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ABSTRACT. There are given necessary and sufficient conditions for a subset of a directed partially ordered group G to be a direct factor of G. The main result is formulated in 2.13.

In the paper [6] of M. Kolibiar there are characterized subsets of a directed distributive multilattice group G which are direct factors of G (see 1.3 below). In This note a generalization of this result is proved for a directed partially ordered group and hence also for a directed multilattice group (without the assumption of the distributivity). The used process is a slight modification of that in [6].

1. PRELIMINARIES

Let $(P, \leq)$ be a partially ordered set. For any $a, b \in P$ denote $(a] = \{x \in P : x \leq \leq a\}$, $[a) = \{x \in P : x \geq a\}$, $L(a,b) = (a] \cap (b]$, $U(a,b) = [a) \cap [b)$; if, in addition, $a \leq b$, let $[a, b] = [a) \cap (b]$. Further let $a \vee b$ denote the set of all minimal elements of the set $U(a,b)$, $a \wedge b$ the set of all maximal elements of the set $L(a,b)$.

A subset A of P is said to be directed if $U(a,b) \cap A \neq \emptyset$ and $L(a,b) \cap A \neq \emptyset$ for all $a, b \in A$. A is called convex if $[a,b] \subseteq A$ whenever $a, b \in A$, $a \leq b$.

A partially ordered set $(P, \leq)$ is said to be a multilattice if for all $a, b \in P$, $h \in U(a,b)$, there exists an element $h_1 \in (a \vee b) \cap (h]$, and dually. If, moreover, $(P, \leq)$ is a directed set, then $(P, \leq)$ is called a directed multilattice.

We say that a partially ordered set $(P, \leq)$ with the least element 0 is an inner direct product of its subsets A, B, and we write $P = A \cdot B$, if there exists a direct product decomposition $f : P \cong A_1 \times B_1$ with $A = f^{-1}(\{(a, 0) : a \in A_1\})$, $B = f^{-1}(\{(0, b) : b \in B_1\})$. (For characterizations of inner direct products see [4] and [6].)

In what follows we will deal with directed partially ordered groups and with directed multilattice groups, as their special case.

By a directed group we will mean a partially ordered group $(G, +, \leq)$ with $(G, \leq)$ directed.

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A partially ordered group $(G, +, \leq)$ is said to be a directed multilattice group if the partially ordered set $(G, \leq)$ is a directed multilattice. (For the definition and fundamental properties of a partially ordered group and a multilattice group see e.g. [3] and [1], respectively.)

For a subset A of a partially ordered group G denote by A^+ and A^- the set $\{a \in A : a \geq 0\}$ and $\{a \in A : a \leq 0\}$, respectively.

By a direct factor of a partially ordered group G we mean a subset C of G such that there exists a direct product decomposition $f : G \cong K \times L$ satisfying $f^{-1}(\{(a, 0) : a \in K\}) = C$.

We shall use the following two theorems.

1.1. Theorem. [6; 2.2.A, 2.3]. *There is a bijective correspondence between direct product decompositions of a quasi-ordered set P into two factors and pairs of equivalence relations θ_1, θ_2 in P, satisfying the conditions:*

(*i*) $\theta_1 \cap \theta_2 = id_P$.

(*ii*) $\theta_1 \vee \theta_2 = P \times P$.

(*iii*) θ_1, θ_2 *are permutable.*

(*iv*) $a\theta_i c, c\theta_j b,\ i \neq j\ (i, j \in \{1, 2\})$ *and* $a \leq b$ *imply* $a \leq c \leq b$.

(*v*) *If* $a \leq b, a\theta_i a' (i \in \{1, 2\})$, *then* b' *exists satisfying* $a' \leq b', b\theta_i b'$.

The correspondence is as follows. If $f : P \cong A_1 \times A_2$ is a direct product decomposition, then θ_i $(i = 1, 2)$ defined by $a\theta_i b \Leftrightarrow \pi_i(f(a)) = \pi_i(f(b))$ (π_i is the projection $A_1 \times A_2 \to A_i$) are equivalence relations satisfying $(i) - (v)$. If θ_1, θ_2 are equiovalence relations satisfying $(i) - (v)$, then the map $a \longmapsto ([a]\theta_1, [a]\theta_2)$ is an isomorphism of P onto $P/\theta_1 \times P/\theta_2$.

1.2. Theorem. [4; Theorem 2]. *Let G be a directed group and let $(G^+, \leq)$ be an inner direct product of its subsets A and B, $G^+ = A \cdot B$. Then there exists a direct product decomposition $G \cong K \times L$ such that K, L are partially ordered subgroups of G and $K^+ = A$, $L^+ = B$.*

It is clear from the proof that the mentioned direct product decomposition $f : G \cong K \times L$ satisfies $f^{-1}(\{(k, 0) : k \in K\}) = K$.

In [6] the following theorem is proved.

1.3. Theorem. *Let G be a direct distributive multilattice group. A subset C of G forms a direct factor of G if and only if it satisfies the following conditions:*

(1) $(C, +)$ *is a subgroup of* $(G, +)$.

(2) C *is convex and connected in* $(G, \leq)$.

(3) *For each* $a \in G^+$ *the set* $C \cap [0, a]$ *has the greatest element* .

2. The main Result

Throughout this section, G will be a directed group. Consider the following conditions for a subset C of G:

(1) $(C,+)$ is a subgroup of $(G,+)$.

(2) C is convex and directed in $(G,\leq)$.

(3) For each $a\in G^+$ the set $C\cap[0,a]$ has the greatest element $a(C)$.

(4) For all $a,b\in G^+$ satisfying $a(C)=b(C)$, if $v\in U(a,b)$, then there exists $v_1\in U(a,b)$ with $v_1\leq v, v_1(C)=a(C)$.

We will prove that C is a direct factor of G if and only if C satisfies (1) - (4). Before proving this, we shall make some preliminary considerations assuming that C is a subset of G satisfying (1)-(4). Let us define binary relations θ,ϕ in G^+ as follows:

$$a\theta b \text{ if } a(C)=b(C),$$
$$a\phi b \text{ if } a-b\in C.$$

It is easy to see that θ,ϕ are equivalence relations and that all θ-classes and ϕ-classes are convex subsets of G^+. The aim is to prove that θ and ϕ satisfy the conditions (i) - (v) of 1.1. We begin with some auxiliary lemmas.

2.1. Lemma. *If $a,b\in C$, $t\in U(a,b)$, then there exists $t_1\in C\cap U(a,b)$ with $t_1\leq t$.*

Proof. Let $a,b\in C$, $t\in U(a,b)$. By the directedness of C there exist elements $c_1,c_2\in C$ such that $c_1\leq a\leq c_2$, $c_1\leq b\leq c_2$. It is easy to verify, using the conditions (1) and (3), that the element $t_1=(t-c_1)(C)+c_1$ is the greatest element of the set $C\cap[c_1,t]$. Evidently $t_1\in C$, $t_1\leq t$. Since $a,b\in C\cap[c_1,t]$, we have $a,b\leq t_1$.

2.2. Lemma. *If $a,b\in C$, $r\in L(a,b)$, then there exists $r_1\in C\cap L(a,b)$ with $r_1\geq r$.*

Proof. Let $a,b\in C$, $r\in L(a,b)$. It is easy to see that $t=a-r+b\in U(a,b)$ and by the preceding lemma there exists $t_1\in C\cap U(a,b)$ with $t_1\leq t$. Set $r_1=b-t_1+a$. Then evidently $r_1\in C\cap L(a,b)$, $r_1\geq r$.

2.3. Lemma. *If $a\in G^+$, then $a(C)$ is the greatest element of $C\cap(a]$.*

Proof. Evidently $a(C)\in C\cap(a]$. Now let $x\in C\cap(a]$. We will prove that $x\leq a(C)$. Since $0,x\in C$, $a\in U(0,x)$, there exists $a_1\in C\cap U(0,x)$ with $a_1\leq a$, by 2.1. Then $a_1\in C\cap[0,a]$, so that $a_1\leq a(C)$. We have $x\leq a_1\leq a(C)$.

2.4. Lemma. *Let $a, b \in G^+$, $e = a - b \in C$. Then $e = a(C) - b(C)$*

Proof. We have $-e + a(C) = b - a + a(C) \leq b$, so that $-e + a(C) \in C \cap (b]$. Using the preceding lemma we obtain $-e + a(C) \leq b(C)$, which gives $a(C) \leq e + b(C)$. Further $e + b(C) \leq e + b = a$, so that $e + b(C) \in C \cap (a]$. Using 2.3 once more we get $e + b(C) \leq a(C)$. We have proved $a(C) = e + b(C)$. Hence $e = a(C) - b(C)$.

Analogously can be proved:

2.5. Lemma. *If $a, b \in G^+$ and $e = -a + b \in C$, then $e = -a(C) + b(C)$.*

2.6. Lemma. *If $a, b \in G^+$, $a\phi b$ and $u \in L(a, b)$, $u \geq 0$, then there exists $u_1 \in$ $\in L(a, b)$ such that $u_1 \geq u$, $u_1 \phi a$.*

Proof. Let $a, b \in G^+$, $a\phi b$, $u \in L(a, b)$, $u \geq 0$. It is easy to see that $u - b \in L(a-b, 0)$, where $a - b$, $0 \in C$. By 2.2 there exists $r_1 \in C \cap L(a - b, 0)$ with $r_1 \geq u - b$. Then $u_1 = r_1 + b \in L(a, b)$, $u_1 \geq u$ and evidently $u_1 \phi b$, which follows $u_1 \phi a$, too.

2.7. Lemma. *If $a, b \in G^+$, $a\phi b$ and $v \in U(a, b)$, then there exists $v_1 \in U(a, b)$ such that $v_1 \leq v$, $v_1 \phi a$.*

Proof. Let $a, b \in G^+$, $a\phi b$, and let $v \in U(a, b)$. Then $v - b \in U(a - b, 0)$, where $a - b$, $0 \in C$. By 2.1 there exists $t_1 \in C \cap U(a - b, 0)$ with $t_1 \leq v - b$. Then $v_1 = t_1 + b$ has the required properties.

Now we proceed to the proof that θ and ϕ satisfy the conditions (i) - (v) of 1.1.

2.8. Lemma. *$\theta \cap \phi$ is the least equivalence relation in G^+.*

Proof. Let $a\theta \cap \phi b$. Then $a(C) = b(C)$ and $e = a - b \in C$. Using 2.4 we get $e = a(C) - b(C) = 0$, hence $a = b$.

2.9. Lemma. *$\theta \vee \phi = G^+ \times G^+$.*

Proof. If $(a, b) \in G^+ \times G^+$, then evidently $a\theta a(C)\phi b(C)\theta b$.

2.10. Lemma. *θ, ϕ are permutable.*

Proof. First we show that $\theta \cdot \phi \leq \phi \cdot \theta$. Let $a\theta \cdot \phi b$ for some $a, b \in G^+$. Then there exists $t \in G^+$ with $a\theta t\phi b$. Set $e = b - t$. Since $e \in C$, we have $e = b(C) - t(C)$, by 2.4. Now $a(C) = t(C)$, so that $e = b(C) - a(C)$. Consider the element $e + a$. We are going to prove that $a\phi e + a\theta b$. Evidently $0 \leq b(C) = e + a(C) \leq e + a$, so that $e + a(C) \in C \cap [0, e + a]$. If $x \in C \cap [0, e + a]$, then $-e + x \in C \cap (a]$, which yields $-e + x \leq a(C)$, by 2.3. Hence $x \leq e + a(C)$. We have proved that

$e + a(C)$ is the greatest element of $C \cap [0, e + a]$, i.e $e + a(C) = (e + a)(C)$. We get $b(C) = e + a(C) = (e + a)(C)$, which implies $b\theta e + a$. The relation $a\phi e + a$ is trivial.

To show that $\phi \cdot \theta \leq \theta \cdot \phi$, let $a\phi \cdot \theta b$. Then $b\theta \cdot \phi a$, by the symmetry of ϕ and θ. Using the inequality proved above and the symmetry of ϕ and θ once more, we obtain $a\theta \cdot \phi a$.

2.11. Lemma. *θ, ϕ fulfil the condition (iv) of 1.1.*

Proof. First we show that if $a\theta c\phi b$ for some $a, c, b \in G^+$, $a \leq b$, then $a \leq c \leq b$. Let $a\theta c\phi b$, $a \leq b$. An application of directedness of G and 2.7 yields the existence of an element $v \in U(b, c)$ with $v\phi c$. since $a(C) = c(C)$, there exists $v_1 \in U(a, c)$ such that $v_1 \leq v$, $v_1(C) = c(C)$, by the property (4) of C. In view of the convexity of ϕ-classes we have $v_1\theta \cap \phi c$ and this implies $v_1 = c$, by 2.8. We have proved that $a \leq c$. To show that $c \leq b$, take an element $d \in L(b, c)$ such that $d \geq a$, $d\phi c$. The existence of such an element is guaranteed by 2.6. The convexity of θ-classes yields $d\theta c$. We have again $d\theta \cap \phi c$, hence $c = d \leq b$.

Next we prove that $a\phi c\theta b$, $a \leq b$ imply $a \leq c \leq b$. Supppose that $a\phi c\theta b$, $a \leq b$ for some $a, c, b \in G^+$. Using the properties (2) and (4) of C we obtain that there exists $v \in U(b, c)$ with $v(C) = c(C)$. In view of 2.7 there exists $v_1 \in U(a, c)$ such that $v_1 \leq v$, $v_1\phi c$. On the other hand the relation $v(C) = c(C)$ gives $v\theta c$ and this together with the convexity of θ-classes yields $v_1\theta c$. We have $v_1\theta \cap \phi c$, which implies $c = v_1$. Hence $a \leq c$. Obviously $c - a + b \in U(b, c)$ and since $b\theta c$, there exists $s \in U(b, c)$ such that $s \leq c - a + b$, $s(C) = b(C)$, by (4). Further $b - s + c \in$ $\in L(b, c) \cap [a)$ and hence $b - s + c\phi c$, by convexity of ϕ-classes. But then also $b\phi s$, because $b - s = (b - s + c) - c \in C$. We have $b\theta \cap \phi s$, hence $b = s \geq c$.

2.12. Lemma. *θ, ϕ fulfil the condition (v) of 1.1.*

Proof. First let $a \leq b$, $a\theta a'$ for some $a, b, a' \in G^+$. We are going to prove that there exists $b' \in G^+$ such that $a' \leq b'$, $b\theta b'$. Put $b' = a' - a'(C) + b(C)$. It is easy to see that $a' \leq b'$. Set $e = -a' + b'$. Then the definition of b' gives $e = -a'(C) + b(C) \in C$ and 2.5 ensures $e = -a'(C) + b'(C)$. Therefore $-a'(C) + b(C) = -a'(C) + b'(C)$, which follows $b\theta b'$.

Now let $a \leq b$, $a\phi a'$ fo some $a, b, a' \in G^+$. We are seeking for b' with $a' \leq b'$, $b\phi b'$. Put $b' = a' - a + b$. Evidently $a' \leq b'$ and $b\phi b'$, because $b' - b = a' - a \in C$.

We are ready to prove the main theorem.

2.13. Theorem. *Let G be a directed group, C a subset of G. Then C is a direct factor of G if and only if C fulfils the conditions (1) - (4) mentioned at the beginning of this section.*

Proof. If C is a direct factor of G, we can suppose that $G = C' \times D'$, $C = \{(x, 0) :$

$x \in C'\}$. It is easy to see that C fulfils (1)-(4). (If $a = (x, y) \in G^+$, then the greatest element $a(C)$ of $C \cap [0, a]$ is $(x, 0)$.)

Now let C fulfil (1)-(4). Considering 2.8 – 2.12 we can say that the equivalence relations θ and ϕ satisfy the conditions of Theorem 1.1, so that the map f assigning to any $a \in G^+$ the couple $([a]\theta, [a]\phi)$ is an isomorphism of $(G^+, \leq)$ onto $(G^+/\theta, \leq \leq) \times (G^+/\phi, \leq)$. In accordance with the definition of the inner product given in the section 1, $G^+ = f^{-1}(\{([a]\theta, [0]\phi) : a \in G^+\}) \cdot f^{-1}(\{([0]\theta, [a]\phi) : a \in G^+\})$. There is $f^{-1}(\{([0]\theta, [a]\phi) : a \in G^+\}) = \{x \in G^+ : [x]\theta = [0]\theta\} = [0]\theta$ and analogously $f^{-1}(\{([a]\theta, [0]\phi) : a \in G^+\}) = [0]\phi$. But $[0]\phi = \{x \in G^+ : x\phi 0\} = \{x \in G^+ : x \in \in C\} = C^+$. Hence $G^+ = C^+ \cdot [0]\theta$. By Theorem 1.2 there exists a direct product decomposition $(G, +, \leq) \cong (K, +, \leq) \times (L, +, \leq)$ with K, L being partially ordered subgroups of G satisfying $K^+ = C^+$, $L^+ = [0]\theta$. It is easy to see that K, L are directed groups. Finally we are going to show that $K = C$. From $K^+ = C^+$ we get immediately $K^- = C^-$. Now let a be any element of C, $r \in C \cap L(a, 0)$. (The existence of such an element r is guaranteed by (2) and 2.2.). Hence $r \in C^- = K^-$. Further evidently $t = a - r \in C \cap U(a, 0)$, hence $t \in C^+ = K^+$. We have $a = t + r$ with $t \in K^+$, $r \in K^-$, so that $a \in K$. We have proved $C \subseteq K$. Since K, as a direct factor of G, also satisfies (1)-(4), the inclusion $K \subseteq C$ can be proved anlogously. Hence $C = K$ is a direct factor of G. The proof is finished.

3. Direct factors of directed multilattice groups

In this section G will be any directed multilattice group. Besides (1)-(4), consider also the following conditions for a subset C of G.

(5) For all $a, b \in G^+$, the relations $u \in a \vee b$, $a(C) = b(C)$ imply $u(C) = a(C)$.
(6) For all $a, b \in G^+$, $u \in a \vee b$ implies $u(C) \in a(C) \vee b(C)$.

3.1. Theorem. *Let G be a directed multilattice group, C a subset of G. Then the following conditions are equivalent:*
(I) C is a direct factor of G, i.e. C fulfils (1)-(4).
(II) C fulfils (1)-(3) and (6).
(III) C fulfils (1)-(3) and (5).

Proof. It is easy to verify that if C is a direct factor of G, then C fulfils (1)-(3) and (6).

Evidently (6) implies (5), so that (II) implies (III).

Finally, if C fulfils (5) and we have $a, b \in G^+$ satisfying $a(C) = b(C)$ and $v \in U(a, b)$, then $v_1 \in a \vee b$ with $v_1 \leq v$ fulfils $v_1(C) = a(C)$. Hence (III) implies (I).

Remark. It is easy to verify that the condition of connectedness in (2) of 1.3 can be replaced by the condition of directedness. Taking this into consideration and comparing 1.3 and 3.1 we can see that the distributivity of a directed multilattice group G yields that if a subset C of G fulfils (1)-(3), then it fulfils also (4), (5) and (6).

The following two examples are illustrative. 3.2 shows that the condition (5) (and hence also (4) and (6)) does not follow from (1)-(3), in general.

3.2. Example. Let $G = \{(a,b) \in Z \times Z : a+b \text{ even}\}$, where Z is the additive group of all integers with the natural order. Define in G the operation $+$ and the order relation $\leq$ componentwise. Evidently $(G,+,\leq)$ is a directed multilattice group. We will show that G has no direct factors, but $\{(0,0)\}$ and G. Firstly, the subset $C_1 = \{(a,0) : a \text{ even}\}$ of G is not a direct factor; it satisfies (1)-(3), but it does not satisfy (5). Indeed, if we set $a = (1,1)$, $b = (0,2)$, $u = (2,2)$, then $a,b,u \in G^+$, $u \in a \vee b$, $a(C_1) = b(C_1) = (0,0)$, but $u(C_1) = (2,0)$. Analogously $C_2 = \{(0,b) : b \text{ even}\}$ is not a direct factor of G. Now let us suppose that C is a subset of G satisfying (1)-(3) and (5), $C \neq \{(0,0)\}$. Then one of the three following possibilities occurs:1) $C \subseteq C_1$, 2) $C \subseteq C_2$, 3) there exists $(a,b) \in C$ with $a \neq 0$, $b \neq 0$. If $C \subseteq C_1$, then there exists $(a,0) \in C$ with $a > 0$, which follows $(2,0) \in C$, because $(-a,0) \leq (2,0) \leq (a,0)$. Hence in the case $C \subseteq C_1$ we have $C = C_1$, a contradiction. The seconde possibility can be excluded analogously. Let $(a,b) \in C$, $a \neq 0$, $b \neq 0$. Since C is directed, there exists $(u,v) \in C$ such that $(u,v) \geq (a,b)$, $(u,v) \geq (-a,-b)$. Hence we have $(u,v) \in C$ with $u > 0$, $v > 0$. Then $(-u,-v) \leq (1,1) \leq (u,v)$, which implies $(1,1) \in C$. Consequently $(t,t) \in C$ for each $t \in Z$. Given any $(r,s) \in G$ we have $(-p,-p) \leq (r,s) \leq (p,p)$ for $p = \max(|r|,|s|)$, so that $(r,s) \in C$. We have proved $C = G$.

3.3. Example. Let $G = \{(a,b,c) \in Z \times Z \times Z : b+c \text{ even}\}$ and let the operation $+$ and the order relation $\leq$ be defined componentwise. We obtain a directed multilattice group with nontrivial direct factors $\{(a,0,0) : a \in Z\}$, $\{(0,b,c) : b+c \text{ even}\}$.

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