinear operators (3.11) at the point $f_0 \in H_1$.

Indeed, $A_j'$ are bounded operators for $f_0 \in H_1$, since:

$$\|A_j'(f_0)f\|_1 \leq 3\|f_0\|_1^2\|f\|_1$$

and $\|f_0\|_1 < \infty$, since $f_0 \in H_1$. Also for $f_0 \in H_1$ and $h \in H_1$, we have:

$$\frac{\|K_j'(f_0+h) - K_j'(f_0) - A_j'(f_0)h\|_1}{\|h\|_1} \leq 3\|f_0\|_1^2\|h\|_1 + \|h\|_1^2 \to 0$$

for $\|h\|_1 \to 0$. Thus the linear operators $A_j'(f_0)f$ are the Fréchet derivatives of the nonlinear operators (3.11) at the point $f_0 \in H_1$. Now we can prove that the linear operators (3.19) are the Fréchet derivatives of the nonlinear operators (3.12) at the point $f_0 \in H_1$. Indeed, the linear operators (3.19) are bounded, since:

$$\|A_j(f_0)f\|_1 \leq |d_j(n)| \cdot \|A_j'(f_0)f\|_1 \leq 3\delta_j\|f_0\|_1 \cdot \|f\|_1 \,.$$

Also

$$\|K_j(f_0+h) - K_j(f_0) - A_j(f_0)h\|_1 \leq \delta_i\|K_j'(f_0+h) - K_j'(f_0) - A_j'(f_0)h\|_1$$
$$\Rightarrow \frac{\|K_j(f_0+h) - K_j(f_0) - A_j(f_0)h\|_1}{\|h\|_1} \leq \delta_j(3\|f_0\|_1 \cdot \|h\|_1 + \|h\|_1^2 \to 0$$

for $\|h\|_1 \to 0$. Thus the linear operators (3.19) are the Fréchet derivatives of the nonlinear operators (3.12) at the point $f_0 \in H_1$. □

**Proof of Theorem 3.1.** Consider the function:

$$\phi : \phi(f) = \Gamma^{-1}u + \sum_{i=1}^{N} \Gamma^{-1}V^m N_i(f) + \sum_{j=1}^{M} \Gamma^{-1}V^m K_j(f) \tag{3.20}$$

and assume that

$$\|\Gamma^{-1}\|_1 \leq L\,. \tag{3.21}$$

By Lemma 1 we have from (3.15) and (3.17):

$$\|\phi(f)\|_1 \leq L(|u| + \gamma_i \|f\|_1^2 + \delta_j \|f\|_1^3)\,. \tag{3.22}$$

Let $\|f\|_1 \leq \bar{R}$, where $\bar{R}$ can be as large as we want but not infinite. Then for $\|f\|_1 \leq R \leq \bar{R}$ we have from (3.22):

$$\|\phi(f)\|_1 \leq L(|u| + \gamma_i R^2 + \delta_j R^3)\,. \tag{3.23}$$

By hypothesis $R$ is very large so there exists $R_1 \in [0, \bar{R}]$ such that

$$LR_1(\gamma_i + \delta_j R_1) > 1\,.$$

Thus the function

$$P(R) = 1 - LR(\gamma_i + \delta_j R) \tag{3.24}$$

has a first zero $R_2$ between 0 and $R_1$ since: $P(0) = 1 > 0$ and

$$P(R_1) = 1 - LR_1(\gamma_i + \delta_j R_1) < 0\,.$$

So the continuous function

$$P_1(R) = L^{-1}RP(R) \tag{3.25}$$

satisfies $P_1(0) = P_1(R_2) = 0$ and therefore attains a maximum at a point $R_0 : 0 < R_0 < R_2$. Now for every $\epsilon > 0, R = R_0$ and

$$|u| \leq P_1(R_0) - \frac{\epsilon}{L} \tag{3.26}$$

we find from (3.23):

$$\|\phi(f)\|_1 \leq R_0 - \epsilon < R_0 \tag{3.27}$$

for $\|f\|_1 < R_0$. This means that for

$$|u| < P_1(R_0) = P_0 \tag{3.28}$$

$\phi$ is a holomorphic map from $B(0,R_0)$ strictly inside $B(0,R_0)$. Indeed, it is obvious that $\phi(B(0,R_0)) \subset B(0,R_0)$ and $\phi(B(0,R_0))$ lies strictly inside $B(0,R_0)$, since if $k \in H_1 - B(0,R_0) \Rightarrow \|k\|_1 \geq R_0$ and $g \in \phi(B(0,R_0))$, i.e. there exists an $f \in B(0,R_0) \Rightarrow \|f\|_1 < R_0$ such that $\phi(f) = g$, then we find easily that $\|g-k\|_1 \geq \epsilon > \frac{\epsilon}{2} = \epsilon_1$. Thus applying the fixed point theorem of Earle and Hamilton [2] we find that equation $\phi(f) = f$ has a unique fixed point in $H_1$. This means equivalently that the initial value problem (3.3), (3.4) has a unique solution in $l_1$. □

**Remark 3.1.** Theorem 3.1 predicts local asymptotic stability. However, it is not a purely local result. The proof of the theorem has a constructive character and gives some information about the size of the region of attraction. This is something that one cannot obtain with the classical methods of linearisation [4]. Moreover, we can see in particular cases how the size of the above region depends on the parameters of the equation. Note that the region of attraction may be so small relative to a given application that practically the centre of this region could be considered as unstable. Also an equilibrium point of a difference equation could be unstable, but a very small neighborhood of it could be an attractor and thus from a practical point of view, it could be considered as stable [4].

**Remark 3.2.** The solution that is established by Theorem 3.1 is an element of $l_1$ and thus $\lim_{n\to\infty} f(n) = 0$. Also from (3.5) we obtain:

$$|f(n)| \leq \sum_{n=1}^{\infty} |f(n)| < R_0 \Rightarrow |f(n)| < R_0\,.$$

This means that the solution that is predicted by Theorem 3.1 is a bounded solution of the initial value problem (3.3), (3.4) and $R_0$ is an upper bound of it.

**Remark 3.3.** Theorem 3.1 predicts a unique solution $\{f(n)\}$ of (3.3) in $l_1$. This means that $\lim_{n\to\infty} f(n) = 0$. Thus zero is a locally asymptotically stable equilibrium point of equation (3.3) with region of attraction given by (3.2). In the case where equation (3.3) has nonzero equilibrium points ($\varrho$), we set:

$$f(n) = g(n) + \varrho \tag{3.29}$$

and we apply Theorem 3.1 to the new equation which results from equation (3.3) after the transformation (3.29). If Theorem 3.1 can be applied to this transformed equation, then this equation has a unique solution in $l_1$, i.e. $g(n) \in l_1 \Rightarrow \lim_{n\to\infty} g(n) = 0$ and zero is a locally asymptotically stable equilibrium point with region of attraction given by (3.2). As a consequence, equation (3.3) has a unique solution, not in $l_1$, but in the space $\{\varrho\} + l_1$. Also $\lim_{n\to\infty} f(n) = \varrho$ and thus $\varrho$ is a locally asymptotically stable equilibrium point. Its region of attraction results from the region of attraction for the zero equilibrium point of the

transformed equation using (3.29). Finally the upper bound of f(n) is given by
$$|f(n)| < R_0 + |\varrho|$$

**Remark 3.4.** Theorem 3.1 holds also in the case where $n = 0, 1, 2, \ldots$. In this case the orthonormal basis of $H$ consists of the elements $e_0, e_1, e_2, \ldots$ and the shift operator and its adjoint are defined as follows:

$$V : Ve_n = e_{n+1}\,,\ n = 0, 1, 2, \ldots$$

$$V^* : V^*e_n = e_{n-1}\,,\ n \geq 1\,, V^*e_0 = 0\,.$$

Also the solution $f(n) \in l_1$ of the nonlinear difference equation (3.3) does not satisfy conditions (3.4) anymore, but instead satisfy the following conditions:

$$f(0) = u_0, f(1) = u_1, f(2) = u_2, \ldots, f(m-1) = u_{m-1}\,.$$

Finally the relationship (3.2) should be replaced by the following:

$$|u| = |u_0| + |\alpha_1 u_0 + u_1| + \cdots + |\alpha_{m-1}u_0 + \alpha_{m-2}u_1 + \cdots + u_{m-1}| < P_0\,,$$

where $u = u_0e_0 + (\alpha_1 u_0 + u_1)e_1 + \cdots + (\alpha_{m-1}u_0 + \alpha_{m-2}u_1 + \cdots + u_{m-1})e_{m-1}$.

## 4. Examples

1) Consider the difference equation:

$$f(n+1) + \alpha(n)f(n) = d(n)f(n+2)f(n+1)f(n)\,, \tag{4.1}$$

where $\lim_{n\to\infty} \alpha(n) = \alpha$, $|\alpha| < 1$, $\sup_n |d(n)| \leq \delta$ and $\alpha(n), d(n)$ are in general complex-valued sequences, $\alpha$ is in general a complex number and $\delta$ is a real positive number.

First, we shall show that zero is a locally asymptotically stable equilibrium point of equation (4.1). Moreover we shall find a region of attraction for it, a bound for the solution of equation (4.1) and the radius of convergence for the generating function $f(z)$.

Equation (4.1) can be written as follows:

$$f(n+1) + \alpha f(n) + [\alpha(n) - \alpha]f(n) = d(n)f(n+2)f(n+1)f(n)\,. \tag{4.2}$$

This equation results from equation (3.3) for:

$$m = 1, \alpha_1 = \alpha, \beta_1(n) = \alpha(n) - \alpha, c_i(n) \equiv 0, \gamma_i \equiv 0\,,$$
$$M = 1, d_1(n) = d(n), \delta_1 = \delta, q_{13} = 2, q_{14} = 1, q_{15} = 0\,.$$

In this case the operator $\Gamma$ has the form

$$\Gamma = I + \alpha V + VD_1\,,\ \text{where } D_1 e_n = [\alpha(n) - \alpha]e_n$$

and let $\|\Gamma^{-1}\|_1 \le L$. Also $P(R) = 1 - L\delta R^2$ and $P_1(R) = \frac{R}{L} - \delta R^3$. It follows easily that the region of attraction predicted by Theorem 3.1 is given by:

$$|f(0)| < \frac{2}{3L\sqrt{3\delta L}}\,. \tag{4.3}$$

Thus zero is a locally asymptotically stable equilibrium point of equation (4.1) with region of attraction given by (4.3). Also $f(n)$ is bounded and in particular the following holds:

$$|f(n)| < \frac{1}{\sqrt{3\delta L}}\,. \tag{4.4}$$

Finally from equation (4.1) we have:

$$\frac{f(n+1)}{f(n)} + \alpha(n) = d(n)f(n+2)f(n+1)\,.$$

The solution $\{f(n)\}$ predicted by Theorem 3.1 is an element of $l_1$ and so $\lim\limits_{n\to\infty} f(n) = 0$. Thus

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = -\alpha\,.$$

This means that for every solution $\{f(n)\}$ starting at a point given by (4.3), the generating analytic function $f(z) = \sum_{n=1}^{\infty} f(n)z^{n-1}$ converges absolutely for $|z| < \frac{1}{|\alpha|}$.

If $\alpha(n) = \alpha, |\alpha| < 1$, $\alpha$ complex and $d(n) = d$, $d$ complex, then the equilibrium point zero is a locally asymptotically stable equilibrium point of equation (4.1) with region of attraction given by:

$$|f(0)| < \frac{2}{3|\alpha|\sqrt{3|\alpha|\cdot|d|}}\,, \tag{4.5}$$

an upper bound of $f(n)$ is $\frac{1}{\sqrt{3|\alpha|\cdot|d|}}$, i.e.

$$|f(n)| < \frac{1}{\sqrt{3|\alpha|\cdot|d|}} \tag{4.6}$$

and the corresponding generating analytic function $f(z) = \sum_{n=1}^{\infty} f(n)z^{n-1}$ converges absolutely for $|z| < \frac{1}{|\alpha|}$.

Let us consider now the case where

$$\alpha(n) = \alpha\,,\ d(n) = d\,,$$

$\alpha$, $d$ real numbers. Then equation (4.1) can be written as follows:

$$f(n+1) + \alpha f(n) = df(n+2)f(n+1)f(n)\,. \tag{4.7}$$

The nonzero real equilibrium points ($\varrho$) of equation (4.7) are:

$$\varrho_1 = \sqrt{\frac{1+\alpha}{d}}, \varrho_2 = -\sqrt{\frac{1+\alpha}{d}}, (1+\alpha)d > 0\,. \tag{4.8}$$

In the following we shall show that $\varrho_1$, $\varrho_2$ are locally asymptotically stable equilibrium points of (4.7). Moreover we shall find a region of attraction for each one of them, a bound of the solution of equation (4.1) and the radius of convergence for the corresponding generating analytic function. We set $f(n) = g(n) + \varrho$ and equation (4.7) becomes:

$$\begin{aligned} g(n+2) &+ \frac{\alpha}{1+\alpha} g(n+1) + \frac{1}{1+\alpha} g(n) \\ &= -\frac{d}{1+\alpha} g(n+2)g(n+1)g(n) - \frac{d\varrho}{1+\alpha} g(n+2)g(n+1) \\ &\quad - \frac{d\varrho}{1+\alpha} g(n+2)g(n) - \frac{d\varrho}{1+\alpha} g(n+1)g(n)\,. \end{aligned} \tag{4.9}$$

We shall consider the following three cases for the values of $\alpha$ and $d$:

i) $0 < \alpha < 2 + 2\sqrt{2}, d > 0$,

ii) $\alpha \geq 2 + 2\sqrt{2}, d > 0$,

iii) $\alpha < -1, d < 0$.

In the first case the operator $\Gamma$ has the form

$$\Gamma = (I - r_1 V)(I - r_2 V)\,,$$

where $r_{1,2} = \frac{-\alpha \pm i\sqrt{4+4\alpha-\alpha^2}}{2(1+\alpha)}$ and $|r_{1,2}| = \frac{1}{\sqrt{1+\alpha}} < 1$. In this case $L = (1 - \frac{1}{\sqrt{1+\alpha}})^{-2}$. As before we find:

$$P(R) = 1 - LR(3\sqrt{\frac{d}{1+\alpha}} + \frac{Rd}{1+\alpha})\,,$$

$$P_1(R) = L^{-1}R(1 - LR(3\sqrt{\frac{d}{1+\alpha}} + \frac{Rd}{1+\alpha})\,.$$

Thus point zero in each case ($\varrho = \varrho_1$, $\varrho = \varrho_2$) is locally asymptotically stable equilibrium points of equation (4.9) with region of attraction given by:

$$\begin{aligned} |g(0)| + \left|\frac{\alpha}{1+\alpha} g(0) - g(1)\right| &< \frac{10 + 8\alpha - 4\sqrt{1+\alpha}}{3(1+\alpha)} \sqrt{\frac{4(1+\alpha)}{3d} + \frac{1}{3d} - \frac{2\sqrt{1+\alpha}}{3d}} \\ &\quad -3\sqrt{\frac{1+\alpha}{d}} - \frac{1}{\sqrt{d}\sqrt{1+\alpha}} + 2\,. \end{aligned} \tag{4.10}$$

Also $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| < \sqrt{\frac{4(1+\alpha)}{3d} + \frac{1}{3d} - \frac{2\sqrt{1+\alpha}}{3d}} - \sqrt{\frac{1+\alpha}{d}}\,. \tag{4.11}$$

Thus the equilibrium points $\varrho_1$, $\varrho_2$ are locally asymptotically stable with regions of attraction given respectively by:

$$\begin{aligned}\left|f(0)-\sqrt{\frac{1+\alpha}{d}}\right|&+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)-\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &<-3\sqrt{\frac{1+\alpha}{d}}-\frac{1}{\sqrt{d}\sqrt{1+\alpha}}+2\\ &\quad+\frac{10+8\alpha-4\sqrt{1+\alpha}}{3(1+\alpha)}\sqrt{\frac{4(1+\alpha)}{3d}+\frac{1}{3d}-\frac{2\sqrt{1+\alpha}}{3d}}\,.\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}\left|f(0)+\sqrt{\frac{1+\alpha}{d}}\right|&+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)+\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &<-3\sqrt{\frac{1+\alpha}{d}}-\frac{1}{\sqrt{d}\sqrt{1+\alpha}}+2\\ &\quad+\frac{10+8\alpha-4\sqrt{1+\alpha}}{3(1+\alpha)}\sqrt{\frac{4(1+\alpha)}{3d}+\frac{1}{3d}-\frac{2\sqrt{1+\alpha}}{3d}}\,.\end{aligned}\tag{4.13}$$

Also $\{f(n)\}$ is bounded and in particular the following holds:

$$|f(n)|<\sqrt{\frac{4(1+\alpha)}{3d}+\frac{1}{3d}-\frac{2\sqrt{1+\alpha}}{3d}}\,.\tag{4.14}$$

In the second case the operator $\Gamma$ has the form

$$\Gamma=(I-r_1V)(I-r_2V)\,,$$

where $r_1=\frac{-\alpha+\sqrt{\alpha^2-4\alpha-4}}{2(1+\alpha)}<0$, $r_2=\frac{-\alpha-\sqrt{\alpha^2-4\alpha-4}}{2(1+\alpha)}<0$ and $|r_{1,2}|<1$. In this case $L=\frac{1+\alpha}{2}$. As before we find:

$$P(R)=1-\frac{3R\sqrt{d(1+\alpha)}}{2}-\frac{dR^2}{2}\,,$$

$$P_1(R)=\frac{2R}{1+\alpha}-\frac{3\sqrt{d}R^2}{\sqrt{1+\alpha}}-\frac{dR^3}{1+\alpha}\,.$$

Thus point zero in each case ($\varrho=\varrho_1$, $\varrho=\varrho_2$) is locally asymptotically stable equilibrium point of equation (4.9) with region of attraction given by:

$$|g(0)|+\left|\frac{\alpha}{1+\alpha}g(0)-g(1)\right|<\frac{2}{\sqrt{d}}\left[\frac{(\alpha+\frac{5}{3})^{\frac{3}{2}}}{1+\alpha}-\frac{5\alpha+8}{\sqrt{1+\alpha}}\right]\,.\tag{4.15}$$

Also $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| < \frac{\sqrt{\alpha+\frac{5}{3}}-\sqrt{\alpha+1}}{\sqrt{d}}\,. \tag{4.16}$$

Thus the equilibrium points $\varrho_1$, $\varrho_2$ are locally asymptotically stable with regions of attraction given respectively by:

$$\begin{aligned}&\left|f(0)-\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)-\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &\quad<\frac{2}{\sqrt{d}}\left[\frac{(\alpha+\frac{5}{3})^{\frac{3}{2}}}{1+\alpha}-\frac{5\alpha+8}{\sqrt{1+\alpha}}\right].\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}&\left|f(0)+\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)+\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &\quad<\frac{2}{\sqrt{d}}\left[\frac{(\alpha+\frac{5}{3})^{\frac{3}{2}}}{1+\alpha}-\frac{5\alpha+8}{\sqrt{1+\alpha}}\right].\end{aligned} \tag{4.18}$$

Also $\{f(n)\}$ is bounded and in particular the following holds:

$$|f(n)| < \frac{\sqrt{\alpha+\frac{5}{3}}}{\sqrt{d}}\,. \tag{4.19}$$

In the third case the operator $\Gamma$ has the form

$$\Gamma = (I-r_1V)(I-r_2V)\,,$$

where $r_1 = \frac{-\alpha+\sqrt{\alpha^2-4\alpha-4}}{2(1+\alpha)} > 0$, $r_2 = \frac{-\alpha-\sqrt{\alpha^2-4\alpha-4}}{2(1+\alpha)} > 0$ and $|r_{1,2}| < 1$. In this case $L = \frac{1}{2}$. As before we find:

$$P(R) = 1-\frac{3R}{2}\sqrt{\frac{d}{1+\alpha}}-\frac{dR^2}{2(1+\alpha)}\,,$$

$$P_1(R) = 2R-3\sqrt{\frac{d}{1+\alpha}}R^2-\frac{dR^3}{1+\alpha}\,.$$

Thus point zero in each case ($\varrho=\varrho_1$, $\varrho=\varrho_2$) is locally asymptotically stable equilibrium point of equation (4.9) with region of attraction given by:

$$|g(0)|+\left|\frac{\alpha}{1+\alpha}g(0)-g(1)\right| < \frac{10\sqrt{15}-36}{9}\sqrt{\frac{1+\alpha}{d}}\,. \tag{4.20}$$

Also $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| < \frac{\sqrt{15}-3}{3}\sqrt{\frac{1+\alpha}{d}}\,. \tag{4.21}$$

Thus the equilibrium points $\varrho_1$, $\varrho_2$ are locally asymptotically stable with regions of attraction given respectively by:

$$\begin{aligned}&\left|f(0)-\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)-\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &<\frac{10\sqrt{15}-36}{9}\sqrt{\frac{1+\alpha}{d}}\,.\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\left|f(0)+\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)+\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &<\frac{10\sqrt{15}-36}{9}\sqrt{\frac{1+\alpha}{d}}\,.\end{aligned} \tag{4.23}$$

Also $\{f(n)\}$ is bounded and in particular the following holds:

$$|f(n)| < \frac{\sqrt{15}}{3}\sqrt{\frac{1+\alpha}{d}}\,. \tag{4.24}$$

Finally from equation (4.8) we have (for all these cases):

$$\frac{f(n+1)}{f(n)}+\alpha = df(n+2)f(n+1)\,. \tag{4.25}$$

The solution $\{g(n)\}$ of equation (4.9) which is predicted by Theorem 3.1 is an element of $l_1$ and so $\lim_{n\to\infty} g(n) = 0$. Thus $\lim_{n\to\infty} f(n) = \varrho$ and $f(n)$ is not an element of $l_1$ but it is an element of the space of bounded complex sequences. So from (4.25) we have:

$$\lim_{n\to\infty}\frac{f(n+1)}{f(n)} = -\alpha + d\frac{1+\alpha}{d} = 1\,.$$

This means that for every solution $\{f(n)\}$ starting at a point given by (4.12), (4.13) or (4.17), (4.18) or (4.22), (4.23), depending on the values of $\alpha$ and $d$, the corresponding generating analytic function $f(z) = \sum_{n=1}^{\infty} f(n)z^{n-1}$ converges absolutely for $|z| < 1$.

**Remark 4.1.** The method used in this paper does not give us any information in the case where $-1 < \alpha \leq 0$, as in this case the operator $\Gamma$ has the form $\Gamma = (I - r_1V)(I - r_2V)$ and both $r_1, r_2$ are in modulus greater than 1.

**Remark 4.2.** From (4.3), (4.5), (4.12), (4.13), (4.17), (4.18), (4.22) and (4.23) we see how each region of attraction depends on the parameters $\alpha$ and d of the equation.

**Remark 4.3.** From equation (4.1) for $\alpha(n) = A$ and $d(n) = 1$ we obtain the following difference equation:

$$f(n+2)f(n+1)f(n) = Af(n) + f(n+1)\,.$$

In the case when this difference equation has positive solutions with positive initial conditions it is known [5] that its positive equilibrium point is globally asymptotically stable for $A \in (0, \infty)$.

2) Consider the difference equation:

$$g(n+3)g(n+2)g(n+1) + g(n+3)g(n) = g(n+2) + g(n+1)g(n), \qquad n = 0, 1, \ldots \tag{4.26}$$

The three equilibrium points ($\varrho$) of equation (4.26) are:

$$\varrho_1 = 0, \varrho_2 = 1, \varrho_3 = -1\,.$$

We shall show that the three equilibrium points of equation (4.26) are locally asymptotically stable. Moreover we shall find a region of attraction for each one of them and a bound of the solution of equation (4.26).

We set $g(n) = f(n) + \varrho$ into equation (4.26) and we obtain, after some manipulation:

$$\begin{aligned}(\varrho^2+\varrho)f(n+3) + (\varrho^2-1)f(n+2) + (\varrho^2-\varrho)f(n+1)&\\ = -\, f(n+3)f(n+2)f(n+1) - \varrho f(n+3)f(n+2)&\\ - \varrho f(n+3)f(n+1) - \varrho f(n+2)f(n+1)&\\ - f(n+3)f(n) + f(n+1)f(n)\,.&\end{aligned} \tag{4.27}$$

For the equilibrium point $\varrho_1 = 0$ equation (4.27) becomes:

$$f(n+2) = f(n+3)f(n+2)f(n+1) + f(n+3)f(n) - f(n+1)f(n)\,. \tag{4.28}$$

This equation results from equation (3.3) for:

$$m = 2, \alpha_1 = \alpha_2 = 0, \beta_1(n) = \beta_2(n) = 0\,,$$

$$N = 2, c_1(n) = 1, c_2(n) = -1, q_{11} = 3, q_{12} = q_{22} = 0, q_{21} = 1, \gamma_1 = \gamma_2 = 1$$

$$M = 1, d_1(n) = 1, q_{13} = 3, q_{14} = 2, q_{15} = 1, \delta_1 = 1\,.$$

In this case the operator $\Gamma$ has the form:

$$\Gamma = I^2 = I \text{ and } \|\Gamma^{-1}\|_1 = 1 = L.$$

Also $P(R) = 1 - 2R - R^2$ and $P_1(R) = R - 2R^2 - R^3$.

It follows easily that the attraction region of the equilibrium point zero of equation (4.27), predicted by Theorem 3.1 is given by:

$$|f(0)| + |f(1)| < 0.11261179 \tag{4.29}$$

Thus $\varrho_1 = 0$ is a locally asymtotically stable equilibrium point of equation (4.26) and its region of attraction is given by:

$$|g(0)| + |g(1)| < 0.11261179 \tag{4.30}$$

Also $\{f(n)\}$ and thus $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| = |f(n)| < 0.215250437 \tag{4.31}$$

For the equilibrium point $\varrho_2 = 1$ equation (4.27) becomes:

$$\begin{aligned} f(n+3) = &-\frac{1}{2}f(n+3)f(n+2)f(n+1) - \frac{1}{2}f(n+3)f(n+2) \\ &-\frac{1}{2}f(n+3)f(n+1) - \frac{1}{2}f(n+2)f(n+1) \\ &-\frac{1}{2}f(n+3)f(n) + \frac{1}{2}f(n+1)f(n)\,. \end{aligned} \tag{4.32}$$

In a similar way it follows easily that the attraction region of the equilibrium point zero of equation (4.32), predicted by Theorem 3.1 is given by:

$$|f(0)| + |f(1)| + |f(2)| < 0.096322046 \tag{4.33}$$

Thus $\varrho_2 = 1$ is a locally asymtotically stable equilibrium point of equation (4.26) and its region of attraction is given by:

$$|g(0) - 1| + |g(1) - 1| + |g(2) - 1| < 0.096322046 \tag{4.34}$$

Also $\{f(n)\}$ and thus $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| = |f(n) + 1| \le |f(n)| + 1 < 1.189254787 \tag{4.35}$$

For the equilibrium point $\varrho_3 = -1$ equation (4.27) becomes:

$$\begin{aligned} f(n+1) = &-\frac{1}{2}f(n+3)f(n+2)f(n+1) + \frac{1}{2}f(n+3)f(n+2) \\ &+\frac{1}{2}f(n+3)f(n+1) + \frac{1}{2}f(n+2)f(n+1) \\ &-\frac{1}{2}f(n+3)f(n) + \frac{1}{2}f(n+1)f(n)\,. \end{aligned} \tag{4.36}$$

In a similar way it follows easily that the attraction region of the equilibrium point zero of equation (4.36), predicted by Theorem 3.1 is given by:

$$|f(0)| < 0.096322046 \tag{4.37}$$

Thus $\varrho_3 = -1$ is a locally asymtotically stable equilibrium point of equation (4.26) and its region of attraction is given by:

$$|g(0) + 1| < 0.096322046 \tag{4.38}$$

Also $\{f(n)\}$ and thus $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| = |f(n) - 1| \leq |f(n)| + 1 < 1.189254787 \tag{4.39}$$

**Remark 4.4.** In the case when equation (4.26) has positive solutions with positive initial conditions, it is proved recently in [1] that its positive equilibrium point is globally asymptotically stable.

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