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# ON–LINE PACKING REGULAR BOXES IN THE UNIT CUBE

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Abstract. We describe a class of boxes such that every sequence of boxes from this class of total volume smaller than or equal to 1 can be on-line packed in the unit cube.

Let $C$ be a subset of Euclidean $d$-space $E^d$ and let $(C_n)$ be a finite or infinite sequence of $d$-dimensional convex bodies. We say that $(C_n)$ *can be packed* in $C$ if there exist rigid motions $\sigma_i$ such that sets $\sigma_i C_i$, where $i = 1, 2, \ldots$, have pairwise disjoint interiors and are subsets of $C$. By an *on-line packing* we mean a packing in which we are given every $C_i$, where $i > 1$, only after the motion $\sigma_{i-1}$ has been provided. We are given $C_1$ at the beginning. In other words, in the on-line packing each set must be irreversibly put before the next set appears. A survey of results about packing (respectively: on-line packing) sequences of convex bodies is given in [1] (respectively: in [5]).

By a *box* we understand any set of the form

$$\{(x_1, \ldots, x_d);\ t_j \le x_j \le u_j \text{ for } j = 1, \ldots, d\},$$

where $t_j < u_j$ for $j = 1, \ldots, d$. The number $w_j = u_j - t_j$ is called the *$j$-th width* of this box. By the *unit cube $I^d$* we mean the set

$$\{(x_1, \ldots, x_d);\ 0 \le x_j \le 1 \text{ for } j = 1, \ldots, d\}.$$

The aim of this paper is to present a class of boxes such that each sequence of boxes from this class of total volume smaller than or equal to 1 can be on-line packed in the unit cube.

Let $q \ge 2$ be a positive integer. By a *$q$-regular box* we mean a box of the $j$-th widths of the form $w_j = q^{-m-1}$ for $j \le k$ and $w_j = q^{-m}$ for $j = k+1, \ldots, d$, where $k \in \{0, \ldots, d-1\}$, and where $m \in \{0, 1, \ldots\}$. If $k = 0$ in this formula, then such a $q$-regular box is called a *$q$-regular cube*.

In the paper [4] it is shown that every sequence of $q$-regular cubes of total volume not greater than 1 can be on-line packed in the unit cube. In Theorems 1 and 2 we generalize this result.

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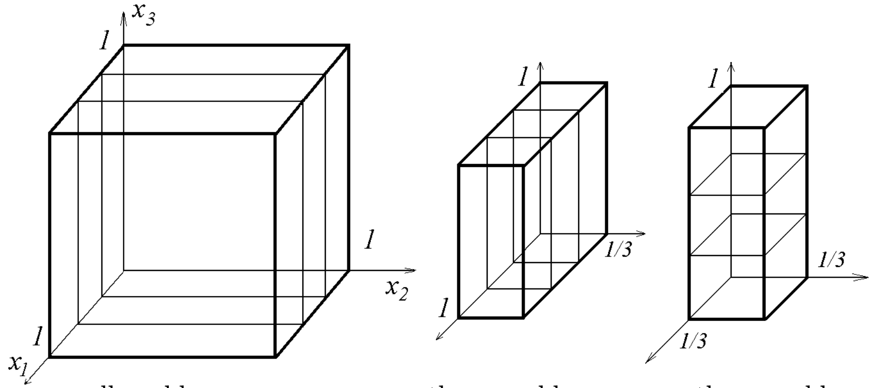
**Theorem 1.** *Let $q \geq 2$ be a fixed integer. Every sequence of $q$-regular boxes of total volume smaller than or equal to* 1 *can be on-line packed in the unit cube $I^d$.*

**Proof.** Let $(R_n)$ be a sequence of $q$-regular boxes of total volume not greater than 1. Let $m$ be a non-negative integer and let $k \in \{0, \ldots, d-1\}$. By a *subbox of type* $(m, k)$ (or by a *subbox*, for short) we mean the set

$$\{(x_1, \ldots, x_d);\ a_j q^{-m-1} \leq x_j \leq (a_j+1) q^{-m-1} \text{ for } j \leq k \\ \text{and } a_j q^{-m} \leq x_j \leq (a_j+1) q^{-m} \text{ for } j = k+1, \ldots d\},$$

where $a_j \in \{0, \ldots, q^{m+1}-1\}$ for $j \leq k$ and $a_j \in \{0, \ldots, q^m - 1\}$ for $j = k+1, \ldots, d$ (see Fig. 1, where $d = 3$, $q = 3$). Obviously, each subbox is a $q$-regular box.



all subboxes of type (0,1) three subboxes of type (0,2) three subboxes of type (1,0)

Fig. 1

We enumerate all the subboxes of type $(m, k)$ by integers $1, \ldots, q^{md+k}$ in such a way that:

(i) for $k \in \{1, \ldots, d-1\}$ the integers $(\lambda - 1)q + 1, \ldots, (\lambda - 1)q + q$ are given to the subboxes of type $(m, k)$ being subsets of the subbox of type $(m, k-1)$ whose number is $\lambda$,

(ii) for $m \geq 1$ the integers $(\mu - 1)q + 1, \ldots, (\mu - 1)q + q$ are given to the subboxes of type $(m, 0)$ being subsets of the subbox of type $(m-1, d-1)$ whose number is $\mu$.

Now, we describe the packing method. We pack $R_1$ in the first subbox congruent to it. Let $k > 1$. By a *k-free* subbox we mean a subbox whose interior has an empty intersection with $\sigma_1 R_1 \cup \cdots \cup \sigma_{k-1} R_{k-1}$. We pack each box $R_k$ from our sequence in the congruent $k$-free subbox with the smallest possible number.

We will show that $(R_n)$ can be on-line packed in $I^d$ by this method. Assume the opposite; let $R_i$ be a box from $(R_n)$ which cannot be packed in $I^d$. Obviously, there exists no $i$-free subbox congruent to $R_i$. Consider the family $\mathcal{R}$ of all $i$-free subboxes of maximal volume. (In other words, $S \in \mathcal{R}$ if and only if $S$ is $i$-free and if there does not exist an $i$-free subbox $S_1$ such that $S \subset S_1$ and $S \neq S_1$.) Subboxes from $\mathcal{R}$ have the volumes of the form $q^{-1}$ Vol$(R_i)$, $q^{-2}$ Vol$(R_i), \dots$.

We show that there are at most $q-1$ subboxes of a fixed type in $\mathcal{R}$. Assume the opposite: there are at least $q$ subboxes of type $(m,k)$ in $\mathcal{R}$. Let us denote these subboxes by $Q_1, \dots, Q_z$. Consider the case when $k \geq 1$. From the description of the packing method we conclude that $q$ subboxes from among $Q_1, \dots, Q_z$ are contained in an $i$-free subbox of type $(m, k-1)$. Consider the case when $k=0$. We can assume that $m \geq 1$, because the box of type $(0,0)$ is nothing else but $I^d$. In this case $q$ subboxes from among $Q_1, \dots, Q_z$ are contained in an $i$-free subbox of type $(m-1, d-1)$. From the above consideration we conclude that $q$ subboxes from among $Q_1, \dots, Q_z$ do not belong to $\mathcal{R}$ because they are not maximal, a contradiction.

A finite number of boxes has been packed in $I^d$. Consequently, there exists a finite number of subboxes in $\mathcal{R}$. This means that the total volume of subboxes in $\mathcal{R}$ is smaller than

$$\mathrm{Vol}(R_i)\big[(q-1)q^{-1} + (q-1)q^{-2} + \dots\big].$$

This value is equal to Vol$(R_i)$. Hence, the total volume of boxes $R_1, \dots, R_{i-1}$ is greater than $1-$ Vol$(R_i)$. Consequently, the total volume of boxes in $(R_n)$ is greater than 1, a contradiction. □

**Remark.** Let $(S_n)$ be a sequence of boxes. The method of packing of $(S_n)$ is called *$q$-adic* (see [2-4]), if for each positive integer $n$, $\sigma_n S_n$ has edges parallel to the axes of the coordinate system and if for each $j \in \{1, \dots, d\}$ the projection of $\sigma_n S_n$ on the $j$-th axis is a segment whose both endpoints are multiples of the $j$-th width of $S_n$. Observe, that the packing method from Theorem 1 is $q$-adic.

Obviously, the estimate 1 in Theorem 1 cannot be improved. It is an open question how to extend our class of $q$-regular boxes. For example, we cannot add the cube of the width $\frac{1}{2}$ to the class of 4-regular boxes. The reason is that one cube of the width $\frac{1}{2}$ and three boxes of the widths $w_1 = \frac{1}{4}$ and $w_2 = \dots = w_d = 1$ cannot be packed in $I^d$. We show in Propositions 1 and 2 that some extensions are possible. Probably, the class of $q$-regular boxes is however the best possible in the sense given in Conjecture.

**Proposition 1.** *Let $q \geq 2$ be an integer. Moreover, let $\mathcal{F}$ be the family of boxes such that each box $B \in \mathcal{F}$ is either $q$-regular or the widths of $B$ are of the form $w_1 = nq^{-1}$, $w_2 = \dots = w_d = 1$, where $n \in \{2, \dots, q\}$. Then each sequence of boxes from $\mathcal{F}$ of total volume smaller than or equal to 1 can be on-line packed in $I^d$.*

**Proof.** We proceed analogously as in the proof of Theorem 1. We can regard a box $B$ with $w_1 = nq^{-1}$, $w_2 = \dots = w_d = 1$ as the union of $n$ $q$-regular boxes with $w_1 = q^{-1}$, $w_2 = \dots = w_d = 1$. We pack such a box $B$ in $I^d$ similarly like in the method from Theorem 1. Just in the first free place. If such a box

$B$ cannot be packed, then the total volume of boxes preceding $B$ is greater than $1 - nq^{-1} = 1 - \text{Vol}(B)$. □

Let $q \geq 2$ and let $p_1, \dots, p_d$ be positive integers. By a $(q, p_1, \dots, p_d)$*-regular box* we mean a box of the widths of the form $w_j = p_j^{-1}q^{-m-1}$ for $j \leq k$ and $w_j = p_j^{-1}q^{-m}$ for $j = k+1, \dots, d$, where $k \in \{0, \dots, d-1\}$, and where $m \in \{0, 1, \dots\}$. Denote by $B_p$ the box

$$\{(x_1, \dots, x_d);\ 0 \leq x_j \leq p_j^{-1} \text{ for } j = 1, \dots, d\}.$$

Obviously, there is an affine image $T(I^d)$ equal to $B_p$. Observe that if a box $B$ is $q$-regular, then the affine image $T(B)$ is $(q, p_1, \dots, p_d)$-regular. Thus, from Theorem 1 we conclude that each sequence of $(q, p_1, \dots, p_d)$-regular boxes of total volume smaller than or equal to $\prod_{i=1}^{d} p_i^{-1}$ can be on-line packed in $B_p$. Consequently, we obtain the following result.

**Proposition 2.** *Every sequence of* $(q, p_1, \dots, p_d)$*-regular boxes of total volume smaller than or equal to* 1 *can be on-line packed in the unit cube.*

Denote by $w(B)$ the greatest width of a box $B$.

**Conjecture.** Let $\mathcal{F}$ be a family of boxes such that: $(i)$ $\mathcal{F}$ contains a cube; $(ii)$ for each $\epsilon > 0$ and for each box $B \in \mathcal{F}$ there exists a homotetic copy $k_1B$ of $B$ such that $k_1B \in \mathcal{F}$ and $w(k_1B) < \epsilon$; $(iii)$ for each box $B \in \mathcal{F}$ there exists a homotetic copy $k_2B$ such that $k_2B \in \mathcal{F}$ and $w(k_2B) = 1$, $(iv)$ each sequence of boxes from $\mathcal{F}$ of total volume not greater than 1 can be on-line packed in $I^d$. Then there exists an integer $q \geq 2$ such that all the boxes from $\mathcal{F}$ are $q$-regular.

Another interesting question is about the conection between usual packing and on-line packing in the unit cube sequences of boxes of total volume not greater than 1.

**Problem 1.** Let $\mathcal{F}$ be a family of boxes such that the conditions $(i) - (iii)$ from Conjecture are satisfied and such that each sequence of boxes from $\mathcal{F}$ of total volume not greater than 1 can be packed in $I^d$. Does there exist an integer $q \geq 2$ such that all the boxes from $\mathcal{F}$ are $q$-regular?

**Problem 2.** Let $\mathcal{F}$ be a family of boxes such that each sequence of boxes from $\mathcal{F}$ of total volume not greater than 1 can be packed in $I^d$. Let $(S_n)$ be a sequence of boxes from $\mathcal{F}$ of total volume smaller than or equal to 1. Can $(S_n)$ be on-line packed in $I^d$?

Finally, we present a theorem about packing another class of boxes. Let $p_1, \dots, p_d$ and $q_1, \dots, q_d$ be positive integers. Let $m \in \{0, 1, \dots\}$. By $(p_1, q_1, \dots, p_d, q_d)$*-regular box* we mean a box of the $j$-th widths of the form $w_j = p_j^{-1}q_j^{-m}$, for $j = 1, \dots, d$.

**Theorem 2.** *Every sequence of* $(p_1, q_1, \ldots, p_d, q_d)$*-regular boxes of total volume not greater than 1 can be on-line packed in the unit cube.*

**Proof.** The proof is similar to the proof of Theorem 1. We can divide the unit cube into regular subboxes. Let $m \in \{0, 1, \ldots\}$. By a *regular subbox of size* $m$ we mean the set

$$\{(x_1, \ldots, x_d);\ a_j p_j^{-1} q_j^{-m} \le x_j \le (a_j + 1) p_j^{-1} q_j^{-m} \ \text{ for } j = 1, \ldots, d\},$$

where $a_j \in \{0, \ldots, p_j q_j^m - 1\}$. We enumerate all the subboxes. The subboxes of size 0 are enumerated from 1 to $\prod_{i=1}^{d} p_j$. We enumerate other subboxes in such a way that the integers $(\lambda - 1) \prod_{i=1}^{d} q_i + 1, \ldots, \lambda \prod_{i=1}^{d} q_i$ are given to the subboxes of size $m \ge 1$ being subsets of the subbox of size $m - 1$ whose number is $\lambda$.

Let $(S_n)$ be a sequence of $(p_1, q_1, \ldots, p_d, q_d)$-regular boxes of total volume smaller than or equal to 1. We pack $S_1$ in the first subbox congruent to it. Let $k > 1$. By a *k-free* subbox we mean a subbox with such a property that no interior point of it is covered by $\sigma_1 S_1 \cup \cdots \cup \sigma_{k-1} S_{k-1}$. We pack each box $S_k$ from our sequence in the congruent $k$-free subbox with the smallest possible number. We can now proceed analogously to the proof of Theorem 1. Consequently, $(S_n)$ can be on-line packed in $I^d$. □

## References


[1] Groemer, H., *Covering and packing by sequences of convex sets*, Discrete Geometry and Convexity, Annals of the New York Academy of Science **440** (1985), 262-278.

[2] Januszewski, J., Lassak, M., Rote, G., Woeginger, G., *On-line q-adic covering by the method of the n-th segment and its application to on-line covering by cubes*, Beitr. Alg. Geom. **37** (1996) No. 1, 51-56.

[3] Kuperberg, W., *Problem 74: Ein Intervallüberdeckungsspiel*, Math. Semesterber. **41** (1994), 207-210.

[4] Lassak, M., *On-line packing sequences of segments, cubes and boxes*, Beitr. Alg. Geom. **38** (1997), 377–384.

[5] Lassak, M., *A survey of algorithms for on-line packing and covering by sequences of convex bodies*, Bolyai Society Mathematical Studies **6** (1997), 129–157.



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