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PROLOGATION OF TANGENT VALUED FORMS TO WEIL BUNDLES

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Abstract. We prove that the so-called complete lifting of tangent valued forms from a manifold M to an arbitrary Weil bundle over M preserves the Frölicher-Nijenhuis bracket. We also deduce that the complete lifts of connections are torsion-free in the sense of M. Modugno and the second author.

It has been pointed out recently that the Weil functors represent a unified technique for studying a large class of geometric spaces. Moreover, the general results from [4] enable us to clarify that certain procedures can be applied precisely to Weil bundles. In [7], A. Morimoto introduced the so-called complete lifting of tensor fields of type $(1,1)$ from a manifold M to any Weil bundle T^AM by using the canonical exchange isomorphism between T^ATM and TT^AM. A special case of such a construction is the lifting of arbitrary connections from a fibered manifold $E \to B$ to $T^AE \to T^AB$ by J. Slovák, [8]. The problem of lifting tensor fields of type $(1,k)$ was studied by J. Gancarzewicz, [1] and by himself, W. Mikulski and Z. Pogoda, [2]. We present their construction of the complete lift of such a tensor field in Section 2 below, but we add a justification of the fact that such a procedure works for Weil bundles only, provided we accept the standard assumption of the so-called point property. A special case of tensor fields of type $(1,k)$ on M are the tangent valued k-forms on M. Using some results from [2] and the expression of the Frölicher-Nijenhuis bracket of tangent valued forms in terms of the bracket of vector fields by P. W. Michor, [4], and M. Modugno, [6], we prove that the complete lifting preserves the Frölicher-Nijenhuis bracket. In our setting this is a consequence of a more general formula deduced in Section 4. This general formula enables us to study the torsions of connections on Weil bundles introduced by M. Modugno and the second author, [5]. In particular we deduce that all torsions of the complete lift of every connection vanish.

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Al manifolds and mappings are assumed to be infinitely differentiable and all manifolds are paracompact.

1. Weil bundles

We recall the definition of a Weil bundle over a manifold M in a form generalizing the classical concept of the jet functor T^r_k of k-dimensional velocities of order r, $T^r_k M = J^r_0(\mathbb{R}^k, M)$. Let $\langle x_1, \dots, x_k\rangle \subset \mathbb{R}[x_1, \dots, x_k]$ be the ideal of all polynomials without absolute term in the algebra of all polynomials in k variables and $\langle x_1, \dots, x_k\rangle^r$ be its r-th power. By a Weil ideal in $\mathbb{R}[x_1, \dots, x_k]$ we mean an ideal $\mathcal{A}$ satisfying $\langle x_1, \dots, x_k\rangle^{r+1} \subset \mathcal{A} \subset \langle x_1, \dots, x_k\rangle^2$. The factor algebra $A = \mathbb{R}[x_1, \dots x_k]/\mathcal{A}$ is called a Weil algebra; the number k is said to be the width of A and the minimum of the r's is called the depth of A. If we consider the algebra $E(k)$ of all germs of smooth functions on $\mathbb{R}^k$ at zero, then $\mathcal{A}$ generates an ideal $\widetilde{\mathcal{A}} \subset E(k)$. Clearly, we have $A = E(k)/\widetilde{\mathcal{A}}$ as well.

Definition 1. Two maps g, $h : \mathbb{R}^k \to M$, $g(0) = h(0) = x$ are said to be A-equivalent, if $\varphi \circ g - \varphi \circ h \in \widetilde{\mathcal{A}}$ for every germ φ of a smooth function on M at x. Such an equivalence class will be denoted by $j^A g$ and called an A-velocity on M. The point $g(0)$ is said to be the target of $j^A g$.

Denote by $T^A M$ the set of all A-velocities on M. It is easy to see that $T^A\mathbb{R} = A$. The target map is a bundle projection $T^A M \to M$. Further, for every $f : M \to N$ we define $T^A f : T^A M \to T^A N$ by $T^A f(j^A g) = j^A(f \circ g)$. Then T^A is a functor on the category $\mathcal{M}f$ of all manifolds with values in the category $\mathcal{FM}$ of smooth fibered manifolds, which is called the Weil functor corresponding to A. Clearly, $T^A(M \times N) = T^A M \times T^A N$, so that T^A preserves products. In particular, for $\mathcal{A} = \langle x_1, \dots, x_k\rangle^{r+1}$ we obtain the functor T^r_k and the tangent functor T corresponds to the algebra $\mathbb{D} = \mathbb{R}[x]/\langle x\rangle^2$ of the so-called dual (or Study) numbers.

Let $B = \mathbb{R}[x_1, \dots x_k]/\mathcal{B}$ be another Weil algebra and $H : A \to B$ be an algebra homomorphism. Then H is the factor map of an algebra homomorphism $\psi : \mathbb{R}[x_1, \dots x_k] \to \mathbb{R}[x_1, \dots x_l]$ satisfying $\psi(\mathcal{A}) \subset \mathcal{B}$ and ψ is generated by a polynomial map $h : \mathbb{R}^m \to \mathbb{R}^k$, $x_i = \psi(x_i)$, $i = 1, \dots, k$. In [3] it is proved that the maps $\tau^H_M : T^A M \to T^B M$,

$$\tau^H_M(j^A g) = j^B(g \circ h), \qquad g : \mathbb{R}^k \to M$$

define a natural transformation $\tau^H : T^A \to T^B$.

The important role of Weil functors in differential geometry has been clarified by a recent result, which reads that every product preserving bundle functor on $\mathcal{M}f$ is a Weil functor and every natural transformation of two product preserving bundle functors is determined by a homomorphism of the corresponding Weil algebras, see [4] for a survey. In particular, the iteration $T^A \circ T^B$ of two Weil bundles corresponds to the tensor product $A \otimes B$ of Weil algebras, $T^A(T^B M) = T^{A\otimes B} M$. The exchange algebra homomorphism $A \otimes B \to B \otimes A$ defines a natural equivalence $\kappa^{A,B}_M : T^A(T^B M) \to T^B(T^A M)$ which generalizes the canonical involution of the second tangent bundle TTM. Furthermore, if $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ or $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the addition or the multiplication of reals, then $T^A a : A\times A \to A$ or $T^A m : A\times A \to A$ is the vector addition or the algebra multiplication in $A = T^A\mathbb{R}$, respectively.

2. Complete lifts

A tensor field D of type $(1,k)$ on M can be interpreted as a map

$$D : TM \underbrace{\times_M \cdots \times_M}_{k\text{-times}} TM \to TM\,.$$

Applying the functor T^A, we obtain

$$T^A D : T^A TM \times_{T^A M} \cdots \times_{T^A M} T^A TM \to T^A TM\,.$$

If we add the above mentioned exchange map $\kappa : T^A TM \to TT^A M$, we construct

$$\begin{aligned} &\mathcal{T}^A D := \kappa \circ T^A D \circ (\kappa^{-1} \times \cdots \times \kappa^{-1}) : \\ &TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M \to TT^A M \end{aligned} \tag{1}$$

This is a tensor field of type $(1,k)$ on $T^A M$, which is called the complete lift of D to $T^A M$, [2]. In the special case $k = 0$, we have a vector field $D = X : M \to TM$. Then $\mathcal{T}^A X$ coincides with the flow prolongation of X, i.e

$$\mathcal{T}^A X = \frac{\partial}{\partial t}\Big|_0 T^A(\exp tX) \tag{2}$$

where $\exp tX$ is the flow of vector field X, [4]. If $X_1, \ldots X_k \in C^\infty TM$ are vector fields on M, then $D(X_1, \ldots X_k)$ is a vector field on M as well. From (1) we deduce directly

$$\mathcal{T}^A D(\mathcal{T}^A X_1, \ldots \mathcal{T}^A X_k) = \mathcal{T}^A(D(X_1, \ldots X_k)) \tag{3}$$

We remark that such a construction of an induced tensor field of type $(1,k)$ can be applied to Weil bundles only. We recall that a bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ is said to have the point property, if $F(pt) = pt$ for each one point set pt. From Proposition 38.8 in [4] it follows easily: If F has the point property and there exists a natural equivalence $FT \to TF$, then F preserves products, i.e. F is a Weil functor.

By [7], every $a \in A$ determines a tensor $L(a)$ of type $(1,1)$ on $T^A M$ as follows. The multiplication of the tangent vectors of M by reals is a map $\mu : \mathbb{R}\times TM \to TM$. Applying the functor T^A, we obtain $T^A\mu : A \times T^A TM \to T^A TM$. Then

$$\mathcal{T}^A\mu := \kappa \circ T^A\mu \circ (\mathrm{id}_A \times \kappa^{-1}) : A \times TT^A M \to TT^A M \tag{4}$$

and we define $L(a) = \mathcal{T}^A\mu(a, -)$. Since the multiplication in A is induced from the multiplication of reals, it holds

$$L(a_1) \circ L(a_2) = L(a_1 a_2) \qquad a_1, a_2 \in M\,.$$

Clearly, $L(1) = \mathrm{id}$. If we need to underline the manifold M, we shall also write $L_M(a)$.

The following lemma is due to Gancarzewicz, Mikulski and Pogoda, [2], but we sketch its proof for the sake of completeness.

Lemma 1. *Let C and $\bar{C}$ be two tensor fields of type $(1,k)$ on T^AM. If it holds*

$$C(L(a_1)\mathcal{T}^A X_1, \dots, L(a_k)\mathcal{T}^A X_k) = \bar{C}(L(a_1)\mathcal{T}^A X_1, \dots, L(a_k)\mathcal{T}^A X_k)$$

for all $X_1, \dots X_k \in C^\infty TM$ and all $a_1, \dots a_k \in A$, then $C = \bar{C}$.

Proof. It suffices to consider $M = \mathbb{R}^m$ and the constant vector fields on $\mathbb{R}^m$. Let $1, e_1, \dots, e_n$ be a basis of the vector space A with nilpotent $e_1, \dots, e_n$ and $x^i, y_1^i, \dots y_n^i$ be the induced coordinates on $T^A\mathbb{R}^m = A^m$. Since the flow of a constant vector field $X = \xi^i \partial/\partial x^i$ is formed by translations, we have $\mathcal{T}^A X = \xi^i \partial/\partial x^i + 0.\partial/\partial y_1^i + \dots + 0.\partial/\partial y_n^i$. Then $L(e_p)\mathcal{T}^A X = \xi^i \partial/\partial y_p^i$, $p = 1, \dots, n$. But ξ^i are arbitrary and this implies the coordinate form of our assertion. □

3. Some lemmas

Every function $f : M \to \mathbb{R}$ induces a vector valued function $T^A f : T^A M \to A$. Every vector field Y on $T^A M$ determines the Lie derivative $YT^A f : T^A M \to A$ of such a vector valued function. Given $a \in A$, we define $aT^A f : T^A M \to A$ by multiplying in A.

Lemma 2. *If two vector fields Y and $\widetilde{Y}$ on $T^A M$ satisfy $Y(aT^A f) = \widetilde{Y}(aT^A f)$ for all $f : M \to \mathbb{R}$ and all $a \in A$, then $Y = \widetilde{Y}$.*

Proof. The proof is quite similar to the proof of Lemma 1. If suffices to take in account the linear functions $f : \mathbb{R}^m \to \mathbb{R}$. □

Lemma 3. *It holds $T^A(Xf) = \mathcal{T}^A X(T^A f)$ for every vector field X on M and every $f : M \to \mathbb{R}$.*

Proof. The derivative Xf is the second projection of $Tf \circ X : M \to T\mathbb{R}$. Then $T^A(Xf) = T^A(pr_2) \circ T^A f \circ T^A X$. We have $\mathcal{T}^A X = \kappa_M \circ T^A X$ by definition and $T^A Tf \circ \kappa_M^{-1} = \kappa_{\mathbb{R}}^{-1} \circ TT^A f$ by naturality of κ. But $T^A(pr_2) \circ \kappa_{\mathbb{R}}$ is the second projection $A \times A \to A$. □

Lemma 4. *For every $X \in C^\infty TM$, every $f : M \to \mathbb{R}$ and every $a \in A$ it holds $\mathcal{T}^A X(aT^A f) = aT^A(Xf)$ and $(L(a)\mathcal{T}^A X)T^A f = aT^A(Xf)$.*

Proof. We have $X(tf) = t(Xf)$ for all $t \in \mathbb{R}$. By Lemma 3 we obtain $\mathcal{T}^A X(aT^A f) = aT^A(Xf)$. Further, we have $(tX)f = t(Xf)$ for all $t \in \mathbb{R}$. Using Lemma 3 and the definition of $L(a)$, we obtain $(L(a)\mathcal{T}^A X)T^A f = aT^A(Xf)$. □

The following lemma can be found in [2], but we present another proof, which replaces real-valued functions by A-valued ones.

Lemma 5. *It holds $[L(a_1)\mathcal{T}^A X_1, L(a_2)\mathcal{T}^A X_2] = L(a_1 a_2)\mathcal{T}^A([X_1, X_2])$ for all X_1, $X_2 \in C^\infty TM$ and all a_1, $a_2 \in A$.*

Proof. We know that the flow prolongation $\mathcal{T}^A$ preserves the bracket of vector fields, [4]. For every vector fields Y_1, Y_2 on $T^A M$ and every $F : T^A M \to A$ we have

$[Y_1, Y_2]F = Y_1(Y_2 f) - Y_2(Y_1 F)$ by definition. Using Lemmas 3 and 4, we obtain

$$[L(a_1)\mathcal{T}^A X_1, L(a_2)\mathcal{T}^A X_2](aT^A f) = L(a_1)\mathcal{T}^A X_1(a_2 aT^A(X_2 f)) - L(a_2)\mathcal{T}^A X_2(a_1 aT^A(X_1 f)) = a_1 a_2 a(T^A(X_1 X_2 f) - T^A(X_2 X_1 f)) = a_1 a_2 a\mathcal{T}^A([X_1, X_2])T^A f = L(a_1 a_2)\mathcal{T}^A([X_1, X_2])(aT^A f)\,.$$

Then our assertion follows from Lemma 2. □

Even the following lemma is due to Gancarzewicz, Mikulski and Pogoda, [2].

Lemma 6. *For every tensor fields D of type $(1, k)$ on M, every $X_1, \dots, X_k \in C^\infty TM$ and every $a_1, \dots, a_k \in A$, it holds*

$$\mathcal{T}^A D(L(a_1)\mathcal{T}^A X_1, \dots, L(a_k)\mathcal{T}^A X_k) = L(a_1 \dots a_k)\mathcal{T}^A(D(X_1, \dots, X_k))\,. \tag{6}$$

Proof. We have $D(t_1 X_1, \dots, t_k X_k) = t_1 \dots t_k D(X_1, \dots, X_k)$ for all $t_1, \dots t_k \in \mathbb{R}$. Applying the functor T^A to this relation and using the definition of $L(a)$, we obtain (6). □

4. The Frölicher-Nijenhuis bracket

A tangent valued k-form P on M is an antisymmetric tensor field of type $(1, k)$ on M. If Q is a tangent valued l-form on M, the Frölicher-Nijenhuis bracket $[P, Q]$ is a tangent valued $(k+l)$-form on M, [4], [6]. Given a tangent valued k-form S on $T^A M$ and an element $a \in A$, $L(a)S$ is a tangent valued k-form on $T^A M$ as well. The main result of the present paper is

Proposition 1. *For every tangent valued k-form P and tangent valued l-form Q on M and every $a, b \in A$, it holds*

$$[L(a)\mathcal{T}^A P, L(b)\mathcal{T}^A Q] = L(ab)\mathcal{T}^A([P, Q]) \tag{7}$$

In particular, for $a = b = 1$ we obtain $[\mathcal{T}^A P, \mathcal{T}^A Q] = \mathcal{T}^A([P, Q])$.

Proof. M. Modugno, [6] and P.W. Michor, [4], found the following expression of $[P, Q]$ in terms of the bracket of vector fields

$$\begin{aligned}
[P, Q](X_1, \dots, X_{k+l}) &= \\
&= \frac{1}{k!l!}\sum_\sigma \operatorname{sign}\sigma[P(X_{\sigma 1}, \dots, X_{\sigma k}), Q(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})] \\
&+ \frac{-1}{k!(l-1)!}\sum_\sigma \operatorname{sign}\sigma Q([P(X_{\sigma 1}, \dots, X_{\sigma k}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \dots) \\
&+ \frac{(-1)^{kl}}{(k-1)!l!}\sum_\sigma \operatorname{sign}\sigma P([Q(X_{\sigma 1}, \dots X_{\sigma l}), X_{\sigma(l+1)}], X_{\sigma(l+2)}, \dots) \\
&+ \frac{(-1)^{k-1}}{(k-1)!(l-1)!2}\sum_\sigma \operatorname{sign}\sigma Q(P([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots], X_{\sigma(k+2)}, \dots) \\
&+ \frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2}\sum_\sigma \operatorname{sign}\sigma P(Q([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots], X_{\sigma(l+2)}, \dots)
\end{aligned} \tag{8}$$

with $X_1, \dots X_{k+l} \in C^\infty TM$. Let us express the value of $[L(a)\mathcal{T}^A P, L(b)\mathcal{T}^B Q]$ on $L(a_1)\mathcal{T}^A X_1, \dots, L(a_{k+l})\mathcal{T}^A X_{k+l}$ in this way. Using Lemmas 5 and 6 and (3), we deduce that each term of such a modification of (8) is equal to the value of $\mathcal{T}^A$ on the corresponding term of (8) multiplied by $L(aba_1 \dots a_{k+l})$. Hence we obtain $L(ab)\mathcal{T}^A([P,Q])(L(a_1)\mathcal{T}^A X_1, \dots, L(a_{k+l}\mathcal{T}^A X_{k+l}))$. Then Lemma 1 yields (7). □

Given an arbitrary fibered manifold $p : E \to B$, a connection on E can be studied either as a lifting map $\gamma : E \times_B TB \to TE$ or as the horizontal projection $\Gamma : TE \to TE$, which is a special tangent valued 1-form on E. Clearly, it holds $\Gamma = \gamma \circ Tp$. Using the first approach, Slovák defined the induced connection $\mathcal{T}^A\gamma$ on $T^A E \to T^A B$ by $\mathcal{T}^A\gamma = \kappa_E \circ T^A\gamma \circ \kappa_B^{-1}$, [8]. Under the second approach, we have $\mathcal{T}^A\Gamma = \kappa_E \circ T^A\Gamma \circ \kappa_E^{-1}$ according to (1). But $T^A Tp \circ \kappa_E^{-1} = \kappa_B^{-1} \circ TT^A p$ by naturality, so that $\mathcal{T}^A\Gamma = (\kappa_E \circ T^A\gamma \circ \kappa_B^{-1}) \circ TT^A p$. Hence the results of both approaches coincide.

Consider two connections Γ and Δ on E in the second form of tangent valued 1-forms. The Frölicher-Nijenhuis bracket $[\Gamma, \Delta]$ is called the mixed curvature of Γ and Δ, [4], p. 232. Then Proposition 1 yields the following formula for the mixed curvature of $\mathcal{T}^A\Gamma$ and $\mathcal{T}^A\Delta$.

Proposition 2. *It holds* $[\mathcal{T}^A\Gamma, \mathcal{T}^A\Delta] = \mathcal{T}^A([\Gamma, \Delta])$.

In the special case $\Gamma = \Delta$ we obtain the curvature $[\Gamma, \Gamma]$ of Γ. We remark that this case has been studied in [2].

5. Torsions

In [5], M. Modugno and the second authors deduced that all natural tensors (in the sense of [4]) of type (1,1) on $T^A M$ are of the form $L_M(a)$, $a \in A$. For example, in the special case $A = \mathbb{D}$ of the tangent bundle, the class $\{x\} \in \mathbb{R}[x]/\langle x\rangle^2$ determines the well known vertical operator on TTM. Given a connection Γ on $T^A M \to M$, the Frölicher-Nijenhuis bracket $[\Gamma, L(a)]$ is called the $L(a)$-torsion of Γ, [5]. This idea can be modified to the case of connections on $T^A p : T^A E \to T^A B$ as well.

Definition 2. Let Γ be a connection on $T^A p : T^A E \to T^A B$ and $a \in A$. Then the Frölicher-Nijenhuis bracket $[\Gamma, L_E(a)]$ will be called the a-torsion of Γ.

A natural question is to study the torsions of the connection $\mathcal{T}^A\Gamma$ induced from a connection Γ on $E \to B$. The answer is a corollary of the following more general assertion.

Proposition 3. *For every tangent valued k-form P on a manifold M and every $a \in A$, it holds* $[\mathcal{T}^A P, L_M(a)] = 0$.

Proof. We have $L_M(a) = L(a)I_{T^A M}$, where $I_{T^A M}$ is the identity of $TT^A M$. Then Proposition 1 yields $[\mathcal{T}^A P, L(a)I_{T^A M}] = L(a)\mathcal{T}^A([P, I_M])$. But $[P, I_M] = 0$ is a well known formula. □

Corollary. *For every connection Γ on $E \to B$, all a-torsions of the induced connection $\mathcal{T}^A\Gamma$ vanish.*

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