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RELATIONS BETWEEN LINEAR CONNECTIONS ON THE TANGENT BUNDLE AND CONNECTIONS ON THE JET BUNDLE OF A FIBRED MANIFOLD

JOSEF JANYŠKA AND MARCO MODUGNO

To Ivan Kolář, on the occasion of his 60th birthday.

ABSTRACT. All natural operations transforming linear connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle are classified. It is proved that such operators form a 2-parameter family (with real coefficients).

Introduction

This paper is motivated by the bijective relation between time-preserving linear connections on space-time with absolute time and affine connections on 1-jet bundle of space-time, [1], [2], [3]. We would like to know if similar relation holds also for a general fibred manifold and so we study all natural operations transforming linear connections on the tangent bundle of a fibred manifold to connections on the 1-jet bundle. We prove that such operators form a 2-parameter family (with real coefficients) and we give its coordinate and geometric expressions.

Our operator are natural in the sense of [4] and [5].

All manifolds and mappings are assumed to be smooth.

1. Linear connections

Let $:\ \to$ be a fibred manifold with a local fibred coordinate chart $(\ ) = (\ ),\ = 1\ \dim\ =\ ,\ = 1\ \dim\ - \dim\ =\ ,\ = 1\ \dim\ =\ +\ .$

A linear connection Λ on the bundle $:\ \to$ and a linear connection on the bundle $:\ \to$ can be expressed, respectively, by tangent valued

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forms

$$\Lambda : \quad \to \;^{*} \quad \otimes$$

$$: \quad \to \;^{*} \quad \otimes$$

with coordinate expressions, respectively,

(1.1) $\Lambda = \quad \otimes(\quad + \Lambda \quad) \qquad \Lambda \quad \in \;^{\infty}(\;)$

(1.2) $= \quad \otimes(\quad + \quad) \qquad \in \;^{\infty}(\;)$

where ( ) and ( ) are the induced coordinate charts on and , respectively. The connections Λ and are also characterised by the vertical projections $_{\Lambda}$: → and : → , respectively, or equivalently by the forms $_{\Lambda}$: → * ⊗ and : → * ⊗ with coordinate expressions, respectively,

(1.3) $_{\Lambda} = (\quad - \Lambda \quad)\otimes$

(1.4) $= (\quad - \quad)\otimes$

Let us denote by ⊗ Λ* the tensor product of the connection and the pullback of the dual connection Λ* with respect to , i.e.

$$\otimes \; \Lambda^{*} : \;^{*} \; \otimes \quad \to \;^{*} \underset{Y}{\overset{*}{\otimes}} \; (\;^{*} \; \otimes \;)$$

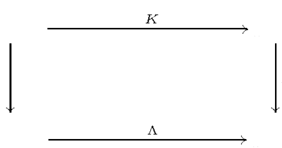
with coordinate expression, in the induced fibred coordinate chart ( ) on : * ⊗ → ,

(1.5) $\otimes \; \Lambda^{*} = \quad \otimes(\quad + (\quad - \quad \Lambda \quad) \quad)$

where we put Λ = 0. The connection ⊗ Λ* can be defined by the vertical projection $_{\otimes_Y \Lambda^{*}}$: (* ⊗) → * ⊗ . We have the coordinate expression

(1.6) $_{\otimes_Y \Lambda^{*}} = (\quad - (\quad - \quad \Lambda \quad) \quad)\otimes$

A linear connection on is said to be *projectable* on a linear connection Λ on if the following diagram commutes



A pair of linear connections (Λ) is said to be *fibre preserving* if the covariant derivative of with respect to ⊗ Λ* vanishes, i.e. $\nabla_{\otimes_Y \Lambda^{*}}(\;) = 0$.

Lemma 1.1. *Let be a linear connection on and* Λ *a linear connection on . The following three conditions are equivalent*

i) *is projectable on* Λ.

ii) *The pair* (Λ) *is fibre preserving.*

iii) *In a fibred coordinate chart* $= \quad = 0$ *and* $= \Lambda$.

Proof. It can be proved by using (1.3), (1.4) and (1.6). □

2. Contact mappings

We deal with the natural complementary contact maps

$$д : \; _1 \times \quad \to \qquad : \; _1 \times \quad \to$$

or equivalently

$$д : \; _1 \to \; ^* \; \otimes \qquad : \; _1 \to \; ^* \; \otimes$$

which split the natural exact sequence

(2.1) $$0 \to \quad \to \quad \longrightarrow \quad \to 0$$

through the exact sequence over $_1$

(2.2) $$0 \to \; _1 \times \quad \xrightarrow{д} \; _1 \times \quad \to \; _1 \times \quad \to 0$$

We have the coordinate expressions

(2.3) $д = \quad \otimes д \quad = \quad \otimes(\quad + \quad) \qquad = \quad \otimes \quad = (\quad - \quad) \otimes$

where (;) is the induced coordinate chart on $_1$.

We recall the canonical isomorphism

$$_1 \simeq \; _1 \times (\; ^* \; \otimes \quad)$$

given by

$$\mapsto \quad \otimes$$

3. Induced connection

A connection Γ on the affine bundle $^1_0 : \; _1 \to$ can be expressed by a tangent valued form

$$\Gamma : \; _1 \to \; ^* \; \underset{1}{\otimes} \; _1$$

with coordinate expression

(3.1) $$\Gamma = \quad \otimes(\quad + \Gamma \quad) \qquad \Gamma \quad \in \; ^\infty(\; _1 \;)$$

Using the identification of $_1$ and $^* \otimes$, the connection Γ can be characterised by the vertical projection $_\Gamma :\ _1 \to\ ^* \otimes _1$, or equivalently by the form $_\Gamma :\ _1 \to\ ^* \otimes\ ^*{}_1 \otimes {}_1$. In coordinates we have

$$(3.2)\qquad {}_\Gamma = \otimes(- \Gamma)\otimes$$

The connection Γ is affine if and only if its coordinate expression is of the type

$$\Gamma = \Gamma + \Gamma \qquad \Gamma \quad \Gamma \in {}^\infty( )$$

Theorem 3.1. *Let Λ be a linear connection on and a linear connection on . The map*

$$\Gamma = \circ (\otimes\Lambda^*) \circ Д$$

given by the following diagram

$$\begin{array}{ccccc} _1 & \xrightarrow{\Gamma} & _1 & \xrightarrow{\simeq} & _1 \times (^* \otimes) \\ \downarrow (_{J_1Y} \ Д) & & & & \uparrow (\mathrm{id}_{J_1Y} \times (\mathrm{id}_{T^*X} \otimes)) \\ _1 \times (^* \otimes) & & \xrightarrow{(\mathrm{id}_{J_1Y} \times \ _{K\otimes\Lambda^*})} & & _1 \times (^* \otimes) \end{array}$$

turns out to be a connection on the bundle $^1_0 :\ _1 \to$. *Moreover, we have the coordinate expression*

$$(3.3)\qquad \Gamma = + - (+)$$

i.e. the connection Γ is independent of Λ.

Thus, we have obtained a natural operator

$$: \ \mapsto \Gamma$$

transforming linear connections on into connections on $_1$.

Proof. It can be proved in coordinates by using (2.3), (1.6) and (3.2). □

Lemma 3.1. *If (Λ) are fibre preserving, then the induced connection ( ) on* $_1$ *is affine.*

Proof. From the coordinate expression (3.3), for a pair of fibre preserving connections and Λ, we get

$$(3.4)\qquad \Gamma = (-) +$$

where we put $= 0$ and $= \Lambda$. □

Remark 3.1. In Galilei relativistic theory [1], [2], [3], the base manifold (time) is assumed to be 1-dimensional and affine. A linear connection on space–time is said to be *time–preserving* if it is projectable on the canonical flat connection on the base. (3.4) then implies that the relation between time–preserving linear connections on space–time and affine connections on its 1-jet bundle is bijective. But for dim 1 and the flat connection on an affine base manifold this relation is not one-to-one.

4. Curvature

The curvatures of a linear connection on and of a connection Γ on $_1$ are, respectively, the 2–forms

$$= \frac{1}{2}[\quad] : \quad \to \wedge^2 \ ^* \quad \otimes$$

$$_\Gamma = \frac{1}{2}[\Gamma \ \Gamma] : \ _1 \quad \to \wedge^2 \ ^* \quad \otimes (\ ^* \quad \otimes \quad)$$

with coordinate expressions

(4.1) $$= (\quad) \quad \wedge \quad \otimes \quad =$$

$$= (\frac{}{} + \quad) \quad \wedge \quad \otimes$$

and

(4.2) $$_\Gamma = (\ _\Gamma) \quad \wedge \quad \otimes \quad \otimes \quad =$$

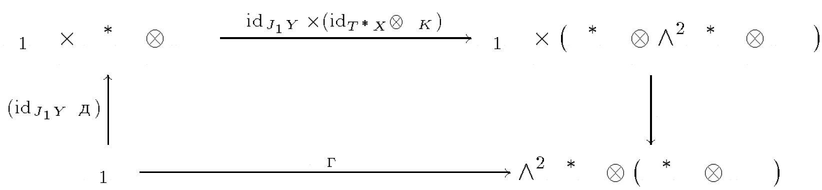
$$= (\frac{\Gamma}{} + \Gamma \ \frac{\Gamma}{}) \quad \wedge \quad \otimes \quad \otimes$$

respectively.

Theorem 4.1. *If* Γ *is the connection on* $_1$ *induced by a linear connection on , then we have*

$$_\Gamma = \ \circ \quad \circ д$$

according to the following commutative diagram



i.e in coordinates

$$(\ _\Gamma) \quad = (\quad) \quad + (\quad) \quad - \quad ((\quad) \quad + (\quad) \quad)$$

Proof. It can be proved by using (3.3), (4.1) and (4.2). □

5. Main theorem

Let us denote by $(\ )$, ≥ 0, the group of -order jets of diffeomorphisms of $\mathbb{R}^{+}$ which preserve the origin and the fibration $\mathbb{R}^{+} \to \mathbb{R}$, i.e $(\ )$ is the subgroup in $_{+}$ given by $_{1} \ _{r} = 0$, $= 0$ -1. We have the canonical group homomorphism $: (\ ) \to (\ )$, , and we denote by $^{(\ )}_{(\ )}$ its kernel.

Let us denote by $= \mathbb{R} \otimes \mathbb{R}^{*} \times \mathbb{R}^{+} \otimes \otimes^{2}\mathbb{R}^{(+)*}$ the $^{2}_{(\ )}$-space with coordinates ( ) and the left action of the group $^{2}_{(\ )}$ given by

$$= \ + \ $$

$$= \ + \ $$

Let us denote by $\tilde{} = \mathbb{R} \otimes \mathbb{R}^{*} \times \mathbb{R}^{+} \otimes \Lambda^{2}\mathbb{R}^{(+)*}$ the $^{1}_{(\ )}$-space with coordinates ( ) and the tensor action of the group $^{1}_{(\ )}$. We denote by $: \to \tilde{}$ the $^{2}_{(\ )}$-equivariant mapping given by the antisymmetrisation of subindices , i.e.

$$= \ = 1 \ 2(\ - \ )$$

Let us consider the space $= \mathbb{R}^{(+)*} \otimes \mathbb{R}^{*} \otimes \mathbb{R}$ with coordinates ( ) and the action of the group $^{1}_{(\ )}$ given by

$$= $$

Lemma 5.1. *All* $^{2}_{(\ )}$*-equivariant mappings from to are of the form*

$$
\begin{aligned}
&= \ _{1}(\ + \ - \ - \ ) \\
(5.1) \qquad & + \ _{2}(\ + \ - \ - \ )
\end{aligned}
$$

where $= 1 \ 2(\ - \ )$.

Proof. The proof uses the standard techniques of computation of $^{2}_{(\ )}$-equivariant mappings, [4], and we can divide it into three steps. We omit technical computations.

Step 1. Let $: \to$ be a $^{2}_{(\ )}$-equivariant mapping. From the equivariancy of with respect to $^{(2\ 1)}_{(\ )}$ we get that is of the form $= \tilde{} \circ$, where $\tilde{} : \tilde{} \to$ is a $^{1}_{(\ )}$-equivariant mapping, so it is sufficient to classify all mappings $\tilde{}$.

Step 2. Let us denote by the homotheties of $\mathbb{R}$. From the equivariancy of $\tilde{}$ with respect to $(\ \times \mathrm{id}_{\mathbb{R}^m})$ and $(\mathrm{id}_{\mathbb{R}^n} \times \ )$ we get that $\tilde{}$ is polynomial and any monomial is linear in and of maximum degree 3 in . Coefficients are absolute invariant tensors and we have a polynomial with 33 coefficients.

Step 3. Finally, using equivariancy with respect to diffeomorphisms $(\ ) \mapsto (\ + \ )$, we find relations between coefficients of $\tilde{}$ and we get (5.1). □

Theorem 5.1. *All natural operations transforming a linear connection on into connections on* $_1$ *form the following 2–parameter family*

$$(\ ) + (\mathrm{id}\otimes д^{*}\otimes\ )(\ _1\ \ +\ _2 \otimes \hat{}\ \ ) \tag{5.2}$$

where $_1\ \ _2 \in \mathbb{R}$, *is the torsion tensor of* , $\hat{}$ *denotes the contraction and is the identity tensor on* .

Proof. Any natural connection on $_1$ is of the form $(\ ) + \Phi(\ )$, where Φ is an operator (over $_1$) transforming into a section of $^{*} \otimes\ ^{*} \otimes$. So it is sufficient to classify all operators Φ. The generalized Peetre theorem implies that any operator Φ is of finite order, [4], [8].

Using homogeneity conditions, [4, Proposition 25.2], we get that all finite order operators Φ are of order 0 ($\Phi(\ )$ depends only on coefficients of and not on their derivatives).

All 0-order operators Φ are in a bijective correspondence with $^2(\ )$-equivariant mappings from to and it is easy to see that the operator corresponding to the mapping of Lemma 5.1 is $(\mathrm{id}\otimes д^{*}\otimes\ )(\ _1\ \ +\ _2 \otimes \hat{}\ \ )$. □

Corollary 5.1. *For a torsion free connection the connection* $(\ )$ *is the unique natural connection on* $_1$ *given by* . □

Another geometrical description of Theorem 5.1 is based on the following theorem, [4, Proposition 25.2].

Theorem 5.2. *All natural operations transforming a linear connection on into linear connections on form the following 3–parameter family*

$$+\ _1\ \ +\ _2 \otimes \hat{}\ \ +\ _3 \hat{}\ \ \otimes$$

where $_1\ \ _2\ \ _3 \in \mathbb{R}$. □

Theorem 5.1 now can be interpreted by applying the operator on the family of connections from Theorem 5.2. Then the resulting connection on $_1$ does not depend on $_3$ and it is easy to see that

$$(\ +\ _1\ \ +\ _2 \otimes \hat{}\ \ +\ _3 \hat{}\ \ \otimes\ ) = (\ ) + (\mathrm{id}\otimes д^{*}\otimes\ )(\ _1\ \ +\ _2 \otimes \hat{}\ \ )$$

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