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ON FRAME CONGRUENCES GENERATED BY FRAME TOLERANCES

Josef Niederle

Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. Nuclei of frame congruences generated by frame tolerances and by lattice congruences are constructed.

A *frame* is a complete lattice $(L, \leq)$ in which $x \wedge \bigvee\{x_i | i \in I\} = \bigvee\{x \wedge x_i | i \in I\}$ holds for any nonempty index set I and arbitrary elements $x, x_i \in L$ $(i \in I)$. Johnstone's book [1] is our general reference about frames. A *tolerance* of the frame $(L, \leq)$ is a reflexive and symmetric relation on L compatible with operations $\wedge$ and $\vee$ induced by the order $\leq$. A transitive tolerance is said to be a *congruence*. A *frame tolerance* is a tolerance T such that $[\bigvee\{x_i | i \in I\}, \bigvee\{y_i | i \in I\}] \in T$ whenever $[x_i, y_i] \in T$ $(i \in I)$. A *frame congruence* is a transitive frame tolerance. A mapping $f\colon (L, \leq) \to (L, \leq)$ is called *extensive* if $x \leq f(x)$ for each $x \in L$, *idempotent* if $f(f(x)) = f(x)$ for each $x \in L$, and a *meet-endomorphism* if $f(x \wedge y) = f(x) \wedge f(y)$ for each $x, y \in L$. An element $x \in L$ is a *fixpoint* of f if $f(x) = x$. Recall that a tolerance (congruence) of $(L, \leq)$ is a frame tolerance (frame congruence) if and only if each of its polars possesses a top element. Further, there is a one-to-one correspondence between frame tolerances of $(L, \leq)$ and extensive meet-endomorphisms of $(L, \leq)$ such that the extensive meet-endomorphism associated with a tolerance T, which will be referred to as the *nucleus* of T, assigns to each element $x \in L$ the top element of the polar $T(x) := \{y \in L | [x, y] \in T\}$ of the element x in the frame tolerance T. A frame tolerance is a frame congruence if and only if its nucleus is idempotent. We know that frame tolerances of a frame form a frame with respect to set inclusion, the same applies to frame congruences. The operator FC assigning to any relation R the least frame congruence that includes R is a frame homomorphism (that is a join-complete lattice homomorphism) of the frame of all frame tolerances onto the frame of all frame congruences. See [2], [3] and [4] for these and further results. The aim of this paper is to provide a

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more detailed analysis how FC works. The axiom of choice is a prerequisite of our considerations.

We know that the transitive closure of a tolerance is a congruence, see [5]. Therefore the transitive closure of a frame tolerance is a (lattice) congruence, but it may not be a frame congruence.

Example 1. Let α be an arbitrary infinite ordinal, consider $L := \alpha + 1$ with the usual order. Since it is a complete chain (with the greatest element α), it is a frame. Define $P := \{[\beta, \beta + 1] | \beta < \alpha\}$, and $T := P \cup \check{P} \cup \Delta_L$, where $\check{P}$ is the reverse relation to P and Δ_L is the identity relation on L. It is obvious that T is a tolerance of $(L, \leq)$. As $T(\beta) = \{\beta - 1, \beta, \beta + 1\}$ for $0 < \beta < \alpha$, β being a successor ordinal, $T(\beta) = \{\beta, \beta + 1\}$ for $0 < \beta < \alpha$, β being a limit ordinal, $T(0) = \{0, 1\}$, $T(\alpha) = \{\alpha - 1, \alpha\}$, α being a successor ordinal, and $T(\alpha) = \{\alpha\}$, α being a limit ordinal, the polar of each element of L contains a greatest element, and we may conclude that T is a frame tolerance. Now, the polar of the element 0 in the smallest frame congruence FC(T) that includes T contains with each element $\beta \neq \alpha$ its successor, and simultaneously it has a greatest element. It is clear that this greatest element is exactly α, and FC(T) is the universal relation on L. In contrast, the polar of the element 0 in the smallest congruence $\Theta(T)$ that includes T consists of only (at most) finite ordinal numbers.

This example was not chosen by chance. In a similar manner we can "construct" frame congruences generated by frame tolerances in the general case. This fact follows from the well-known fixpoint theorems. We will utilize the construction of least fixpoints as "limits" of monotone transfinite sequences.

Lemma 1. *Let $(L, \leq)$ be a frame, $x \in L$ and $f\colon L \to L$ an extensive mapping. Then there exists precisely one order-preserving mapping f_x from the class of all ordinals to L such that $f_x(0) = x$, $f_x(\beta) = f(f_x(\beta - 1))$ if β is a successor ordinal, and $f_x(\beta) = \bigvee\{f_x(\gamma) | \gamma < \beta\}$ if β is a limit ordinal.*

Proof. Follows immediately from the transfinite recursion theorem.

Lemma 2. *Let $(L, \leq)$ be a frame, $x \in L$, and $f\colon L \to L$ an extensive mapping. Then there exists an ordinal β_x such that $f_x(\gamma) = f_x(\beta_x)$ whenever $\beta_x \leq \gamma$. Moreover, if f is order-preserving, $f_x(\beta_x)$ is the least fixpoint of the mapping f in the principal dual ideal $\langle x)$.*

Proof. Straightforward and therefore omitted.

Definition. The ordinal β_x just defined is said to be the *closure ordinal* of f in x.

Next we will investigate some properties of the mapping f^β defined by $f^\beta(x) := f_x(\beta)$, where f is an extensive meet-endomorphism of the frame $(L, \leq)$.

Lemma 3. *Let $(L, \leq)$ be a frame, let $f\colon L \to L$ be an extensive meet-endomorphism and β be an ordinal. Then the mapping f^β is an extensive meet-endomorphism of the frame $(L, \leq)$ and each fixpoint of f is also a fixpoint of f^β.*

Proof. We first show by transfinite induction on β that f^β is a meet-endomorphism for all ordinals β. For $\beta = 0$ we have $f^0(x) \wedge f^0(y) = f_x(0) \wedge f_y(0) = x \wedge y = f_{x\wedge y}(0) = f^0(x \wedge y)$. Suppose the assertion is valid for all $\gamma < \beta$. If β is a successor ordinal, we have $f^\beta(x) \wedge f^\beta(y) = f_x(\beta) \wedge f_y(\beta) = ff_x(\beta - 1) \wedge ff_y(\beta - 1) = f(f_x(\beta-1) \wedge f_y(\beta-1)) = f(f^{\beta-1}(x) \wedge f^{\beta-1}(y)) = ff^{\beta-1}(x \wedge y) = ff_{x\wedge y}(\beta-1) = f_{x\wedge y}(\beta) = f^\beta(x \wedge y)$, using the definition of f^β, the definition of $f_-(\beta)$, the meet-compatibility of f, the definition of $f^{\beta-1}$, the induction hypothesis, the definition of $f^{\beta-1}$, the definition of $f_-(\beta)$ and the definition of f^β. If β is a limit ordinal, we obtain $f^\beta(x) \wedge f^\beta(y) = f_x(\beta) \wedge f_y(\beta) = \bigvee\{f_x(\gamma)|\gamma < \beta\} \wedge \bigvee\{f_y(\gamma)|\gamma < \beta\} = \bigvee\{f_x(\gamma_1) \wedge f_y(\gamma_2)|\gamma_1,\gamma_2 < \beta\} = \bigvee\{f_x(\max(\gamma_1,\gamma_2)) \wedge f_y(\max(\gamma_1,\gamma_2))|\gamma_1,\gamma_2 < \beta\} = \bigvee\{f_x(\gamma) \wedge f_y(\gamma)|\gamma < \beta\} = \bigvee\{f^\gamma(x) \wedge f^\gamma(y)|\gamma < \beta\} = \bigvee\{f^\gamma(x \wedge y)|\gamma < \beta\} = \bigvee\{f_{x\wedge y}(\gamma)|\gamma < \beta\} = f_{x\wedge y}(\beta) = f^\beta(x\wedge y)$, using the definition of f^β, the definition of $f_-(\beta)$, the infinite distributivity, the monotonicity of f_x and f_y, the definition of f^γ, the induction hypothesis, the definition of f^γ, the definition of $f_{x\wedge y}$ and the definition of f^β. This completes the induction argument. Extensivity of f^β is obvious because $f^\beta(x) = f_x(\beta) \geq f_x(0) = x$. Now let z be a fixpoint of f. We show by transfinite induction that z is a fixpoint of f^β for all β. For $\beta = 0$ clearly $f^0(z) = f_z(0) = z$. Suppose $f^\gamma(z) = z$ for all ordinals $\gamma < \beta$. If β is a successor ordinal, $f^\beta(z) = f_z(\beta) = ff_z(\beta-1) = ff^{\beta-1}(z) = f(z) = z$. If β is a limit ordinal, $f^\beta(z) = f_z(\beta) = \bigvee\{f_z(\gamma)|\gamma < \beta\} = \bigvee\{f^\gamma(z)|\gamma < \beta\} = \bigvee\{z|\gamma < \beta\} = z$.

Notation. The set of all fixpoints of f will be denoted by $\mathrm{fix}(f)$.

Proposition 1. *Let $(L, \leq)$ be a frame, $f\colon L \to L$ an extensive meet-endomorphism of $(L, \leq)$. Define $g(x) := \min(\mathrm{fix}(f) \cap \langle x))$. Then g is the least idempotent extensive meet-endomorphism of $(L, \leq)$ such that $f \leq g$.*

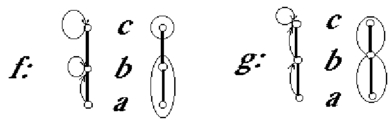
Proof. We first show that g is indeed an idempotent extensive meet-endomorphism of the frame $(L, \leq)$. The closure ordinal of f in x will be denoted by β_x. There exists an ordinal α such that $\beta_x \leq \alpha$ for each $x \in L$ because β_x-s form a set of ordinals. So we have $f^\alpha(x) = f_x(\alpha) = f_x(\beta_x) = \min(\mathrm{fix}(f) \cap \langle x))$. Hence $g(x) = f^\alpha(x)$. By the preceding lemma, g is an extensive meet-endomorphism. As $g(x)$ is a fixpoint of f for each $x \in L$, it is also a fixpoint of $f^\alpha = g$. Consequently g is idempotent. Clearly $f(x) \leq g(x)$. If h is also an idempotent extensive meet-endomorphism of $(L, \leq)$ such that $f \leq h$, then $x \leq h(x) \leq f(h(x)) \leq h(h(x)) = h(x)$, hence $h(x)$ is a fixpoint of f in $\langle x)$. Therefore $g(x) \leq h(x)$.

Consider the set $\mathrm{Ext}(L, \leq)$ of all extensive meet-endomorphisms of the frame $(L, \leq)$. For $f, g \in \mathrm{Ext}(L, \leq)$ define $L^f(g) := fg$. We know that $\mathrm{Ext}(L, \leq)$ is a frame with respect to the pointwise order (cf. [1]) and it is easy to see that L^f is an extensive meet-endomorphism of $\mathrm{Ext}(L, \leq)$. Hence there exists a least fixpoint of L^f in $\langle f)$ according to lemma 2. We can thus rewrite the statement of the preceding proposition as follows.

Corollary. *Let f be the nucleus of a frame tolerance T of $(L, \leq)$. Then the least fixpoint of L^f in $\langle f)$ is identical with the nucleus of* $\mathrm{FC}(T)$.

Note that fixpoints of L^f in $\langle f)$ are not idempotent in general.

Example 2. Consider the following situation:



Both f and g are nuclei, $f \leq g$, $fg = g$, $gg \neq g$.

Conversely, idempotents are obviously fixpoints of L^f.

In view of Proposition 1, we can construct FC(T) as follows. First, take the transitive closure of T, which is equal to $\Theta(T)$. Second, find congruence blocks of $\Theta(T)$ with top elements. Third, to each $x \in L$ assign the least top element $\tilde{x}$ of a block such that $x \leq \tilde{x}$. Fourth, take the quotient congruence of $^\sim$. It is FC(T). More precisely, $\mathrm{FC}(T) = \{[x,y] | \tilde{x} = \tilde{y}, \tilde{x} = \bigwedge\{t \in \langle x) | [s,t] \in \Theta(T) \implies s \leq t\}, \tilde{y} = \bigwedge\{t \in \langle y) | [s,t] \in \Theta(T) \implies s \leq t\}, \Theta(T) = \{[s,t] | \exists n \in \mathbb{N}\, \exists s_0, \ldots, s_n \in L, s = s_0, t = s_n, [s_{i-1}, s_i] \in T (i = 1, \ldots, n)\}\}$.

Not every congruence of a frame is a transitive closure of a frame tolerance.

Example 3. Take $L := \omega_1 + 1$. It is a frame. Let $C := \{[\omega_1, \omega_1]\} \cup \{[\alpha, \beta] | \alpha < \omega_1, \beta < \omega_1\}$. It is a congruence of $(L, \leq)$ with precisely two blocks ω_1 and $\{\omega_1\}$, which is not a frame congruence because ω_1 fails to have a top element. Suppose T is a frame tolerance on $(L, \leq)$ such that $C = \Theta(T)$, let f be its nucleus. Then for any $\alpha < \omega_1$ there exists $n < \omega$ such that $\alpha < f^n(0)$, hence ω_1 possesses a cofinal subset of type ω, which is a contradiction. (We of course accept the axiom of choice.)

Nevertheless, we are still able to show that the preceding construction works also in this more general case.

Definition. Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$. We say that an element $x \in L$ is C-*good* if it is a top element of a block of C.

The set of all C-good elements forms a closure system in $(L, \leq)$.

Lemma 4. *Let $(L, \leq)$ be a frame, C a congruence on $(L, \leq)$. Then the top element in $(L, \leq)$ is C-good, and if x_i $(i \in I)$ are C-good, $\bigwedge\{x_i | i \in I\}$ is C-good as well.*

Proof. The top element $\top$ in L is obviously a top element in the block $[\top]_C$. Suppose that x_i $(i \in I)$ are C-good, that is x_i is a top element in $[x_i]_C$ for every index $i \in I$. But then $x_i \vee x \in [x_i]_C$ whenever $x \in [\bigwedge\{x_i | i \in I\}]_C$. This yields that $\bigwedge\{x_i | i \in I\}$ is a top element in $[\bigwedge\{x_i | i \in I\}]_C$.

Notation. The least C-good element of $(L, \leq)$ in $\langle x)$ will be denoted by $\nu_C(x)$.

Lemma 5. *Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$, $x, y \in L$, $x \leq y$, x being C-good. Then $z \leq \nu_C(y)$, $y \wedge z = x$ implies $z = x$.*

Proof. For each $a \in L$ put $A(a) := \{b \in L | a < b, [a,b] \in C\}$ if a is not C-good, and $A(a) := \{a\}$ otherwise. Let f be a choice function associated with A. It is obviously extensive, and its fixpoints are precisely C-good elements. We know that there exists a transfinite sequence $\{y_\beta\}$ and an element $\tilde{y}$ such that $y_0 = y$, $y_\beta = f(y_{\beta-1})$ if β is a successor ordinal, $y_\beta = \bigvee\{y_\gamma | \gamma < \beta\}$ if β is a limit ordinal, and $y_\beta = \tilde{y}$ for every ordinal β greater than the closure ordinal β_y of f in y. We first show by transfinite induction that $z \wedge y_\beta = x$ for any ordinal β. Since $y_0 = y$, $z \wedge y_0 = z \wedge y = x$. Now, suppose β is the least ordinal with $z \wedge y_\beta > x$. If β is a successor ordinal, we have $y_{\beta-1} < y_\beta$, and $[y_{\beta-1}, y_\beta] \in C$. Therefore $x < z \wedge y_\beta$ and $[x, z \wedge y_\beta] \in C$, which is a contradiction. If β is a limit ordinal, we have $y_\beta = \bigvee\{y_\gamma | \gamma < \beta\}$, hence $z \wedge y_\beta = z \wedge \bigvee\{y_\gamma | \gamma < \beta\} = \bigvee\{z \wedge y_\gamma | \gamma < \beta\} = \bigvee\{x | \gamma < \beta\} = x$, which is again a contradiction. This completes the induction argument. It is easy to see that $y_\beta \leq \nu_C(y)$ for all ordinals β, thus $\tilde{y} \leq \nu_C(y)$. The element $\tilde{y}$ is, however, a fixpoint of f, and therefore C-good. We may conclude that $\tilde{y} = \nu_C(y)$. It follows that $x = z \wedge \tilde{y} = z \wedge \nu_C(y) = z$.

Proposition 2. *Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$. Then ν_C is an extensive meet-endomorphism of $(L, \leq)$.*

Proof. Extensivity is obvious. To show that ν_C is a meet-endomorphism, we shall use the preceding lemma twice. Clearly $\nu_C(a \wedge b) \leq \nu_C(a) \wedge \nu_C(b)$. Denote $a' = \nu_C(a \wedge b) \vee a$, $b' = \nu_C(a \wedge b) \vee b$. Then $\nu_C(a') = \nu_C(a)$, $\nu_C(b') = \nu_C(b)$ and $a' \wedge b' = \nu_C(a \wedge b)$. First, $\nu_C(a \wedge b) \leq b'$, $\nu_C(b) \wedge a' \leq \nu_C(b) = \nu_C(b')$ and $(\nu_C(b) \wedge a') \wedge b' = \nu_C(b) \wedge \nu_C(a \wedge b) = \nu_C(a \wedge b)$. By the lemma, $\nu_C(b) \wedge a' = \nu_C(a \wedge b)$. Second, $\nu_C(a \wedge b) \leq a'$, $\nu_C(a) \wedge \nu_C(b) \leq \nu_C(a) = \nu_C(a')$, $\nu_C(a) \wedge \nu_C(b) \wedge a' = a' \wedge \nu_C(b) = \nu_C(a \wedge b)$. By the lemma, $\nu_C(a) \wedge \nu_C(b) = \nu_C(a \wedge b)$.

Theorem. *Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$. Then ν_C is the nucleus of* FC(C).

Proof. Immediate and therefore omitted.

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