Archivum Mathematicum

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Archivum Mathematicum, Vol. 29 (1993), No. 3-4, 169--176

Persistent URL: http://dml.cz/dmlcz/107480

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 29 (1993), 169 – 176

A REMARK ON SECOND ORDER FUNCTIONAL DIFFERENTIAL SYSTEMS

Valter Šeda, Štefan Belohorec

Abstract. It is proved that under some conditions the set of solutions to initial value problem for second order functional differential system on an unbounded interval is a compact R_δ-set and hence nonvoid, compact and connected set in a Fréchet space. The proof is based on a Kubáček's theorem.

In the paper the initial value problem for second order functional differential system on an unbounded interval is studied. By means of the Kubáček's theorem in [2] which guarantees that the set of all fixed points of a compact map in a Fréchet space is a compact R_δ-set it is shown that under some restrictions on the growth of the right-hand side of that system the set of the solutions to that initial value problem is a compact R_δ-set in a properly chosen Fréchet space of C^1-functions. The result extend a similar theorem for first order functional differential systems which is proved in [2].

1. Introduction

Throughout the whole paper we shall use the following denotations and assumptions:

Let $h > 0$, $b \in R$, $\nu \in N$ and let $|\cdot|$ be a norm in R^ν. Further let $H = C^1([-h,0], R^\nu)$ be provided with the norm $\|x\| = \max\{|x(s)| + |x'(s)| : -h \leqq s \leqq 0\}$ for each $x \in H$ and let $H_0 = C([-h,0], R^\nu)$.

Let $X = C^1([b,\infty), R^\nu$ be equipped with the topology of locally uniform convergence of the functions and of their derivatives on $[b,\infty)$. The topology on the Fréchet space X is given by the metrics

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)}\,,$$

1991 *Mathematics Subject Classification*: 34K05, 34K25.
Key words and phrases: initial value problem, functional differential system, R_δ-set, Kubáček's theorem, Fréchet space.
Received September 15, 1992.

where $p_m(x) = \sup\{|x(t)| + |x'(t)| : b \leqq t \leqq b+m\}$, $x, y \in X$, $m \geqq 1$.

Let $X^* = C^1([b-h,\infty), R^\nu)$ be the Fréchet space provided with the seminorms $p_m^*(x) = \sup\{|x(t)| + x'(t)| : b-h \leqq t \leqq b+m\}$, $x \in X^*$, $m \geqq 1$.

For $x \in C([b-h,\infty), R^\nu)$ we denote by $x_t \in H_0$ the function $x_t(s) = x(t+s)$, $s \in [-h,0]$, $t \geqq b$. Clearly $(x_t)'(s) = (x')_t(s)$, $s \in [-h,0]$ and $x \in X^*$, $t \geqq b$.

Let $f \in C([b,\infty) \times H \times H_0, R^\nu)$, $\psi \in H$. We shall consider the following initial value problem

(1) $x''(t) = f(t, x_t, x_t')$, $b \leqq t < \infty$

(2) $x_b = \psi$, $x_b' = \psi'$.

A solution x of (1), (2) is a function $x \in X^* \cap C^2([b,\infty), R^\nu)$ which satisfies (2) and the functional differential system (1) at each point $t \geqq b$.

Now we shall state the Kubáček theorem in [2] as Lemma 1. In that lemma a compact R_δ-set in a metric space (E,ϱ) means a nonempty subset F of E which is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. By [1], p. 92, a metric space G is called an absolute retract when for each metric space H and each closed $K \subset H$, each continuous map $f : K \to G$ has a continuous extension $g : H \to G$. E.g. a nonempty convex subset of a Fréchet space is an absolute retract.

Lemma 1. *Let M be a nonempty closed set in a Fréchet space (E,ϱ), $T : M \to E$ a compact map (i.e. T is continuous and $T(M)$ is a relatively compact set). Denote by S the map $I - T$, where I is the identity map on E. Let there exist a sequence $\{U_n\}$ of closed convex sets in E fulfilling*

(i) *$0 \in U_n$ for each $n \in N$;*

(ii) $\lim_{n\to\infty} \operatorname{diam} U_n = 0$

and a sequence $\{T_n\}$ of maps $T_n : M \to E$ fulfilling

(iii) *$T(x) - T_n(x) \in U_n$ for each $x \in M$ and each $n \in N$;*

(iv) *the map $S_n = I - T_n$ is a homeomorphism of the set $S_n^{-1}(U_n)$ onto U_n.*

Then the set F of all fixed points of the map T is a compact R_δ-set.

In a special case, when $E = X$, $\varrho = d$, Lemma 1 implies the following lemma.

Lemma 2. *Let (X,d) be the Fréchet space given above, let φ, $\varphi_n \in C([b,\infty)$, $(0,\infty))$, $n \in N$ and let the following condition be satisfied:*

(v) *For each $t \in [b,\infty)$ the sequence $\{\varphi_n(t)\}$ is nonincreasing and $\lim_{n\to\infty} \varphi_n(t) = 0$.*

Let r, $s \in R^\nu$ and let

$$M = \{x \in X : |x(t) - r| + |x'(t) - s| \leqq \varphi(t), \quad t \geqq b, \quad x(b) = r, \quad x'(b) = s\}.$$

Suppose that $T : M \to X$ is a compact map with the property $T(x)(b) = r$, $(T(x))'(b) = s$ for each $x \in M$ and there exists a sequence $\{T_n\}$ of compact maps $T_n : M \to X$ such that $T_n(x)(b) = r$, $(T_n(x))'(b) = s$ for each $x \in M$ and

(vi) $|T_n(x)(t) - T(x)(t)| + |(T_n(x))'(t) - T(x)'(t)| \leqq \varphi_n(t), \quad x \in M, \ t \geqq b;$

(vii) *for every* $n \in N$ *there exists a function* $\varphi_{*n} \in C([b,\infty),(0,\infty))$ *such that*

$$\varphi_{*n} + \varphi_n \leqq \varphi \text{ on } [b,\infty)$$

and

$$|T_n(x)(t) - r| + |(T_n(x))'(t) - s| \leqq \varphi_{*n}(t), \quad x \in M,\ t \geqq b;$$

(viii) *the map* $S_n = I - T_n$ *is injective on* M *where* I *is the identity on* X.

Then the set F *of all fixed points of the map* T *is a compact* R_δ.

Proof. The set

$$U_n = \{x \in X : |x(t)| + |x'(t)| \leqq \varphi_n(t), \quad t \geqq b, x(b) = 0, x'(b) = 0\}$$

is convex and closed in X for each $n \in N$. We shall show that the sequence $\{U_n\}$ satisfies all conditions of Kubáček's theorem when $E = X$, $\varrho = d$. Then the statement of Lemma 2 follows from Lemma 1.

Clearly the condition (i) is fulfilled. Further for a given $\varepsilon > 0$ there exists an $m_0 \in N$ such that $\sum_{m=m_0+1}^{\infty} (1/2^m) < \varepsilon/2$. (v) and the Dini theorem imply that the sequence $\{\varphi_n\}$ converges on $[b,\infty)$ locally uniformly to 0, therefore for $\varepsilon > 0$ and $m_0 \in N$ there exists an $n_0 \in N$ such that $q_m(\varphi_n) = \sup\{|\varphi_n(t)| : b \leqq t \leqq b+m\} \leqq \varepsilon/4m_0$ for $n \geqq n_0$ and $m = 1,2,\dots,m_0$. Thus for $n \geqq n_0$ and $x, y \in U_n$ we have

$$\begin{aligned} d(x,y) &= \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)} \leqq \sum_{m=1}^{m_0} p_m(x-y) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \\ &= \sum_{m=1}^{m_0} 2\ q_m(\varphi_n) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \leqq 2m_0\ \frac{\varepsilon}{4m_0} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

This implies that the condition (ii) is satisfied. The assumption (iii) follows from (vi) and from the definition of T, T_n, $n \in N$. To show that (iv) is fulfilled it suffices to prove the inclusion $U_n \subset S_n(M)$. (viii) then implies that S_n is a bijection of $S_n^{-1}(U_n)$ onto U_n and the continuity of $S_n^{-1}\big|_{U_n}$ is then a consequence of the compactness of T_n.

Thus we have to prove that for each $y \in U_n$ there is an $x_y \in M$ such that $x_y - T_n(x_y) = y$. This means that for every $y \in U_n$ the map $P_n(x) = y + T_n(x)$ has a fixed point. (vii) implies that

$$\begin{gathered} |y(t) + T_n(x)(t) - r| + |y'(t) + (T_n(x))'(t) - s| \leqq \\ |y(t)| + |y'(t)| + |T_n(x)(t) - r| + |(T_n(x))'(t) - s| \leqq \\ \varphi_n(t) + \varphi_{*n}(t) \leqq \varphi(t), \quad t \geqq b\,, \end{gathered}$$

and $P_n(x)(b) = r$, $(P_n(x))'(b) = s$ for each $x \in M$. Therefore $P_n(M) \subset M$. As M is a closed, convex bounded set and P_n is a compact map, by the Tichonov fixed point theorem, P_n has a fixed point. This completes the proof of the lemma. □

In what follows the function φ in Lemma 2 will be given as a solution of an integral equation. The existence of a solution to that equation will be discussed in the following two lemmas.

Lemma 3. *Let $\psi \in H$, $\omega \in C([b,\infty),[0,\infty))$, $g \in C([0,\infty),[0,\infty))$ be a nondecreasing function. Further denote*

$$U(x)(t) = |\psi'(0)|(t-b) + \int_b^t (t-s+1)\omega(s)g(\|\psi\| + x(s))\,ds,\ \ b \leqq t < \infty\,, \tag{3}$$

for each $x \in C([b,\infty),[0,\infty))$. Then the following statements are true:

1. A solution $\varphi \in C([b,\infty),[0,\infty))$ of the integral equation

$$\varphi(t) = U(\varphi)(t),\ \ b \leqq t < \infty\,, \tag{4}$$

exists iff there exists a function $\lambda \in C([b,\infty),[0,\infty))$ such that

$$\lambda(t) \geqq U(\lambda)(t),\ \ b \leqq t < \infty\,. \tag{5}$$

2. The solution φ of (4) (whenever it exists) is nondecreasing in $[b,\infty)$.

Proof. 1. Necessary condition is clear. Sufficient condition. We shall proceed by the method of steps. Hence we prove by mathematical induction that for each $m = 1, 2, \dots,$:

a) There exists a solution $y_m \in C([b,b+m],[0,\infty))$ of (4) on $[b,b+m]$ satisfying the inequalities $0 \leqq y_m(t) \leqq \lambda(t)$, $b \leqq t \leqq b+m$ and

b) $y_{m+1}(t) = y_m(t), \quad b \leqq t \leqq b+m.$

Consider the partially ordered Banach space $X_1 = C([b,b+1),R)$ with the sup-norm where $z_1 \leqq z_2$ if and only if $z_1(t) \leqq z_2(t)$ for each $t \in [b,b+1]$ and each pair z_1, z_2 from that space. Then, by definition, the interval $\langle z_1, z_2\rangle = \{y \in X_1 : z_1(t) \leqq y(t) \leqq z_2(t), \quad b \leqq t \leqq b+1\}$. The operator $U : X_1 \to X_1$ defined by (3) on $[b,b+1]$ is completely continuous, nondecreasing and in view of (5), it maps the interval $\langle 0, \lambda|_{[b,b+1]}\rangle$ into itself. Hence, by the Schauder fixed point theorem, there exists a fixed point y_1 of U which satisfies (4) in $[b,b+1]$.

Suppose, now, that there exists a solution y_m of (4) on $[b,b+m]$. Then we consider the space $X_{m+1} = C([b,b+m+1],R)$ with the sup-norm and the natural partial ordering. X_{m+1} is a partially ordered Banach space. The operator U given by (3) on [b, b+m+1] maps X_{m+1} into itself, it is completely continuous, nondecreasing and similarly as before, it maps the interval $\langle 0, \lambda|_{[b,b+m+1]}\rangle = \{y \in X_{m+1} : 0 \leqq y(t) \leqq \lambda(t), \quad b \leqq t \leqq b+m+1\}$ into itself. Moreover, U maps the closed and convex set $Y_{m+1} = \{x \in X_{m+1} : x(t) = y_m(t), \quad b \leqq t \leqq b+m\} \cap \langle 0, \lambda|_{[b,b+m+1]}\rangle$ into itself. Hence there exists a fixed point y_{m+1} of U in Y_{m+1} and this is the searched function y_{m+1} with the properties (a), and (b). Then the function $\varphi(t) = y_m(t)$ for $b \leqq t \leqq b+m$ and $m = 1, 2, \dots,$ is a solution of (4) in $[b,\infty)$ and it satisfies the inequalities

$$0 \leqq \varphi(t) \leqq \lambda(t), \quad b \leqq t < \infty\,.$$

2. The statement follows from (3), (4). □

Lemma 4. *Let $\psi \in H$ and let $g \in C([0,\infty),(0,\infty))$ be a nondecreasing function. Then to each function $\varrho \in C([b,\infty),[0,\infty))$ there exists a unique pair of functions $\varphi \in C([b,\infty),[0,\infty))$, $\omega \in C([b,\infty)),[0,\infty))$ such that*

$$\omega(t) = \frac{\varrho(t)}{g(\|\psi\| + \varphi(t))}, \quad t \geqq b \tag{6}$$

and φ is a solution of the equation (4).

Proof. Define

$$\varphi(t) = |\psi'(0)|(t-b) + \int_b^t (t-s+1)\,\varrho(s)\,ds, \quad b \leqq t < \infty\,. \tag{7}$$

If φ is a solution of (4) and ω satisfies (6), then necessarily φ has the form (7). Hence there exists at most one pair of functions φ, ω satisfying (4), and (6) simultaneously. On the other hand, if φ is determined by (7) and ω satisfies (6), then φ is a solution of (4). The statement of the lemma is proved. □

Example 1. If $\varrho(t) = e^t, \quad b \leqq t < \infty$, then by the last lemma the function

$$\varphi(t) = (|\psi'(0)| - e^b)(t-b) + 2(e^t - e^b), \quad b \leqq t < \infty$$

is increasing in $[b,\infty)$ and if $g(u) = ku + q$, $u \geqq 0$, where $k \geqq 0$, $q > 0$, then φ is a solution of (4) with

$$\omega(t) = \frac{e^t}{k(\|\psi\| + \varphi(t)) + q}, \quad t \geqq b\,.$$

2. Main Theorem

Now we shall state and prove the main theorem.

Theorem 1. *Let $\psi \in H$, $f \in C([b,\infty)\times H \times H_0, R^\nu)$. Let, further, $\omega \in C([b,\infty),[0,\infty))$, $g \in C([0,\infty),[0,\infty))$ be a nondecreasing function and let*

(ix) $\quad |f(t,\mathcal{X}_t,\mathcal{X}'_t)| \leqq \omega(t)\,g(\|\mathcal{X}_t\|)$ *for each* $(t,\mathcal{X}) \in [b,\infty)\times M^*$
where

$$M^* = \{x \in X^* : |x(t)-\psi(0)| + |x'(t)-\psi'(0)| \leqq \varphi(t) \quad \text{for} \quad t \geqq b \\ \text{and } x_b = \psi, \quad x'_b = \psi'\}\,,$$

φ is a solution of the equation (4) in $[b,\infty)$.

Then the problem (1), (2) has a solution x satisfying the inequality

$$|x(t)-\psi(0)| + |x'(t)-\psi'(0) \leqq \varphi(t), \quad t \geqq b$$

and the set F^* of all such solutions is a compact R_δ-set in the space X^*.

Proof. Consider the set

$$M = \{x \in X : |x(t) - \psi(0)| + |x'(t) - \psi'(0)| \leqq \varphi(t) \text{ for } t \geqq b \\ \text{and } x(b) = \psi(0), x'(b) = \psi'(0)\}.$$

Evidently the map $P : X^* \to X$ determined by $P(x) = x\big|_{[b,\infty)}$ is a homeomorphism of M^* onto M. Let the map $T : M \to X$ be defined by

$$T(x)(t) = \psi(0) + \psi'(0)(t-b) + \int_b^t (t-s) f[s, (P^{-1}x)_s, (P^{-1}x)'_s]\, ds\,, \\ x \in M,\ t \geqq b\,. \tag{8}$$

Then $F^* = P^{-1}(F)$ where F is the set of all fixed points of the map T. As a homeomorphic image of a compact R_δ-set is again a compact R_δ-set, it suffices to prove that F is a compact R_δ-set in the space X. This will be done by using Lemma 2 where we put $r = \psi(0)$, $s = \psi'(0)$. The maps $T_n : M \to X$ defined by

$$T_n(x)(t) = \begin{cases} \psi(0) + \psi'(0)(t-b) & \text{if } \ b \leqq t \leqq b + \frac{1}{n} \\ \psi(0) + \psi'(0)(t-b) + \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s) \\ \qquad f[s, (P^{-1}x)_s, (P^{-1}x)'_s]\, ds & \text{if } \ t \geqq b + \dfrac{1}{n}, \quad n \in N \end{cases} \tag{9}$$

as well as T are compact due to (ix) and, again by that assumption,

$$|T_n(x)(t) - T(x)(t)| + |(T_n(x))'(t) - (T(x))'(t)| \leqq \\ = \begin{cases} \int_b^t (t-s+1)\omega(s) g(\|(P^{-1}x)_s\|)\, ds, & b \leqq t \leqq b + \frac{1}{n} \\ \int_b^t (t-s+1)\omega(s) g(\|(P^{-1}x)_s\|)\, ds - \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s + 1)\omega(s) g(\|(P^{-1}x)_s\|)\, ds, \\ \qquad b + \dfrac{1}{n} \leqq t < \infty\,. \end{cases}$$

As φ is nondecreasing in $[b, \infty)$, $\|(P^{-1}x)_s\| \leqq \|\psi\| + \varphi(s)$ and thus,

$$|T_n(x)(t) - T(x)(t)| + |(T_n(x))'(t) - (T(x))'(t)| \leqq \varphi_n(t), \\ x \in M, \quad t \geqq b$$

where

$$\varphi_n(t) = \begin{cases} \int_b^t (t-s+1)\omega(s) g(\|\psi\| + \varphi(s))\, ds, & b \leqq t \leqq b + \frac{1}{n} \\ \int_b^t (t-s+1)\omega(s) g(\|\psi\| + \varphi(s))\, ds - \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s + 1)\omega(s)\,. \\ \qquad g(\|\psi\| + \varphi(s))\, ds, \quad b + \dfrac{1}{n} \leqq t < \infty, \quad n \in N\,. \end{cases}$$

Clearly that $\varphi_n(t) \in C([b,\infty),[0,\infty))$, and $\lim_{n\to\infty} \varphi_n(t) = 0$, $\varphi_{n+1}(t) \leqq \varphi_n(t)$ for each $t \geqq b$ can be proved. Hence these functions satisfy the assumptions (v), (vi) of Lemma 2.

Further

$$|T_n(x)(t) - \psi(0)| + |T_n(x)'(t) - \psi'(0)| \leqq \varphi_{*n}(t)$$
$$t \geqq b, \quad x \in M$$

where

$$\varphi_{*n}(t) = \begin{cases} |\psi'(0)|(t-b), & b \leqq t \leqq b + \frac{1}{n} \\ |\psi'(0)|(t-b) + \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s + 1)\,\omega(s) g(\|\psi\| + \varphi(s))\,ds \end{cases}$$
$$b + \frac{1}{n} \leqq t < \infty, \quad n \in N,$$

are nonnegative continuous functions on $[b,\infty)$ and with respect to the meaning of φ we have that $\varphi_n(t) + \varphi_{*n}(t) = \varphi(t)$, $t \geqq b$, $n \in N$. Thus (vii) in Lemma 2 is satisfied, too. So it remains to show that the assumption (viii) in that lemma holds.

If $x, y \in M$, $x \neq y$, then there is a $t_0 \in [b,\infty)$ such that $x(t_0) \neq y(t_0)$. If $b \leqq t_0 \leqq b + \frac{1}{n}$, then $x(t_0) - T_n(x)(t_0) = x(t_0) - \psi(0) - \psi'(0)(t_0 - b) \neq y(t_0) - \psi(0) - \psi'(0)(t_0 - b) = y(t_0) - T_n(y)(t_0)$. In the other case there exists a $t_1 \geqq b + \frac{1}{n}$ such that $t_1 = \sup\{\tau > b : x(t) = y(t) \text{ for } b \leqq t < \tau\}$. Then there exists a $t_0 \in (t_1, t_1 + \frac{1}{n})$ such that $x(t_0) \neq y(t_0)$. This gives that

$$\begin{aligned} T_n(x)(t_0) &= \psi(0) + \psi'(0)(t_0 - b) + \int_b^{t_0 - \frac{1}{n}} (t_0 - \frac{1}{n} - s) f[s, (P^{-1}x)_s\,, (P^{-1}x)'_s\,ds \\ &= \psi(0) + \psi'(0)(t_0 - b) + \int_b^{t_0 - \frac{1}{n}} (t_0 - \frac{1}{n} - s) f[s, (P^{-1}y)_s\,, (P^{-1}y)'_s]\,ds \\ &= T_n(y)(t_0) \end{aligned}$$

and hence, $x(t_0) - T_n(x)(t_0) \neq y(t_0) - T_n(y)(t_0)$, $n \in N$. Thus all assumptions of Lemma 2 are fulfilled and the statement of Theorem 1 follows from that lemma.□

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