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3–CONFIGURATIONS WITH SIMPLE EDGE BASIS AND THEIR CORRESPONDING QUASIGROUP IDENTITIES

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Abstract. There is described a procedure which determines the quasigroup identity corresponding to a given 3-coloured 3-configuration with a simple edge basis.

Part 1 3-configurations

1.1. Main notations

Under a *(3)- configuration* we shall understand a finite incidence structure $(\mathcal{V}, \mathcal{E}, \mathrm{I})$ such that $\{y \in \mathcal{E} \,|\, x \,\mathrm{I}\, y\} \leqq 3$ for all $x \in \mathcal{V}$. If, moreover, $\{y \in \mathcal{E} \,|\, x \,\mathrm{I}\, y\} = 3$ for all $x \in \mathcal{V}$ and $\{x \in \mathcal{V} \,|\, x \,\mathrm{I}\, y\} \geqq 2$ for all $y \in \mathcal{E}$, then we get a *3-configuration*. If one (3)-configuration is an incidence substructure of another (3)-configuration we speak of a *subconfiguration*. Elements of $\mathcal{V}$ will be called *vertices* and elements of $\mathcal{E}$ *edges* of a given (3)-configuration $(\mathcal{V}, \mathcal{E}, \mathrm{I})$. After all, such (3)-configuration are hypergraphs with vertices of degree ≤ 3 and with the property that two distinct vertices cannot lie simultaneously on two distinct edges.

If $x \;\mathrm{I}\; y$ then x, y are said to be *adjacent*. Two distinct vertices adjacent to the same edge; respectively two or three mutually distinct edges adjacent to the same vertex, are said to be *neighbouring*. A (3)-configuration is called *connected* if for any distinct vertices a, b there is a finite sequence of edges $e\ , e\ , \dots, e_k$ such that $a \,\mathrm{I}\, e\ $, $b \,\mathrm{I}\, e_k$ and that e_i, e_i are neighbouring for all $i \in \{1, \dots, k-1\}$.
Troughout the paper every 3-configuration will be supposed to be connected.

Let there be given a 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$. We shall define a halfoperation on $\mathcal{E}$: for every couple (x, y) of neighbouring edges x, y their product $x \cdot y$ is equal to the remaining edge which forms together with x and y a neighbouring triple. We obtain the *edge halfgroupoid* $(\mathcal{E}, \cdot)$ *associated* to $\mathfrak{C}$.

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1.2. Bases of 3-configurations

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-configuration. If $\mathcal{X}$ is a non-void set of edges then denote by $[\mathcal{X}]$ the set of such vertices which are adjacent to least two edges from $\mathcal{X}$.

Now start with a non-void set $\mathcal{A}$ of edges, construct a sequence $\mathcal{A} \ := \mathcal{A}$, $\mathcal{A} \ := [\mathcal{A} \ ]$, $\mathcal{A} \ := [\mathcal{A} \ ]$, $\mathcal{A} \ := [\mathcal{A} \ ]$, ... and form the unions $\mathcal{E}_{\mathcal{A}} := \bigcup_i^\infty \mathcal{A}_{\ i}$, $\mathcal{V}_{\mathcal{A}} := \bigcup_i^\infty \mathcal{A}_{\ i}$. Then $(\mathcal{V}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}}, \mathrm{I})$ is a subconfiguration in $\mathfrak{C}$ *generated* by $\mathcal{A}$; we denote it by $\langle\mathcal{A}\rangle$. We shall call the sequence $(\mathcal{A}_i)_i^\infty$ the *pillar* over $\mathcal{A}$. Further let the *stage* of elements of $\mathcal{A}$, respectively $\mathcal{A}$ be defined as 0, respectively 1. If the stages of elements of $\mathcal{A} \ \cup \mathcal{A} \ \cup \cdots \cup \mathcal{A}_i$ are already known then define the stage of elements of$\mathcal{A}_i \ \setminus \mathcal{A}_i$ to be equal to $i+2$. The edge, respectively vertex a of $\langle\mathcal{A}\rangle$ is said to be *terminal* if there is no edge, respectively no vertex of greater stage neighbouring to a. Under a *constituent* we shall understand a term expression (by means of $\cdot$) of edges over $\mathcal{A}$ within the framework of the pillar. The set $\mathcal{A}$ will be called *independent* if $\langle\mathcal{A}\setminus\{a\}\rangle \neq \mathcal{A}$ for all $a \in \mathcal{A}$. The set $\mathcal{A}$ is called an *edge basis* of $\mathfrak{C}$ if it is independent and if $\langle\mathcal{A}\rangle = \mathfrak{C}$. The set of all element bases of a given 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ cannot be void: we can namely delete, step by step, all dependent elements from the full edge set $\mathcal{E}$. If $\mathcal{B}$ is an edge basis and $\mathcal{T}$ the set of all terminal elements (vertices, as well as edges, if they exist) then $\mathcal{V} + \ \mathcal{B} = \ \mathcal{E} + \ \mathcal{T}$ (a well-known condition, cf. [1], pp. 50-51). Different edge bases of $\mathfrak{C}$ can we have different number of edges. Bases which lead to just one terminal vertex or edge will be denoted as *simple*. Thus for simple bases we have

$$\mathcal{T} = \ \mathcal{V} + \ \mathcal{B} - \ \mathcal{E} = 1, \quad \text{i.e.} \quad \mathcal{B} = \ \mathcal{E} - \ \mathcal{V} + 1 .$$

If $\mathcal{B}$ is a simple basis of $\mathfrak{C}$ then in the corresponding sequence $(\mathcal{B}_i)_i^\infty$ the terminal vertex t lies on two edges e, e' with maximal stage (equal to stage of t minus one) and still on the third edge e''. This gives the equation $e \cdot e' = e''$ and consequently the equation between corresponding terms over $\mathcal{B}$ in the halfgroupoid $(\mathcal{B}, \cdot)$. Analogously, if there is the terminal edge t then it has maximal stage and determines an equality between two corresponding terms over $\mathcal{B}$ in $(\mathcal{E}, \cdot)$.

We give two examples: First let us investigate the 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ on Fig. 1 with $\mathcal{V} = 10$, $\mathcal{E} = 12$, $\mathcal{B} = \{x, y, z\}$.

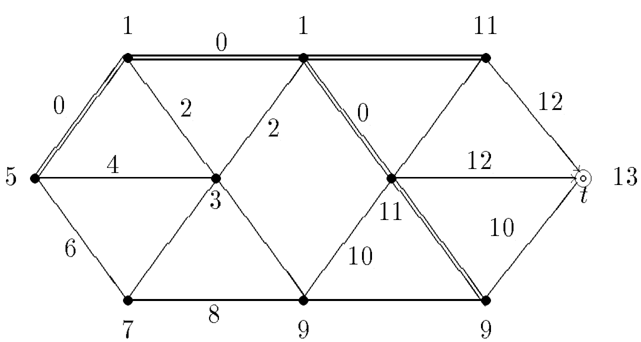
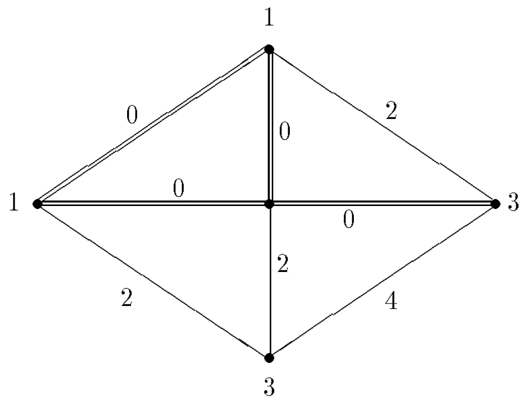


Fig.1

Here: 0 $\ldots x$; 0 $\ldots y$; 0 $\ldots z$; 2 $\ldots xy$; 2 $\ldots yz$; 4 $\ldots (xy)(yz)$; 6 $\ldots x((xy)(yz))$; 8 $\ldots (x((xy)(yz)))(yz)$; 10 $\ldots ((x((xy)(yz)))(yz))(xy)$; 10 $\ldots ((x((xy)(yz))(yz))z$; 12 $\ldots (((x((xy)(yz)))(yz))(xy))$; 12 $\ldots (((x((xy)(yz)))(yz))(xy))y$.

This basis is simple and determines just one terminal vertex t of stage 13, which results as the vertex adjacent to two neighbouring edges of stages 12. The remaining edge through t is one of edges of stage 10. The corresponding term equality is thus $((((x((xy)(yz))(yz))(xy))y) \cdot (((x((xy)(yz))(yz))xy))z) = ((x((xy)(yz))(yz))z$.

The second example deals with a 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ on Fig. 2 with $\mathcal{V} = 5$, $\mathcal{E} = 7$, $\mathcal{B} = \{x, y, z\}$. There is just one terminal element t. It has stage 4 and determines the equality among terms $(xy)(yz)$, $(xy)z$.



0 $\ldots x$, 0 $\ldots y$, 0 $\ldots z$, 2 $\ldots xy$, 2 $\ldots yz$, 2 $\ldots xz$, 4 $\ldots (xy)z \ldots (xy)(yz)$

Fig. 2

Now let $\mathcal{B}$ be a simple edge basis of a given 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ and $t = t$ the term equality over $\mathcal{B}$ in $(\mathcal{E}, \cdot)$ corresponding to the terminal element (vertex or edge).

We shall find some properties of the equality $t = t$:

(i) The set $\mathcal{B}$ must contain at least three edges:
In fact, by the existence of just one edge in $\mathcal{B}$ or just two edges in $\mathcal{B}$, respectively, the pillar over $\mathcal{B}$ should have no vertices of stage 1 respectively 3 contrary to further size of the pillar.

(ii) Every $x \in \mathcal{B}$ occurs at least twice in $t = t$:
By exactly one occurence (in the subterm $u \cdot x$ of the pillar or in $t = x$, respectively $t = x$) we should have on x only one vertex or we should express $x \in \mathcal{B}$ by the edges from $\mathcal{B}$ (which contradicts the independence of edges of the basis).

(iii) No constituent in $(\mathcal{B}_i)_i^\infty$ can be of the form $u(uv)$ with u and v as constituents in $(\mathcal{B}_i)_i^\infty$:
This follows at once from the definition of the pillar over $\mathcal{B}$.

(iv) Equations $t \quad = uv \quad$, $t \quad = uv \quad$, respectively $t \quad = u$, $t \quad = uv$, respectively $t \quad = uv$, $t \quad = u$ with constituents in $(\mathcal{B}_i)_i^{\infty}$ are not possible:
The equality $uv \quad = uv$ or $u = uv$, respectively, should violate the fundamental properties of the 3-configuration ($uv \quad = uv \quad \Rightarrow v \quad = v \quad$, but on the other side, v and v should be distinct edges as constituents in the pillar; moreover, $u = uv$ contradicts the fact that uv must be the third edge neighbour to u and v).

(v) Let t and t be not formed by exactly one edge of $\mathcal{B}$ and let $x \in \mathcal{B}$ enters in $v = ux$, where u, v are constituents of the pillar. Then x enters still in some term $v' = u'x \neq v$ where also u', v' are constituents.

We can define a closure condition in a 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ derived from a 3-configuration $\mathfrak{C}_* = (\mathcal{V}_*, \mathcal{E}_*, \mathrm{I}_*)$ having a simple basis $\mathcal{B}$ with terminal element t and with the *main subconfiguration* $\hat{\mathfrak{C}}_* = (\mathcal{V}_*, \mathcal{E}_* \setminus \{t\}, \mathrm{I}_*)$. The corresponding *closure condition* claims:
For every incidence structure homomorphism $\sigma : \hat{\mathfrak{C}}_* \to \mathfrak{C}$ there exists its prolongation onto a homomorphism $\bar{\sigma} : \mathfrak{C}_* \to \mathfrak{C}$.

In the second part of our considerations we shall adapt this definition for 3-web configurations and 3-webs. As we are familiar with the subject, only less is known about closure conditions in general 3-configurations (or, more largely, in webs of arbitrary number of parallel line pencils).

Part 2 Closure conditions in 3-webs

2.1. Coloured configurations

We say that a (3)-configuration is 3-coloured if for every vertex x an injective map of the set of all edges through x into the colour set $\{1, 2, 3\}$ is given such that for all neigbouring vertices x, y the colour assigned to the common adjacent edge is the same. The edges with colour 1, 2, or 3, respectively will be called *vertical*, *horizontal* or *skew*, respectively.

A 3-coloured (3)-configuration $\mathfrak{C}$ is a *subconfiguration* of a coloured (3)-configuration $\mathfrak{C}$ if it is an incidence substructure preserving the colour of every edge from $\mathfrak{C}$.

Now we shall define a 3-web independently on 3-configurations though it is possible to take a (finite) 3-web as a special 3-configuration.
A *3-web* is defined as an incidence structure $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathcal{P} \leqq 2$, together with a decomposition of the set $\mathcal{L}$ onto mutually disjoint sets $\mathcal{L}$, $\mathcal{L}$, $\mathcal{L}$ such that for every $x \in \mathcal{P}$ and every $i \in \{1, 2, 3\}$ there is just one $y \in \mathcal{L}_i$ such that $x \,\mathrm{I}\, y$ and for every $x \in \mathcal{L}_i$ and every $y \in \mathcal{L}_j$ for distinct $i, j \in \{1, 2, 3\}$ there is just one $z \in \mathcal{P}$ such that $z \,\mathrm{I}\, x$ and $z \,\mathrm{I}\, y$. Elements of $\mathcal{P}$ are *points*, elements of $\mathcal{L}$ *lines*, $\mathcal{L}$, $\mathcal{L}$ and $\mathcal{L}$ are *pencils* (of *parallel lines*). Lines from $\mathcal{L}$, $\mathcal{L}$ or $\mathcal{L}$, respectively, are said to be *vertical*, *horizontal* or *skew*, respectively. If $z \,\mathrm{I}\, x$ and $z \,\mathrm{I}\, y$ is true for distinct lines x, y then z is called the *intersection point* of both lines.

A *homomorphism* of a 3-coloured (3)-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ into a 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is an incidence structure homomorphism which maps every vertical, horizontal or skew edge, respectively, onto a vertical, horizontal or skew line, respectively.

On Fig. 3 we present a non-3-colourable 3-configuration $(\mathcal{V}, \mathcal{E}, \text{ I })$,

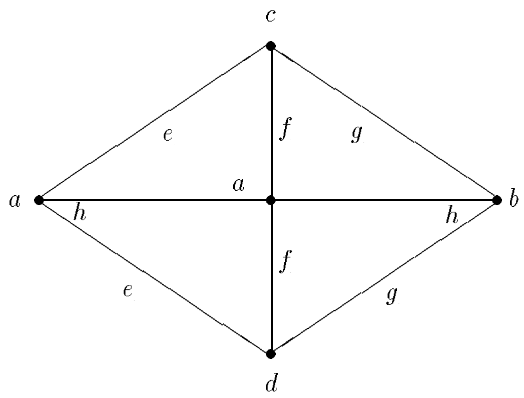


Fig. 3

with $\mathcal{V} = \{a\ , a, b, c, d\}$, $\mathcal{E} = \{e\ , f\ , g\ , e\ , f\ , g\ , h\}$ having one edge adjacent to just three vertices. If the colours of $e\ , f\ , g\ , h$ are $i, j, k, l \in \{1, 2, 3\}$ then $i \neq j \neq k \neq i$ so that every possibility for l leads to a contradiction.

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \text{ I })$ be a 3-coloured 3-configuration and $\mathcal{E}\ , \mathcal{E}\ , \mathcal{E}$ sets of all its horizontal, vertical or complementar edges, respectively. Then define a half-operation $\binom{k}{i\,j} : \mathcal{E}_i \times \mathcal{E}_j \to \mathcal{E}_k$ such that $\binom{k}{i\,j}(x_i, x_j) = x_k$ is defined if and only if $x_i \in \mathcal{E}_i$, $x_j \in \mathcal{E}_j$, $x_k \in \mathcal{E}_k$ are neighbouring edges ($\binom{}{i\ j\ k}$ is an arbitrary permutation of the set $\{1, 2, 3\}$). So we obtain six *parastrophic* halfoperations associated to $\mathfrak{C}$ which from a *three-sorted edge halfquasigroup* $(\mathcal{E}\ , \mathcal{E}\ , \mathcal{E}\ , \{\binom{k}{i j} \big| \binom{}{i\ j\ k} \in \mathcal{S}\ \})$ *associated* to $\mathfrak{C}$.

Let $\mathcal{B}$ be a basis of a simple 3-coloured 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \text{ I })$ with the terminal element t. We express the edges of $\mathfrak{C}$ as constituents in the pillar $(\mathcal{B}_i)_i^\infty$ over $\mathcal{B}$ with use of halfoperations $\binom{k}{i\,j}$. The existence of t leads to an equality between constituents of greatest stage.

From the successive construction of constituents in $(\mathcal{B}_i)_i^\infty$ we obtain the validity of the *Position Property*: In every occurence of the edge e as a constituent $\binom{k_1}{i_1 j_1}(\cdot, \cdot)$ or as a member of other constituents $\binom{k_2}{i_2 j_2}(e, \cdot)$, $\binom{k_3}{i_3 j_3}(\cdot, e)$ the equalities $k\ = i\ = j$ must hold.

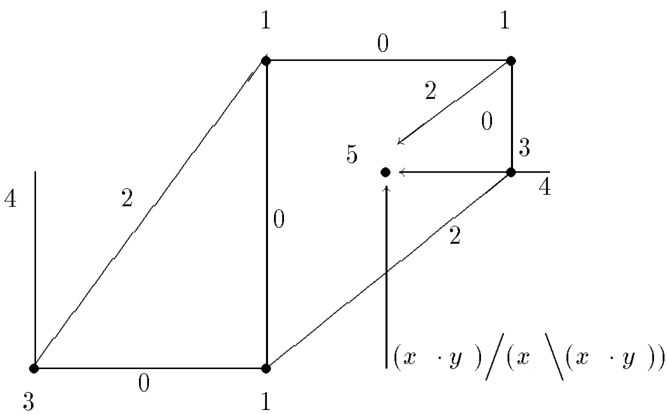
Now we see that Position Property implies that constituents of the form $\binom{k_1}{i_1 j_1}$ $(u, \binom{k_2}{i_2 j_2}(u, v))$ with $i\ \neq i$ do not occur. Property (iii) affirms that constituents of the form $\binom{k_1}{i j_1}(u, \binom{k_2}{i j_2}(u, v))$ do not occur. Property (iv) excludes the equality $t\ = t$ with $t\ = \binom{k}{i\,j}(u, v\ ), t\ = \binom{k}{i\,j}(u, v\ )$ whereas the case $t\ = \binom{k}{i\,j}(t\ , v)$ with $i \neq 3$ is excluded again by Position Property.

If Position Property holds (v) can be reduced onto the occurence of a constituent $\binom{k}{i\,j}(x,u')$, $u' \neq u$ when a constituent $\binom{k}{i\,j}(x,u)$ occurs (with t and t different from a single $x \in \mathcal{B}$).

When a 3-sorted quasigroup $\mathbb{Q} = (Q\ ,Q\ ,Q\ ,\ \{\binom{k}{ij} | (_{i\ j\ k}) \in \mathcal{S}\ \})$ is given with $Q\ =\ Q\ =\ Q\ > 1$ then let $t\ = t$ be a quasigroup identity for $\mathbb{Q}$ with a finite set $\mathcal{X}$, $\mathcal{X} \leq 3$ of variables (the formal construction of terms t , t is supposed as known, see e.g.[1], pp. 30-31). The subterms of $t\ = t$ fulfill necessarily this *Position Property*: In every occurence of a subterm $\tau = \binom{k_1}{i_1 j_1}(\cdot,\cdot)$ and subterms $\binom{k_2}{i_2 j_2}(\tau,\cdot)$, $\binom{k_3}{i_3 j_3}(\cdot,\tau)$ the equalities $k\ = i\ = j$ must hold.

We assume that following conditions are fulfilled: every $x \in \mathcal{X}$ occurs in $t\ = t$ at least twice; the case $t\ = \binom{k}{i\,j}(u,v\ )$, $t\ = \binom{k}{i\,j}(u,v\ )$ is excluded; if t , t are not single variables and a subterm $\binom{k}{i\,j}(x,u)$ occurs for $x \in \mathcal{X}$ then also a further subterm $\binom{k}{i\,j}(x,u')$ with $u' \neq u$ occurs; subterms are reduced by fundamental quasigroups identities $\binom{j}{ik}(\xi,\binom{k}{i\,j}(\xi,\eta) = \eta)$ where $(\ \ )(\xi\ ,\xi\ ) = \xi\ \Leftrightarrow \binom{k}{i\,j}(\xi_i,\xi_j) = \xi_k$ for all permutations $(_{i\ j\ k})$. Under these assumptions the subterm buildup of the identity coincides with the edge generation of a convenient 3-coloured 3-configuration over a simple basis corresponding to $\mathcal{X}$. We leave the details aside.

As an example we shall investigate a 3-coloured 3-configuration $\mathfrak{C} = (\mathcal{V},\mathcal{E},\ \mathrm{I})$ on Fig. 4 with a simple basis $\mathcal{B} = \{x\ ,x\ ,y\ ,y\ \}$, where $x\ ,x$ are vertical edges, $y\ ,y$ horizontal edges and the terminal element is the vertex b. The edges of stages 4 determine the remaining edge through b. This leads to the equality $\big((x\ \cdot\ y\ )/y\ \big)\cdot(x\ \backslash(x\ \cdot y\ )\big) = x\ \cdot y$ where $(\ \ ) = \cdot$, $(\ \ ) = /$, $(\ \ ) = \backslash$. We can write equivalently $(x\ \cdot y\ )/y\ = (x\ \cdot y\ )/(x\ \backslash(x\ \cdot y\ ))$.



$0\ \ldots x\ ,\ 0\ \ldots x\ ,\ 0\ \ldots y\ ,\ 0\ \ldots y\ ,\ 1\ \ldots a',\ 2\ \ldots x\ \cdot y\ ,\ 2\ \ldots x\ \cdot y\ ,$
$2\ \ldots x\ \cdot y\ ,\ 3\ \ldots a,\ 3\ \ldots a'',\ 4\ \ldots (x\ \cdot y\ )/y\ ,\ 4\ \ldots x\ \backslash(x\ \cdot y\ ),\ 5\ \ldots b$

Fig. 4

The stages of single edges and vertices are written as labels on Fig. 4 as well as the corresponding constituents. We can readily confirm that properties (i)-(v) are fulfilled.

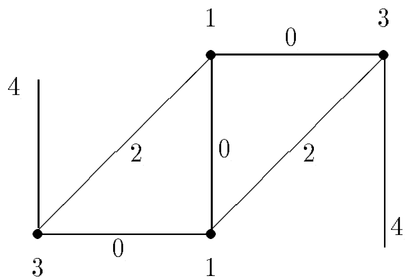
We say that a 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ satisfies the closure condition $\Gamma(\mathfrak{C}, \mathcal{B})$ derived from a 3-coloured 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ with a simple basis $\mathcal{B}$ if every homomorphism $\sigma : \hat{\mathfrak{C}} \to \mathcal{W}$ can be prolonged onto a homomorphism $\bar{\sigma} : \hat{\mathfrak{C}} \to \mathcal{W}$. Here $\hat{\mathfrak{C}} = (\mathcal{V}\backslash\{t\}, \mathcal{E}, \mathrm{I})$ or $\hat{\mathfrak{C}} = (\mathcal{V}, \mathcal{E}\backslash\{t\}, \mathrm{I})$ where t is the terminal vertex or edge.

To a given 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with pencils $\mathcal{L}\ , \mathcal{L}\ , \mathcal{L}\ $ we associate the *coordinatizing 3-sorted quasigroup* $(\mathcal{L}\ , \mathcal{L}\ , \mathcal{L}\ , \{\binom{k}{i\,j} | (_{i\ j\ k}) \in \mathcal{S}\ \})$ which is defined by $\binom{k}{i\,j} : \mathcal{L}\ \times \mathcal{L}_j \to \mathcal{L}_k, (x_i, x_j) \mapsto x_k$ where x_i, x_j, x_k are concurrent lines.

Let a given 3-web $\mathcal{W}$ with pencils $\mathcal{L}\ , \mathcal{L}\ , \mathcal{L}\ $ satisfy the closure condition derived from a 3-coloured 3-configuration $\mathfrak{C}$ with a simple basis $\mathcal{B}$. Then the corresponding term equality in $\mathfrak{C}$ over $\mathcal{B}$ can be interpreted as an identity with respect to the coordinatizing 3-sorted quasigroup $\mathbb{Q}$ and $\mathcal{B}$ as the set of variables. Thus $\mathcal{W}$ determines the corresponding identity over $\mathbb{Q}$.

We shall return to the 3-configuration $\mathfrak{C}$ on Fig. 4 and suppose that a 3-web $\mathcal{W}$ satisfies a closure condition $\Gamma(\mathcal{C}, \mathcal{B})$. Then the coordinatizing 3-sorted quasigroup satisfies the identity $(x\ \cdot y\ )/y\ = (x\ \cdot y\ )\big/(x\ \backslash(x\ \cdot y\ ))$ with variables $x\ , x\ $ of "position" 1 and variables $y\ , y\ $ of "position" 2.

If we take a homomorphism $\varrho : \hat{\mathfrak{C}} \to \mathcal{W}$ with $\varrho(b) = \varrho(a') = \varrho(a'')$ we get a subconfiguration $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ on Fig. 5 with a simple basis $\overline{\mathcal{B}} = \{x\ , y\ , y\ \}$. The closure condition $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ determines the identity $(x\ \cdot y\ )/y\ = (x\ \cdot y\ )/y\ $. Thus in $\mathcal{W}$ from $\Gamma(\mathfrak{C}, \mathcal{B})$ it follows that $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$.



$0\ \ldots x\ ,\ 0\ \ldots y\ ,\ 0\ \ldots y\ ,\ 2\ \ldots x\ \cdot y\ ,\ 2\ \ldots x\ \cdot y\ ,\ 4\ \ldots (x\ \cdot y\ )/y\ ,$
$4\ \ldots (x\ \cdot y\ )/y$

Fig. 5

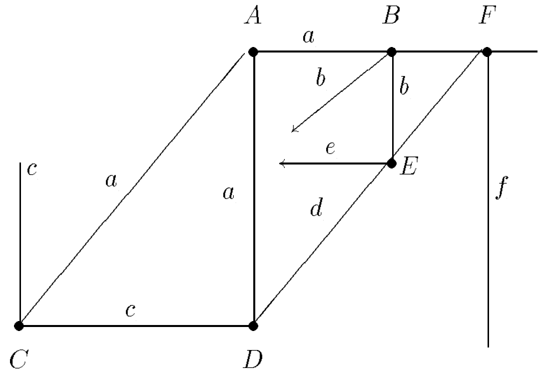


Fig. 6

Now we map $\hat{\mathcal{C}}$ under a homomorphism into $\mathcal{W}$ as on Fig. 6 and suppose that $\Gamma(\overline{\mathfrak{C}},\overline{\mathcal{B}})$ holds in $\boldsymbol{\mathcal{W}}$. We construct the intersection point F of the lines a and d and apply $\Gamma(\overline{\mathfrak{C}},\overline{\mathcal{B}})$ onto the images a , a , a of basis edges so that $c = f$. We apply $\Gamma(\overline{\mathfrak{C}},\overline{\mathcal{B}})$ once more namely onto the images b , a , e of basis elements so that f must go through the intersection point G of b and e . Thus G lies on c and $\Gamma(\mathfrak{C},\mathcal{B})$ holds when applied onto the images a , b , a , c of basis edges. Thus $\Gamma(\overline{\mathfrak{C}},\overline{\mathcal{B}})$ implies $\Gamma(\mathfrak{C},\mathcal{B})$ so that $\Gamma(\mathfrak{C},\mathcal{B}) \Leftrightarrow \Gamma(\overline{\mathfrak{C}},\overline{\mathcal{B}})$ in $\boldsymbol{\mathcal{W}}$ and consequently $(x \cdot y)\big/y = (x \cdot y)\big/(x \backslash(x \cdot y))$, $(x \cdot y)/y = (x \cdot y)\big/y$ are equivalent identities in $\mathbb{Q}$. Note that the second identity has the same left side as the first one whereas on the right side there occur variables x , x , y , y , respectively only variables x , y , y .

These "geometric" considerations will be now complemented by direct algebraic ones.

Let us consider quasigroup identities

(i) $(x\ y)\big/y = (x\ y)\big/(x \backslash(x\ y))$,

(ii) $(x \cdot y)\big/y = (x \cdot y)/y$

once again.

Let (i) be valid in a 3-sorted quasigroup $\mathbb{Q}$. Putting $x\ y = x\ y$, $x = x$ we rewrite the right side of (i): $(x\ y\big/(x \big\backslash(x\ y)) = (x \cdot y)\big/(x \backslash(x\ y)) = (x \cdot y)\big/y$. The left side of (i) is $(x \cdot y)/y$ so that (ii) is valid in $\mathbb{Q}$.

Conversely, let (ii) be valid in $\mathbb{Q}$. Write a duplicate of (ii)

(ii′) $$(x' \cdot y')\big/y' = (x' \cdot y')\big/y'$$

with $x' \cdot y' = x \cdot y$, $y' = y$. Thus $y' = x' \backslash (x \cdot y)$ and, after substitution into (ii$'$), we get $(x' \cdot y' / (x' \backslash (x \cdot y)) = (x' \cdot y) / y'$, i.e. $(x' \cdot y) / (x' \backslash (x \cdot y')) = (x \cdot y) / y$.

Writing here $x = x$, $x' = x$ we get (i).

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