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NATURAL AFFINORS ON TIME–DEPENDENT WEIL BUNDLES

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ABSTRACT. We determine all natural affinors on the product manifolds $T^A M \times \mathbb{R}$, where T^A is the Weil functor corresponding to an arbitrary local algebra A.

It is well-known that there is a natural tensor field of type (1,1) (in other words: an affinor) on the tangent bundle TM of an arbitrary manifold M, which characterizes the canonical almost tangent structure of TM and plays a significant role in the autonomous Lagrangian dynamics. M. de León and P. R. Rodrigues have recently pointed out the importance of the tensor product $L_M \otimes dt$ of the classical Liouville vector field L_M on TM with the canonical 1-form dt of $\mathbb{R}$, $t \in \mathbb{R}$, in the non-autonomous dynamics, [6]. Obviously, $L_M \otimes dt$ is a natural affinor on $TM \times \mathbb{R}$. On the other hand, M. Modugno and the second author determined all natural affinors on an arbitrary Weil bundle $T^A M$, the tangent bundle TM being the simpliest special case, [3]. Thus, in connection with a current research by A. Vondra, [7], there appeared the general problem of finding all natural affinors on the products $T^A M \times \mathbb{R}$. The complete list of them is given in item 5 of the present paper.

All manifolds and maps are assumed to be infinitely differentiable.

1. Let A be a local algebra in the sense of A. Weil,[8]. (We recall that A can be defined as a factor algebra $\mathbb{R}[x_1, \dots, x_k]/\mathbb{A}$, where $\mathbb{A}$ is an ideal of finite codimension in the real polynomial ring with k undetermined.) By [8] or [1], A induces a functor T^A, called the Weil functor corresponding to A, transforming every manifold M into a fibre bundle $T^A M \to M$ and every smooth map $f : M \to N$ into a morphism of fibre bundles $T^A f$; $T^A M \to T^A N$ over f. The velocities functor T^r_k, with the tangent functor $T = T^1_1$ at the first place, are the most familiar examples of Weil functors, [1].

The role of the product $TM \times \mathbb{R}$ in the non-autonomous dynamics leads us to the following general concept.

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Definition. The time-dependent Weil functor $T^A_{\mathbb{R}}$ corresponding to a local algebra A is defined by $T^A_{\mathbb{R}} = T^A M \times \mathbb{R}$ for every manifold M and by $T^A_{\mathbb{R}} f = T^A f \times id_{\mathbb{R}}$ for every smooth map f.

2. We recall that an affinor on a manifold M means a tensor field of type (1,1) on M, which can be interpreted as a linear endomorphism of TM. Let F be a natural bundle over m-manifolds, see e.g. [4]. By [3], a natural affinor Q on F is a system of affinors $Q_M : TFM \to TFM$ for every m-manifold M satisfying

$$TFf \circ Q_M = Q_N \circ TFf$$

for every local diffeomorphism $f : M \to N$.

The restriction of $T^A_{\mathbb{R}}$ to the category of all m-manifolds and their local diffeomorphisms is a natural bundle over m-manifolds, which will be called the natural m-bundle $T^A_{\mathbb{R}}$. In the same way one defines the natural m-bundle T^A.

Now we can formulate precisely our problem: Find all natural affinors on natural m-bundle $T^A_{\mathbb{R}}$.

3. Every element $a \in A$ induces a natural affinor $Q(a)$ on natural m-bundle T^A as follows. The multiplication of tangent vectors by reals is a map $\mu_M : \mathbb{R} \times TM \to TM$. Applying functor T^A, we obtain $T^A \mu_M : T^A\mathbb{R} \times T^A TM \to T^A TM$. But $T^A \mathbb{R} = A$ and there is a canonical exchange map $T^A TM \approx TT^A M$. Hence $T^A \mu_M$ can be interpreted as a map $A \times TT^A M \to TT^A M$. The restriction of the latter map to $a \in A$ defines the natural affinor $Q(a)_M$,[2].

M. Modugno and the second author deduced that all natural affinors on natural m-bundle T^A are of the form $Q(a)$, $a \in A$, [3]. Obviously, $Q(a)$ induces a natural affinor $\tilde{Q}(a)$ by means of the product structure on $T^A M \times \mathbb{R}$. Quite similarly, the identity of $T\mathbb{R}$ defines another natural affinor $\widetilde{id}_{T\mathbb{R}}$ on $T^A_{\mathbb{R}}$.

4. Let X be a vector field on $T^A M$ and dt be the canonical 1-form on $\mathbb{R}$, $t \in \mathbb{R}$. Then the tensor product $X \otimes dt$ defines an affinor on $T^A M \times \mathbb{R}$.

Reformulating Definition 3 from [2], we can define an absolute vector field Y on a natural bundle F over m-manifolds as a system Y_M of vector fields on FM for every m-manifold M satisfying $TFf \circ Y_M = Y_N \circ Ff$ for every local diffeomorphism $f : M \to N$. For example, the Liouville vector field L is an absolute vector field on natural m-bundle T. Let $DerA$ denote the space of all derivations of algebra A. Every element $D \in DerA$ determines an absolute vector field $\tilde{D}$ on natural m-bundle T^A as follows, [2]. There is an identification of $DerA$ with the Lie algebra of the Lie group $AutA$ of all automorphisms of A. Hence D is of the form $\left.\frac{d}{dt}\right|_0 \delta(t)$, where $\delta(t)$ is a curve on $AutA$. By [8] or [1], every $\delta(t)$ determines a natural transformation $\tilde{\delta}(t)_M : T^A M \to T^A M$ and we define $\tilde{D}_M = \left.\frac{d}{dt}\right|_0 \tilde{\delta}(t)_M$. Proposition 1 from [2] reads that all absolute vector fields on T^A are of the form $\tilde{D}$, $D \in DerA$.

Thus, the tensor products $\widetilde{D}_M \otimes dt$ define a natural affinor $\tilde{D} \otimes dt$ on $T^A_{\mathbb{R}}$ for every $D \in DerA$.

5. For the sake of simplicity we shall not distinguish between a real function $\mathbb{R} \to \mathbb{R}$ and its pullback $T^A M \times \mathbb{R} \to \mathbb{R}$ in what follows.

Theorem. *All natural affinors on natural m-bundle $T^A_{\mathbb{R}}$ are linear combinations of*

$$\begin{aligned} (i) &\quad \tilde{Q}(a), \qquad a \in A, \\ (ii) &\quad \tilde{D} \otimes dt, \quad D \in DerA, \\ (iii) &\quad \widetilde{id}_{T\mathbb{R}}, \end{aligned}$$

the coefficients of which are arbitrary smooth functions on $\mathbb{R}$.

Proof. By the general theory, see e.g. [4], it suffices to study the equivariant maps of the standard fibre of $TT^A_{\mathbb{R}}\mathbb{R}^m$ over $0 \in \mathbb{R}^m$ into inself. Write $x \in \mathbb{R}^m$, $y \in T^A_0\mathbb{R}^m$, $t \in \mathbb{R}$, $X = dx$, $Y = dy$, $T = dt$. Then any linear map of $(TT^A_{\mathbb{R}}\mathbb{R}^m)_0$ into inself has the following form

$$\begin{aligned} \bar{X} &= a(y,t)X + b(y,t)Y + c(y,t)T \\ \bar{Y} &= d(y,t)X + e(y,t)Y + f(y,t)T \\ \bar{T} &= g(y,t)X + h(y,t)Y + l(y,t)T \end{aligned} \tag{1}$$

with arbitrary smooth maps $a, \dots, l$ on $T^A_0\mathbb{R}^m \times \mathbb{R}$ valued in the corresponding spaces of linear maps.

I. Consider first the equivariancy of the last row of (1) with respect to the homotheties $\bar{x} = kx$, $0 \neq k \in \mathbb{R}$. Analogously to [2], we obtain

$$\begin{aligned} & g(y,t)X + h(y,t)Y + l(y,t)T = \\ & = g(ky,t)kX + h(ky,t)kY + l(ky,t)T \end{aligned} \tag{2}$$

i.e $g(y,t) = kg(ky,t)$ $h(y,t) = kh(ky,t)$, $l(y,t) = l(ky,t)$. Setting $k \to 0$, the first two relations yield $g = 0$, $h = 0$, while the third one implies $l = \lambda(t)$, where λ is an arbitrary smooth function of one variable. Thus, we have

$$\bar{T} = \lambda(t)T \tag{3}$$

This corresponds to (iii).

II. For $T = 0$, (1) with (3) represent a natural affinor on $T^A M$ for every $t \in \mathbb{R}$. Using item 3 we find that $a(y,t)$, $b(y,t)$, $d(y,t)$ and $e(y,t)$ correspond to (i).

III. Consider (1) diminished by (3) and by the pullback of the results of II. This is an expression of the form $\bar{X} = c(y,t)T$, $\bar{Y} = f(y,t)T$, i. e. $V(t) \otimes dt$, where

$V(t)$ is a vector field on $T_0^A\mathbb{R}^m$ for every $t \in \mathbb{R}$. Since our affinor is natural, every $V(t)$ corresponds to an absolute vector field on T^A, [2]. Using item 4 we prove our Theorem. □

6. We are going to discuss the case of the tangent functor in detail. Writing $x = (x^i) \in \mathbb{R}^m$, we denote by $y^i = dx^i$ the induced coordinates on $T\mathbb{R}^m$ and by $X^i = dx^i$, $Y^i = dy^i$ the additional coordinates on $TT\mathbb{R}^m$. The coordinate expression of the classical natural affinor on TM mentioned in the introduction is

$$\bar{X}^i = 0, \quad \bar{Y}^i = X^i \tag{4}$$

In [3] it is proved that all natural affinors on T form a 2-parameter system linearly generated by the identity affinor and (4). By [2], the absolute vector fields on T are the constant multiples of the Liouville vector field L, the coordinate expression of which is $y^i \frac{\partial}{\partial y^i}$. Hence the coordinate form of all natural affinors on $TM \times \mathbb{R}$ is

$$\begin{aligned} \bar{X}^i &= \varphi(t) X^i \\ \bar{Y}^i &= \psi(t) X^i + \varphi(t) Y^i + \mu(t) y^i T \\ \bar{T} &= \lambda(t) T \end{aligned} \tag{5}$$

with arbitrary smooth functions $\lambda, \mu, \varphi, \psi$ of one variable. We remark that a useful exercise is to derive (5) by direct evaluation of the equivariancy conditions in question.

7. In [3] it was clarified that the natural affinors play a significant role in the theory of torsions of arbitrary connections on natural bundles. The torsion of a connection is defined as the Frölicher - Nijenhuis bracket of the natural affinor with the connection itself. Hence our complete list of all natural affinors on the time-dependent Weil bundles could be useful for a further research in such a direction.

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