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Singular Integral Inequalities and Stability of Semilinear Parabolic Equations

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Abstract. Using a method developed by the author for an analysis of singular integral inequalities a stability theorem for semilinear parabolic PDEs is proved.

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1 Introduction

Integral inequalities play an important role in the theory of differential, integral and integrodifferential equations. One can hardly imagine these theories without the well-known Gronwall inequality and its nonlinear version Bihari inequality [1]. However these inequalities are not directly applicable to integral equations with weakly singular kernels of the form

$$x(t) = \xi(t) + \int_0^t K(t,s) f(s, x(s)) ds, \ x \in X, \tag{1}$$

where X is a Banach space, $K(t,s) : X \to X$ is a linear operator satisfying the condition

$$||K(t,s)|| \leqq \frac{M}{(t-s)^\alpha} ||v||, \ v \in X, \tag{2}$$

for $t > s \geqq 0$, $\alpha > 0, M > 0$ are constants, ξ, f are continuous maps. Such equations appear e.g. in the geometric theory of parabolic differential equations. Basics of this theory are described in the well-known book by D. Henry [4] (see also the book by J. K. Hale [3]).

Many boundary value problems for parabolic PDEs can be written as a Cauchy initial value problem

$$\begin{gathered}\frac{du}{dt} + Au = f(t,u),\ u \in X,\\ u(0) = u_0 \in X,\end{gathered} \tag{3}$$

where X is an appropriate Banach space and $A : X \to X$ is a special linear operator, so called sectorial operator (for the definition see [4, Definition 1.3.1]). For any sectorial operator A there is a real number c such that if $A_1 = A+cI$, where I is the identity mapping, then $\mathrm{Re}\ \sigma(A_1) > 0$ (i.e. $\mathrm{Re}\ \lambda > 0$ for any $\lambda \in \sigma(A_1)$ — the spectrum of the operator A_1). One can define a fractional power A_1^α of A_1 as the inverse of $A_1^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-A_1 t} dt$ for $\alpha > 0$. If $X^\alpha := D(A_1^\alpha)$ — the domain of A_1^α and $||x||_\alpha := ||A_1^\alpha x||, x \in X^\alpha$, then $(X^\alpha, ||.||_\alpha)$ is a Banach space (see [4]).

By [4, Theorem 1.3.4], if A is a sectorial operator then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t \geqq 0}$, $\frac{d}{dt} e^{-tA} = -Ae^{-tA}$ for $t > 0$ and if $\mathrm{Re}\ \sigma(A) > b > 0$ then

$$||e^{-tA}u||_\alpha := ||A_1^\alpha e^{-tA} u|| \leqq \frac{d}{t^\alpha} e^{-bt} ||u||,\ t > 0 \tag{4}$$

for any $u \in X^\alpha$, where $d > 0$ is a constant.

Definition 1 (see [3] and [7]). Let $A : X \to X$ be a sectorial operator and there is an $\alpha \in \langle 0, 1)$ such that the map $f : R \times X^\alpha \to X, (t,u) \mapsto f(t,u)$ is locally Hölder in t and locally Lipschitz in u. A solution of (3) on the interval $\langle 0, T)$ $(0 < T \leqq \infty)$ is a continuous function $u : \langle 0, T) \to X^\alpha$ with $u(0) = u_0 \in X^\alpha$ such that the map $f(., u(.)) : \langle 0, T) \to X, t \to f(t, u(t))$ is continuous, $u(t) \in D(A), t \in \langle 0, T)$ and u satisfies (3) on $(0, T)$.

By M. Miklavčič [7] a solution $u(t)$ of (3) in the sense of Definition 1 coincides with those solutions of the integral equations

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s, u(s)) ds,\ 0 < t \leqq T, \tag{5}$$

for which $u : \langle 0, T) \to X^\alpha$ is continuous and $f(., u(.)) : \langle 0, T) \to X, t \to f(t, u(t))$ is continuous.

If $\mathrm{Re}\ \sigma(A) > b > 0$ then from (4), (5) it follows that

$$||u(t)||_\alpha \leqq \frac{ce^{-bt}}{t^\alpha} ||u_0|| + de^{-bt} \int_0^t \frac{e^{bt}}{(t-s)^\alpha} ||f(s, u(s))|| ds. \tag{6}$$

If
$$||f(v)|| \leqq Q||v||_{\alpha}, v \in X^{\alpha}, \tag{7}$$
$a(t) = \frac{c}{t^{\alpha}}||u_0||, v(t) = ||u(t)||_{\alpha}e^{bt}$ and $c = dQ$, then (6) yields
$$v(t) \leqq a(t) + c\int_0^t (t-s)^{\beta-1}v(s)ds,\ t \in I = \langle 0, T),\tag{8}$$
where $\beta = 1-\alpha,\ \alpha > 0$. By [4, Lemma 7.1.1]
$$v(t) \leqq \Theta \int_0^t E'_{\beta}(\Theta(t-s))a(s)ds,\ t \in I,\tag{9}$$
where $\Theta = (c\Gamma(\beta))^{\frac{1}{\beta}}, R_{\beta}(z) = \sum_{n=0}^{\infty}\frac{z^{n\beta}}{\Gamma(n\beta+1)}, \Gamma$ is the gamma-function and finally $E'_{\beta}(z) = \frac{dE_{\beta}(z)}{dz}$.

The estimate (9) is obviously complicated and it is obtained in [4] by an iterative argument not applicable to the case of nonlinear integral inequalities. In the paper [6] the author developed a new method of a reduction of the inequality (8) as well as some nonlinear singular inequalities to the classical Gronwall and Bihari inequalities, respectively. Using this method we shall analyze an inequality of the form
$$\psi(t) \leqq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\psi(s)^m ds,\ t \in I = \langle 0, T),\tag{10}$$
where $0 < T \leqq \infty$ and $m > 1$ with the aim to prove a stability theorem for the equation (3).

2 Stability theorem

First let us formulate a consequence of a result by G. Butler and T. Rogers published in [2] (see also [5, Theorem 1.3.8]) as the following lemma.

Lemma 2. *Let $a(t), b(t), K(t), \psi(t)$ be nonnegative, continuous function on $I = \langle 0, T)\,(0 < T \leqq \infty), \omega : \langle 0, \infty) \to R$ be a continuous, nonnegative and nondecreasing function, $\omega(0) = 0$, $\omega(u) > 0$ for $u > 0$ and let $A(t) = \max_{0\leqq s\leqq t} a(s)$, $B(t) = \max_{0\leqq s\leqq t} b(s)$. Assume that*
$$\psi(t) \leqq a(t) + b(t)\int_0^t K(s)\omega(\psi(s))ds, t \in I.\tag{11}$$
Then
$$\psi(t) \leqq \Omega^{-1}[\Omega(A(t)) + B(t)\int_0^t K(s)ds],\ t \in \langle 0, T_1\rangle,\tag{12}$$
where $\Omega(v) = \int_{v_0}^{v}\frac{d\sigma}{\omega(\sigma)}, v \geqq v_0 > 0, \Omega^{-1}$ is the inverse of Ω and $T_1 > 0$ is such that $\Omega(A(t)) + B(t)\int_0^T K(s)ds \in D(\Omega^{-1})$ for all $t \in \langle 0, T_1\rangle$.

Lemma 3. *Let $a(t), F(t), \psi(t), b(t)$ be continuous, nonnegative functions on $I = \langle 0, T)$ $(0 < T \leqq \infty), \beta > 0, \gamma > 0, m > 1$ and $\psi(t)$ satisfies the inequality (10). Then the following assertions hold:*

(1) *If $\beta > \frac{1}{2}, \gamma > 1 - \frac{1}{2p}$ for some $p > 1$ and $\varepsilon > 0$ then*
$$\psi(t) \leqq e^{\varepsilon t} \Phi_\varepsilon(t), \tag{13}$$
where $\Phi_\varepsilon(t) = A_1(t)^{\frac{1}{2q}}[1 - (m-1)\Xi_1(t, \varepsilon)]^{\frac{1}{2q(1-m)}}$,
$\Xi_1(t, \varepsilon) = A_1(t)^{m-1} B_1(t, \varepsilon) \int_0^t F(s)^{2q} e^{2qm\varepsilon s} ds$,
$A_1(t) = 2^{2q-1} \max_{0 \leqq s \leqq t} a(s)^{2q}$,
$B_1(t, \varepsilon) = 2^{2q-1} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} \max_{0 \leqq s \leqq t} b(s)^{2q}, K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}$,
$L(\varepsilon) = \frac{\Gamma((2\gamma-2)p+1)}{(p\varepsilon)^{(2\gamma-2)p+1}}, \frac{1}{p} + \frac{1}{q} = 1$ and $t \in I$ is such that $\Phi_\varepsilon(t)$ is defined.

(2) *Let $\beta = \frac{1}{1+z}$ for some $z \geqq 1, \gamma > 1 - \frac{1}{kq}$, where $k > 0, q = z + 2$ and let $\varepsilon > 0$. Then*
$$\psi(t) \leqq e^{\varepsilon t} \Psi_\varepsilon(t), \tag{14}$$
where $\Psi_\varepsilon(t) = A_2(t)^{\frac{1}{rq}}[1 - (m-1)\Xi_2(t, \varepsilon)]^{\frac{1}{rq(1-m}}$,
$\Xi_2(t, \varepsilon) = A_2(t)^{m-1} B_2(t, \varepsilon) \int_0^t F(s)^{rq} e^{mqr\varepsilon s} ds$,
$A_2(t) = 2^{rq-1} \max_{0 \leqq s \leqq t} a(s)^{rq}$,
$B_2(t, \varepsilon) = 2^{rq-1} P(\varepsilon) \max_{0 \leqq s \leqq t} b(s)^{rq}$,
$P(\varepsilon) = (M(\varepsilon)N(\varepsilon))^{rq}, M(\varepsilon) = [\frac{\Gamma(1-\alpha p)}{(p\varepsilon)^{1-\alpha p)}}]^{\frac{1}{p}}$,
$N(\varepsilon) = [\frac{\Gamma(kq(\gamma-1)+1)}{(kq\varepsilon)^{kq(\gamma-1)+1}}]^{\frac{1}{kq}}, \alpha = 1 - \beta, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{k} + \frac{1}{r} = 1, p, q, r, k > 1$ and $t \in I$ is such that $\Psi_\varepsilon(t)$ is defined.

Proof. We shall repeat the same procedure as in the proof of [6, Theorem 4] however instead of inserting $e^t.e^{-t}$ into the integral on the right-hand side of (10) and then applying the Cauchy-Schwarz and Hölder inequality, respectively, we shall insert $e^\varepsilon.e^{-\varepsilon t}$ there. More precisely, under the assumption of the assertion (1) we obtain from (10) that
$$\psi(t) \leqq a(t) + b(t)[\int_0^t (t-s)^{2\beta-2} e^{2\varepsilon s} ds]^{\frac{1}{2}} [\int_0^t s^{2\gamma-2} F(s)^2 e^{-2\varepsilon s} \psi(s)^{2m} ds]^{\frac{1}{2}} \leqq$$
$$\leqq a(t) + b(t) e^{\varepsilon t} K(\varepsilon)^{\frac{1}{2}} [\int_0^t s^{2\gamma-2} F(s)^2 e^{-2\varepsilon s} \psi(s)^{2m} ds]^{\frac{1}{2}},$$
where $K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}$. Using the Hölder inequality with $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ we obtain
$$\psi(t) \leqq a(t) + b(t) e^{\varepsilon t} K(\varepsilon)^{\frac{1}{2}} [\int_0^t s^{(2\gamma-2)p} e^{-\varepsilon ps} ds]^{\frac{1}{2p}} [\int_0^t F(s)^{2q} \psi(s)^{2mq} ds]^{\frac{1}{2q}}$$

and since

$$\int_0^t s^{(2\gamma-2)p} e^{-p\varepsilon s} ds = \frac{1}{(p\varepsilon)^{(2\gamma-2)p+1}} \int_0^{p\varepsilon t} \sigma^{(2\gamma-2)p} e^{-\sigma} d\sigma < \\ < \frac{\Gamma((2\gamma-2)p+1)}{(p\varepsilon)^{(2\gamma-2)p+1}} := L(\varepsilon)$$

$((2\gamma-2)p+1 > [2(1-\frac{1}{2p})-2]p+1 > 0$, i.e. $\Gamma((2p-2)p+1)$ is a positive number) we have

$$\psi(t) \leqq a(t) + b(t)e^{\varepsilon t} K(\varepsilon)^{\frac{1}{2}} L(\varepsilon)^{\frac{1}{2p}} [\int_0^t F(s)^{2q} \psi(s)^{2mq} ds]^{\frac{1}{2q}}. \tag{15}$$

Since $(A_1 + A_2)^r \leqq 2^{r-1}(A_1^r + A_2^r)$ for any nonnegative real numbers A_1, A_2 and any real number $r > 1$ (see [6, (2), (3)]) we obtain from (15) that

$$\psi(t)^{2q} \leqq 2^{2q-1}[a(t)^{2q} + b(t)^{2q} e^{2q\varepsilon t} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} \int_0^t F(s)^{2q} e^{2qm\varepsilon s} e^{-2q\varepsilon s} \psi(s)^{2q})^m ds]. \tag{16}$$

If

$$v(t) = e^{-2q\varepsilon t} u(t)^{2q}, c(t) = 2^{2q-1} a(t)^{2q}, d(t) = 2^{2q-1} b(t)^{2q} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}}, \tag{17}$$

then (16) yields

$$v(t) \leqq c(t) + d(t) \int_0^t F(s)^{2q} e^{2qm\varepsilon s} v(s)^m ds.$$

Now we can apply Lemma 2, where $\omega(u) = u^m, \Omega(v) = \int_{v_0}^v \frac{dy}{\omega(y)} = \int_{v_0}^v y^{-m} dy = \frac{1}{m-1}[v^{1-m} - v_0^{1-m}], \Omega^{-1}(z) = [(1-m)z + v_0^{1-m}]^{\frac{1}{1-m}}$ and we obtain the inequality

$$v(t) \leqq \Omega^{-1}[\Omega(A_1(t)) + B_1(t,\varepsilon) \int_0^t F(s)^{2q} e^{2qm\varepsilon s} ds] = \\ = A_1(t)[1-(m-1)\Xi_1(t,\varepsilon)]^{\frac{1}{1-m}},$$

where $\Xi_1(t,\varepsilon), A_1(t), B_1(t,\varepsilon)$ are as in theorem. From this inequality and (17) the inequality (13) follows.

The proof of the inequality (14) is similar (see the proof of [6, Theorem 4]).

Theorem 4. *Let $A : X \to X$ be a sectorial operator, Re $\sigma(A) > b > 0, f$ be as in Definition 1 and let*

$$||f(t,u)|| \leqq t^\kappa \eta(t) ||u||_\alpha^m, \quad m > 1, \ \kappa \geqq 0 \tag{18}$$

for all $(t,u) \in R \times X^\alpha$, where $\eta : \langle 0, \infty) \to R$ is a continuous, nonnegative function. Then the following assertions hold:

(1) *Let* $0 < \alpha < \min\{\frac{1}{2}, \frac{\kappa}{m} + \frac{1}{2pm}\}$ *for some* $p > 1$ *and* $b > 0$ *be the number from the inequality* (4). *Let the function*

$$t \mapsto t^{2q\alpha} \int_0^t \eta(s)^{2q} e^{2q[(1-m)b+m\varepsilon]s} ds$$

is bounded on the interval $\langle 0, \infty)$ *for some* $0 < \varepsilon < b$, *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $u(t)$ *be a solution of the equation* (3) *satisfying* $u(0) = u_0 \in X^\alpha$, *where*

$$(m-1)2^{2q-1}(c||u_0||)^{2q(m-1)} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} (ct^\alpha)^{2q} \int_0^t \eta^{2q} e^{2q[(1-m)b+m\varepsilon]s} ds < 1,$$

where

$$K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}, L(\varepsilon) = \frac{\Gamma((2\gamma-2)p+1)}{(2\gamma-2)p+1}, \beta = 1-\alpha.$$

Then $u(t)$ *exists on the interval* $\langle 0, \infty)$ *and* $\lim_{t\to\infty} ||u(t)||_\alpha = 0$.

(2) *Let* $\frac{1}{2} \leqq \alpha < \min\{1, \frac{\kappa}{m} + \frac{1}{kqm}\}$ *for some* $k > 1$, *where* $\beta = 1 - \alpha = \frac{1}{1+z}$, $z \geqq 1$, $q = z + 2$ *and* $b > 0$ *is the number from the inequality* (4). *Assume that the function*

$$t \mapsto t^{rq\alpha} \int_0^t \eta(s)^{rq} e^{rq[(1-m)b+m\varepsilon]s} ds$$

is bounded on the interval $\langle 0, \infty)$ *for some* $0 < \varepsilon < b$, *where* $\frac{1}{k} + \frac{1}{r} = 1$. *Let* $u(t)$ *be a solution of the equation* (3) *satisfying* $u(0) = u_0$, *where*

$$(m-1)2^{rqm}(c||u_0||)^{rq(m-1)} P(\varepsilon) t^{rq\alpha} \int_0^t \eta(s)^{rq[(1-m)b+m\varepsilon]s} ds \begin{cases} < 1 & \text{for } rq(m-1) \text{ even}, \\ \neq 1 & \text{for } rq(m-1) \text{ odd}, \end{cases}$$

where $P(\varepsilon)$ *is the number defined in Lemma* 3. *Then* $u(t)$ *exists on the interval* $\langle 0, \infty)$ *and* $\lim_{t\to\infty} ||u(t)||_\alpha = 0$.

Proof. Under the assumptions of theorem there exists a solution of the equation (3) on an interval $I = \langle 0, T)(0 < T \leqq \infty)$ satisfying the condition $u(0) = u_0$. This solution satisfies the equation (5) and for $\alpha > 0$ the inequality (6) is satisfied. This inequality and the condition (18) yield

$$||u(t)||_\alpha \leqq \frac{ce^{-bt}}{t^\alpha}||u_0|| + ce^{-bt}\int_0^t \frac{e^{bs}s^\kappa \eta(s)}{(t-s)^\alpha}||u(s)||_\alpha^m ds, \quad t > 0$$

and if $\psi(t) = e^{bt}t^\alpha ||u(t)||_\alpha$ then

$$\psi(t) \leqq a(t) + b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\psi(s)^m ds, \tag{19}$$

where $a(t) = c||u_0||, b(t) = ct^{\alpha}, \beta = 1-\alpha, \kappa = 1+\kappa-\alpha m, F(t) = e^{(1-m)bt}\eta(t)$.

Let us prove the assertion (1). From the assumption it follows that $\alpha < \frac{1}{2}$, i.e. $\beta = 1-\alpha > \frac{1}{2}$ and $-\alpha m > -\kappa - \frac{1}{2p}$, i.e. $\gamma = 1+\kappa-\alpha m > 1-\frac{1}{2p}$. Thus the assumptions of Lemma 3 are satisfied. By the assertion (1) of this lemma we obtain that $\psi(t) \leqq e^{\varepsilon t}\Phi(t,\varepsilon)$, where

$$\Phi(t,\varepsilon) = A_1(t)^{\frac{1}{2q}}[1-(m-1)\Xi_1(t,\varepsilon)]^{\frac{1}{2q(1-m)}},$$

$$A_1(t,\varepsilon) = 2^{2q-1}(c||u_0||)^{2q},$$

$$\Xi_1(t,\varepsilon) = 2^{2q(m-1)}(c||u_0||)^{2q(m-1)}K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}}.t^{2q\alpha}\int_0^t \eta(s)^{2q}e^{2q[(1-m)b+m\varepsilon]s}ds,$$

$K(\varepsilon), L(\varepsilon)$ are defined in Lemma 3. Under the assumptions of theorem the function $\Phi(t,\varepsilon)$ is bounded on the interval $(0,\infty)$. Since $\psi(t) = e^{bt}t^{\alpha}||u(t)||_{\alpha}$, $0<\varepsilon<b$, we obtain that

$$||u(t)||_{\alpha} \leqq \frac{e^{-(b-\varepsilon)t}}{t^{\alpha}}\Phi(t,\varepsilon).$$

Thus the solution $u(t)$ of (3) exists on the interval $\langle 0,\infty)$ and $\lim_{t\to\infty}||u(t)||_{\alpha} = 0$. From the assumption of the assertion (2) it follows that $\beta = 1-\alpha \leqq \frac{1}{2}, -\alpha m > -\kappa - \frac{1}{kq}$, i.e. $\gamma = 1+\kappa-\alpha m > 1-\frac{1}{kq}$ and thus the assumptions of the assertion (2) of Lemma 3 are satisfied. Applying this lemma in the same way as in the proof of the assertion (1) one can prove the assertion (2).

Remark 5. M. Miklavčič in his paper [7] proved that if for some $0 < \omega \leqq 1$, $0 < \alpha < 1$, $\alpha\omega p > 1$, $\gamma > 1$, $C > 0$, $||t^{\omega}Ae^{-At}|| \leqq C$, $t \geqq 1$,

$$||f(t,x)|| \leqq C[||A^{\alpha}x||^p + (1+t)^{-\gamma}], \quad t \geqq 0,$$

whenever $||A^{\alpha}x|| + ||x||$ is small enough, then for small initial data there exist stable global solutions. Moreover, if the space X is reflexive (in this case $X = N(A) \oplus \overline{R(A)}$), then there exists $y \in N(A)$ such that $\lim_{t\to\infty}||x(t)-y||_{\alpha} = 0$. These results are obviously proved under different assumptions from those in our theorem.

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