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The Boundary-Value Problems for Laplace Equation and Domains with Nonsmooth Boundary

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Abstract. Dirichlet, Neumann and Robin problem for the Laplace equation is investigated on the open set with holes and nonsmooth boundary. The solutions are looked for in the form of a double layer potential and a single layer potential. The measure, the potential of which is a solution of the boundary-value problem, is constructed.

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Suppose that $G \subset R^m$ $(m \geq 2)$ is an open set with a non-void compact boundary ∂G such that $\partial G = \partial(\mathrm{cl}\; G)$, where cl G is the closure of G. Fix a nonnegative element λ of $\mathcal{C}'(\partial G)$ (the Banach space of all finite signed Borel measures supported in ∂G with the total variation as a norm) and suppose that the single layer potential $\mathcal{U}\lambda$ is bounded and continuous on ∂G. (In R^2 it means that $\lambda = 0$. If $G \subset R^m, (m > 2), \partial G$ is locally Lipschitz, $\lambda = f\mathcal{H}$, where $\mathcal{H}$ is the surface measure on ∂G and f is a nonnegative bounded measurable function, then $\mathcal{U}\lambda$ is bounded and continuous.) Here

$$\mathcal{U}\nu(x) = \int_{R^m} h_x(y)\; d\nu(y),$$

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where $\nu \in \mathcal{C}'(\partial G)$,

$$h_x(y) = (m-2)^{-1} A^{-1} |x-y|^{2-m} \text{ for } m > 2,$$

$$A^{-1} \log |x-y|^{-1} \text{ for } m = 2,$$

A is the area of the unit sphere in R^m.

If G has a smooth boundary,$u \in \mathcal{C}^1(\text{cl } G)$ is a harmonic function on G and

$$\frac{\partial u}{\partial n} + fu = g \text{ on } \partial G,$$

where $f, g \in \mathcal{C}(\partial G)$ (the space of all bounded continuous functions on ∂G equipped with the maximum norm) and n is the exterior unit normal of G then for $\phi \in \mathcal{D}$ (the space of all compactly supported infinitely differentiable functions in R^m)

$$\int_{\partial G} \phi g \; d\mathcal{H}_{m-1} = \int_G \nabla \; \phi \cdot \nabla \; u \; d\mathcal{H}_m + \int_{\partial G} \phi f u \; d\mathcal{H}_{m-1}. \tag{1}$$

Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized such that $\mathcal{H}_k$ is the Lebesgue measure in R^k. If we denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ on ∂G and by $N^G u$ the distribution

$$\langle \phi, N^G u \rangle = \int_G \nabla \phi \cdot \nabla u \; d\mathcal{H}_m \tag{2}$$

then (1) has the form

$$N^G u + fu\mathcal{H} = g\mathcal{H}. \tag{3}$$

Here $N^G u$ is a characterization in the sense of distributions of the normal derivative of u.

The formula (3) motivates the following definition of the solution of the Robin problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \text{ in } G, \\ N^G u + u\lambda &= \mu, \end{aligned} \tag{4}$$

where $\mu \in \mathcal{C}'(\partial G)$.

We introduce in R^m the fine topology, i.e. the weakest topology in which all superharmonic functions in R^m are continuous. This topology is stronger than the ordinary topology.

If u is a harmonic function on G such that

$$\int_H |\nabla u| \; d\mathcal{H}_m < \infty \tag{5}$$

for all bounded open subsets H of G we define the weak normal derivative $N^G u$ of u as a distribution

$$\langle \varphi, N^G u \rangle = \int_G \nabla\varphi \cdot \nabla u \, d\mathcal{H}_m$$

for $\varphi \in \mathcal{D}$.

Let $\mu \in \mathcal{C}'(\partial G)$.Now we formulate the Robin problem for the Laplace equation (4) as follows: Find a function $u \in L^1(\lambda)$ on cl G,the closure of G, harmonic on G and fine continuous in λ-a. a. points of ∂G for which ∇u is integrable over all bounded open subsets of G and $N^G u + u\lambda = \mu$.

The single layer potential $\mathcal{U}\nu$, where $\nu \in \mathcal{C}'(\partial G)$, has all these properties and if we look for a solution of the Robin problem in the form of the single layer potential we obtain the equation

$$N^G \mathcal{U}\nu + (\mathcal{U}\nu)\lambda = \mu.$$

It was shown by J. Král for $\lambda = 0$ (see [10]) and independently by Yu. D. Burago, V. G. Maz'ya (see [2]) and by I. Netuka ([20]) for a general λ that $N^G\mathcal{U}\nu + (\mathcal{U}\nu)\lambda \in \mathcal{C}'(\partial G)$ for each $\nu \in \mathcal{C}'(\partial G)$ if and only if $V^G < \infty$,where

$$V^G = \sup_{x \in \partial G} v^G(x),$$

$$v^G(x) = \sup\{\int_G \nabla\ \phi \cdot \nabla\ h_x \ d\mathcal{H}_m; \phi \in \mathcal{D}, |\phi| \leq 1, spt\ \phi \subset R^m - \{x\}\}.$$

There are more geometrical characterizations of $v^G(x)$ which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \cup_{i=1}^k L_i$,where L_i are $(m-1)$-dimensional Ljapunov surfaces (i.e. of class $\mathcal{C}^{1+\alpha}$). Denote

$$\partial_e G = \{x \in R^m; \bar{d}_G(x) > 0, \bar{d}_{R^m - G}(x) > 0\}$$

the essential boundary of G, where

$$\bar{d}_M(x) = \limsup_{r \to 0_+} \frac{\mathcal{H}_m(M \cap U(x;r))}{\mathcal{H}_m(U(x;r))}$$

is the upper density of M at x, $U(x;r)$ is the open ball with the centre x and the radius r. Then

$$v^G(x) = \frac{1}{A} \int_{\partial U(0;1)} n(\theta, x)\ d\mathcal{H}_{m-1}(\theta),$$

where $n(\theta, x)$ is the number of all points of $\partial_e G \cap \{x + t\theta; t > 0\}$ (see [7]). It means that $v^G(x)$ is the total angle under which G is visible from the point x. This expression is a modification of the similar expression in [9]. Let us recall another characterization of $v^G(x)$ using a notion of an interior normal in Federer's sense.

If $z \in R^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in R^m; (x - z) \cdot \theta > 0\}$ has m-dimensional density zero at z

then $n^G(z) = \theta$ is termed the interior normal of G at z in Federer's sense. (The symmetric difference of B and C is equal to $(B - C) \cup (C - B)$.) If there is no interior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in R^m. The set $\{y \in R^m; |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial} G$. Clearly $\widehat{\partial} G \subset \partial_e G$.

If $\mathcal{H}_{m-1}(\partial_e G)$, the perimeter of G, is finite, then $\mathcal{H}_{m-1}(\partial_e G - \widehat{\partial} G) = 0$ and

$$v^G(x) = \int_{\widehat{\partial} G} |n^G(y) \cdot \nabla h_x(y)| \, d\mathcal{H}_{m-1}(y)$$

for each $x \in R^m$.

If G has a piecewise-$\mathcal{C}^{1+\alpha}$ boundary, then $V^G < \infty$. But there is a domain G with $\mathcal{C}^1$ boundary and $V^G = \infty$ (see [18]). On the other hand there is a domain G with $V^G < \infty$ and $\mathcal{H}_m(\partial G) > 0$. So open sets with a locally Lipschitz boundary and open sets with $V^G < \infty$ are incomparable.

Suppose now that $V^G < \infty$.Then the operator

$$\tau : \nu \mapsto N^G(\mathcal{U}\nu) + (\mathcal{U}\nu)\lambda$$

is a bounded linear operator on $\mathcal{C}'(\partial G)$ and

$$\tau\nu(M) = \int_{\partial G \cap M} \mathcal{U}\nu \, d\lambda + \int_{\partial G \cap M} d_G(x) \, d\nu(x) - \int_{\partial G} \int_{\partial G \cap M} n^G(y) \cdot \nabla h_x(y) \, d\mathcal{H}_{m-1}(y) \, d\nu(x).$$

The Robin problem $N^G(\mathcal{U}\nu) + (\mathcal{U}\nu)\lambda = \mu$ leads to the equation

$$\tau\nu = \mu.$$

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ on $\widehat{\partial} G$. Then $\mathcal{H}(\partial G) < \infty$. If $\lambda = f\mathcal{H}$, $\nu = h\mathcal{H} \in \mathcal{C}'(\partial G)$, then

$$\tau(h\mathcal{H}) = (Th)\mathcal{H},$$

where

$$Th(x) = \frac{1}{2} h(x) - \int_{\partial G} n^G(x) \cdot \nabla h_y(x) h(y) \, d\mathcal{H}(y) + f(x)\mathcal{U}(h\mathcal{H})(x).$$

Theorem 1. *Let the Fredholm radius of $\big(\tau - (1/2)I\big)$ be greater than 2, $\mu \in \mathcal{C}'(\partial G)$. Then there is a harmonic function u on G, which is a solution of the Robin problem*

$$N^G u + u\lambda = \mu, \tag{6}$$

if and only if $\mu \in \mathcal{C}_0'(\partial G)$ *(the space of such* $\nu \in \mathcal{C}'(\partial G)$ *that* $\nu(\partial H) = 0$ *for each bounded component* H *of* cl G *for which* $\lambda(\partial H) = 0$*). If* $\mu \in \mathcal{C}_0'(\partial G)$ *then there is a unique* $\nu \in \mathcal{C}_0'(\partial G)$ *such that*

$$\tau\nu = \mu \tag{7}$$

and for such ν *the single layer potential* $\mathcal{U}\nu$ *is a solution of* (6)*. If*

$$\beta > \frac{1}{2}\,(V^G + 1 + \sup_{x \in \partial G} \mathcal{U}\lambda(x)), \tag{8}$$

then

$$\nu = \sum_{n=0}^{\infty} \left(\frac{\beta I - \tau}{\beta} \right)^n \frac{\mu}{\beta} \tag{9}$$

and there are $q \in (0,1),\ C \in < 1, \infty)$ *such that*

$$\left\| \left(\frac{\tau - \beta I}{\beta} \right)^n \mu \right\| \leq C q^n \|\mu\|$$

for $\mu \in \mathcal{C}_0'(\partial G)$ *and a natural number* n*. If* $\lambda = 0$ *then*

$$\nu = \mu + \sum_{n=0}^{\infty} (2\tau - I)^n (2\tau)\mu \tag{10}$$

and there are $q \in (0,1), C \in < 1, \infty)$ *such that*

$$\|(2\tau - I)^n (2\tau)\mu\| \leq C q^n \|\mu\|$$

for $\mu \in \mathcal{C}_0'(\partial G)$ *and a natural number* n*.*

Remark 2. The condition that the Fredholm radius of $\big(\tau - (1/2)I\big)$ is greater than 2 does not depend on λ. In [15] it was shown that this condition has a local character. It is well-known that this condition is fulfilled for sets with a smooth boundary (of class $\mathcal{C}^{1+\alpha}$) (see [10]) and for convex sets (see [23]). J. Radon ([27]) proved this condition for open sets with "piecewise-smooth" boundary without cusps in the plane. R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in R^3 have this property (see [1], [12]). A. Rathsfeld showed in [28], [29] that polyhedral cones in R^3 have this property. (By a polyhedral cone in R^3 we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to R^2) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in R^3 we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons.) N. V. Grachev and V. G. Maz'ya obtained independently analogical result for polyhedral open sets with bounded boundary in R^3 (see [6]). (Remark that there is a polyhedral open set in R^3 which has not a locally Lipschitz boundary, for example $G = \{[x_1, x_2, x_3]; |x_1| < 3, |x_2| < 3, -3 < x_3 < 0\} \cup \{[3t, ty_2, ty_3]; 0 < t < 1, 1 < |y_2| < 2, 0 \leq y_3 < 1\}$. (The boundary of this set is not a graph of a function in a neighbourhood of the point $[0, 0, 0]$.)) The condition that the Fredholm radius of $\big(\tau - (1/2)I\big)$ is greater than 2 is fullfiled for $G \subset R^3$ with "piecewise-smooth" boundary, i.e. such that for each $x \in \partial G$ there are $r(x) > 0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x : U(x; r(x)) \to R^3$ of class $\mathcal{C}^{1+\alpha}$,$\alpha > 0$, such that $\psi_x(G \cap U(x; r(x))) = D_x \cap \psi_x(U(x; r(x)))$ (see [15]). N. V. Grachev and V. G. Maz'ya proved this condition for several types of sets with "piecewise-smooth" boundary in general Euclidean space (see [3,4,5]).

Remark 3. Let the Fredholm radius of $\big(\tau - (1/2)I\big)$ be greater than 2. Then it holds $\mathcal{H}_{m-1}(\partial G) < \infty$ and $\mathcal{H}$ is the restriction of $\mathcal{H}_{m-1}$ on ∂G. If $\lambda = f\mathcal{H}, \mu = g\mathcal{H} \in \mathcal{C}'(\partial G)$, then $\nu = h\mathcal{H}$, where $h \in L^1(\mathcal{H})$. If

$$\beta > \frac{1}{2}\,(V^G + 1 + \sup_{x \in \partial G} \mathcal{U}\lambda(x)),$$

then

$$h = \sum_{n=0}^{\infty} \Big(\frac{\beta I - T}{\beta}\Big)^n \frac{g}{\beta}$$

and there are $q \in (0, 1)$, $C \in< 1, \infty)$ such that

$$\left\| \Big(\frac{T - \beta I}{\beta}\Big)^n g \right\| \leq C q^n \|g\|$$

for a natural number n and $g \in L^1(\mathcal{H})$ such that $g\mathcal{H} \in \mathcal{C}'_0(\partial G)$. If $f = 0$, then

$$h = g + \sum_{n=0}^{\infty} (2T - I)^n (2T) g$$

and there are $q \in (0, 1), C \in< 1, \infty)$ such that

$$\|(2T - I)^n (2T) g\| \leq C q^n \|g\|$$

for a natural number n and $g \in L^1(\mathcal{H})$ such that $g\mathcal{H} \in \mathcal{C}'_0(\partial G)$.

Now, let us concentrate on the Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \text{ in } G, \\ u &= g \text{ on } \partial G, \end{aligned} \tag{11}$$

where $g \in \mathcal{C}(\partial G)$ is a continuous function on the boundary of G. Looking for a solution in the form of the double layer potential

$$Wf(x) = \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y)\; d\mathcal{H}_{m-1}(y) \tag{12}$$

is a classical method. It was shown by J. Král and independently by Yu. D. Burago and V. G. Maz'ya that it is possible to define the double layer potential (12) on G as a continuously extendable function on cl G for each density $f \in \mathcal{C}(\partial G)$ if and only only if $V^G < \infty$. (This condition we obtained for the Robin problem, too.) Under this condition $n^G(y)$ in the expression (12) is the interior normal of G at y in Federer's sense. If we look for the solution of the Dirichlet problem (11) in the form of the double layer potential (12) with a continuous density on the boundary of G we obtain the integral operator

$$Df(x) = (1 - d_G(x))f(x) + \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y)\, d\mathcal{H}_{m-1}(y).$$

on $\mathcal{C}(\partial G)$. The adjoint operator of D is the operator corresponding to the Neumann problem for the Laplace operator on the complementary domain to G. Noting that the Fredholm radius of $(D - \frac{1}{2}I)$ is equal to the Fredholm radius of $\big(\tau - (1/2)I\big)$ we obtain as a consequence of the theorem for the Neumann problem the following result:

Theorem 4. *Let $V^G < \infty$, the Fredholm radius of $(D - \frac{1}{2}I)$ be greater than 2. If the set $R^m - G$ is unbounded and connected and $g \in \mathcal{C}(\partial G)$, then the double layer potential*

$$Wf(x) = \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y)\, d\mathcal{H}_{m-1}(y)$$

is a solution of the Dirichlet problem for the Laplace equation with the boundary condition g, where

$$f = g + \sum_{j=0}^{\infty} (2D - I)^j 2Dg.$$

The condition that the set $R^m - G$ is unbounded and connected is necessary for expressing the solution of the Dirichlet problem for the Laplace equation in the form of the double layer potential for each boundary condition. If we want to calculate the solution for an open set with holes we must modify a double layer potential. Suppose now that the dimension of the space R^m is greater than 2. If we look for a solution of the Dirichlet problem in the form of the sum of the single layer potential and the double layer potential with the same density we obtain the integral operator on the space of all continuous functions in ∂G the adjoint operator of which is the operator corresponding to some Robin problem for the Laplace equation on the complementary domain and we obtain the following result as a consequence of the theorem on the Robin problem.

Theorem 5. *Let $m > 2, V^G < \infty$, the Fredholm radius of $(D - \frac{1}{2}I)$ be greater than 2. If $g \in \mathcal{C}(\partial G)$ then $Wf + \mathcal{U}(f\mathcal{H})$ is a solution of the Dirichlet problem for the Laplace equation with the boundary condition g, where*

$$f = \sum_{n=0}^{\infty} \left(\frac{\beta I - V}{\beta} \right)^n \frac{g}{\beta},$$

$$Vg = Dg + \mathcal{U}(g\mathcal{H}),$$
$$\beta > \frac{1}{2}(V^G + 1 + \sup_{x\in\partial G} \mathcal{U}\mathcal{H}(x)).$$

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